Solving the Vasicek model for reversion to the mean of interest rates.

Reminder: Ito Lemma: If
\[ dX = a(X, t)dt + b(X, t)dW \]
Then
\[ dg(X, t) = \left( ag_x + \frac{1}{2} b^2 g_{xx} + g_t \right) dt + bg_x dW . \]

The Vasicek model is
\[ dX = \alpha(r - X)dt + sdW \]
Look at \( g(X, t) = e^{\alpha t}(X - r) \). From Ito:
\[ dg = (\alpha(r - X)e^{\alpha t} + \alpha e^{\alpha t}(X - r))dt + se^{\alpha t}dW = se^{\alpha t}dW . \]

Integrating, we have
\[
e^{\alpha t}(X(t) - r) - (X(0) - r) = s \int_0^t e^{\alpha u}dW(u) \\
= s \left( e^{\alpha t}W(t) - \alpha \int_0^t W(u)e^{\alpha u}du \right) .
\]

To get the last line we have used Ito again, with “\( g \)” equal to \( e^{\alpha t}W \) (and \( X = W \)). Rearranging gives
\[ X(t) = r + (X(0) - r)e^{-\alpha t} + s \left( W(t) - \alpha \int_0^t W(u)e^{-\alpha(t-u)}du \right) . \]

Notes:

1. There is a problem in this model that \( X \) can become negative.

2. Everyone not in finance calls the process the “Ohrnstein-Uhlenbeck” process — it has many applications outside finance.