Chapter 1

Differential Equations in Economics

Applications of differential equations are now used in modeling motion and change in all areas of science. The theory of differential equations has become an essential tool of economic analysis particularly since computer has become commonly available. It would be difficult to comprehend the contemporary literature of economics if one does not understand basic concepts (such as bifurcations and chaos) and results of modern theory of differential equations.

A differential equation expresses the rate of change of the current state as a function of the current state. A simple illustration of this type of dependence is changes of the Gross Domestic Product (GDP) over time. Consider state x of the GDP of the economy. The rate of change of the GDP is proportional to the current GDP

$$\dot{x}(t) = gx(t),$$

where t stands for time and $\dot{x}(t)$ the derivative of the function x with respect to t. The growth rate of the GDP is $\dot{x}/x$. If the growth rate g is given at any time t, the GDP at t is given by solving the differential equation. The solution is

$$x(t) = x(0)e^{gt}.$$

The solution tells that the GDP decays (increases) exponentially in time when g is negative (positive).

We can explicitly solve the above differential function when g is a constant. It is reasonable to consider that the growth rate is affected by many factors, such as the current state of the economic system, accumulated knowledge of the economy, international environment, and many other conditions. This means that the growth rate may take on a complicated form $g(x,t)$. The economic growth is described by

$$\dot{x}(t) = g(x(t),t)x(t).$$

In general, it is not easy to explicitly solve the above function. There are various established methods of solving different types of differential equations. This book introduces concepts, theorems, and methods in differential equation theory which are widely used in contemporary economic analysis and provides many simple as well as comprehensive applications to different fields in economics.

This book is mainly concerned with *ordinary differential equations*. Ordinary differential equations are differential equations whose solutions are functions of one independent variable, which we usually denote by t. The variable t often stands for time, and solution we are looking for, $x(t)$, usually stands for some economic quantity that changes with time. Therefore we consider $x(t)$ as a dependent variable. For instance, $\dot{x}(t) = t^2 x(t)$ is an ordinary differential equation. Ordinary differential equations are classified as autonomous and nonautonomous. The equation

$$\dot{x}(t) = ax(t) + b,$$

with a and b as parameters is an *autonomous differential equation* because the time variable t does not explicitly appear. If the equation specially involves t, we call the equation *nonautonomous* or *time-dependent*. For instance,

$$\dot{x}(t) = x(t) + \sin t,$$

is a nonautonomous differential equation. In this book, we often omit "ordinary", "autonomous", or "nonautonomous" in expression. If an equation involves derivatives up to and includes the ith derivative, it is called an *ith order* differential equation. The equation $\dot{x}(t) = ax(t) + b$ with a and b as parameters is a first order autonomous differential equation. The equation

$$\ddot{x} = 3\dot{x} - 2x + 2,$$

is a second order equation, where the second derivative, $\ddot{x}(t)$, is the derivative of $\dot{x}(t)$.[1] As shown late, the solution is

$$x(t) = A_1 e^{2t} + A_2 e^{t} + 1,$$

where A_1 and A_2 are two constants of integration. The first derivative $\dot{x}$ is the only one that can appear in a first order differential equation, but it may enter in various powers: $\dot{x}$, $\dot{x}^2$, and so on. The highest power attained by the derivative in the equation is referred to as the *degree* of the differential equation. For instance,

$$3\dot{x}^2 - 2x + 2 = 0$$

is a second-degree first-order differential equation.

1.1 Differential Equations and Economic Analysis

This book is a unique blend of the theory of differential equations and their exciting applications to economics. First, it provides a comprehensive introduction to most important concepts and theorems in differential equations theory in a way that can be understood by anyone who has basic knowledge of calculus and linear algebra. In addition to traditional applications of the theory to economic dynamics, this book also contains many recent developments in different fields of economics. The book is mainly concerned with how differential equations can be applied to solve and provide insights into economic dynamics. We emphasize "skills" for application. When applying the theory to economics, we outline the economic problem to be solved and then derive differential equation(s) for this problem. These equations are then analyzed and/or simulated.

Different from most standard textbooks on mathematical economics, we use computer simulation to demonstrate motion of economic systems. A large fraction of examples in this book are simulated with Mathematica. Today, more and more researchers and educators are using computer tools such as Mathematica to solve – once seemingly

[1] The *n*th *derivative* of $x(t)$, denoted by $x^{(n)}(t)$, is the derivative of $x^{(n-1)}(t)$.

impossible to calculate even three decades ago – complicated and tedious problems.

This book provides not only a comprehensive introduction to applications of linear and linearized differential equation theory to economic analysis, but also studies nonlinear dynamical systems which have been widely applied to economic analysis only in recent years. Linearity means that the rule that determines what a piece of a system is going to do next is not influenced by what it is doing now. The mathematics of linear systems exhibits a simple geometry. The simplicity allows us to capture the essence of the problem. Nonlinear dynamics is concerned with the study of systems whose time evolution equations are nonlinear. If a parameter that describes a linear system, is changed, the qualitative nature of the behavior remains the same. But for nonlinear systems, a small change in a parameter can lead to sudden and dramatic changes in both the quantitative and qualitative behavior of the system.

Nonlinear dynamical theory reveals how such interactions can bring about qualitatively new structures and how the whole is related to and different from its individual components. The study of nonlinear dynamical theory has been enhanced with developments in computer technology. A modern computer can explore a far wider class of phenomena than it could have been imagined even a few decades ago. The essential ideas about complexity have found wide applications among a wide range of scientific disciplines, including physics, biology, ecology, psychology, cognitive science, economics and sociology. Many complex systems constructed in those scientific areas have been found to share many common properties. The great variety of applied fields manifests a possibly unifying methodological factor in the sciences. Nonlinear theory is bringing scientists closer as they explore common structures of different systems. It offers scientists a new tool for exploring and modeling the complexity of nature and society. The new techniques and concepts provide powerful methods for modeling and simulating trajectories of sudden and irreversible change in social and natural systems.

Modern nonlinear theory begins with Poincaré who revolutionized the study of nonlinear differential equations by introducing the qualitative techniques of geometry and topology rather than strict analytic methods to discuss the global properties of solutions of these systems. He considered it more important to have a global understanding of the gross behavior of all solutions of the system than the local behavior of particular, analytically precise solutions. The study of the dynamic systems was furthered in the Soviet Union, by mathematicians such as Liapunov, Pontryagin, Andronov, and others. Around 1960, the study by Smale in the United States, Peixoto in Brazil and Kolmogorov, Arnol'd and Sinai in the Soviet gave a significant influence on the development of nonlinear theory. Around 1975, many scientists around the world were suddenly aware that there is a new kind of motion – now called chaos – in dynamic systems. The new motion is erratic, but not simply "quasiperiodic" with a large number of periods. [2] What is surprising is that chaos can occur even in a very simple system. Scientists were interested in complicated motion of dynamic systems. But only with the advent of computers, with screens capable of displaying graphics, have scientists been able to see that many nonlinear dynamic systems have chaotic solutions.

As demonstrated in this book, nonlinear dynamical theory has found wide applications in different fields of economics. [3] The range of applications includes many topics, such as catastrophes, bifurcations, trade cycles, economic chaos, urban pattern formation, sexual division of labor and economic development, economic growth, values and family structure, the role of stochastic noise upon socio-economic structures, fast and slow socio-economic processes, and relationship between microscopic and macroscopic structures. All these topics cannot be effectively examined by traditional analytical methods which are concerned with linearity, stability and static equilibria. Nonlinear dynamical theory has changed economists' views about evolution. For instance, the traditional view of the relations between laws and consequences - between cause and effect - holds that simple rules imply simple behavior, therefore complicated behavior must arise from

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[2] In the solar system, the motion traveled around the earth in month, the earth around the sun in about a year, and Jupiter around the sun in about 11.867 years. Such systems with multiple incommensurable periods are known as quasiperiodic.

[3] For recent applications of nonlinear theory to economics, see Rosser (1991), Zhang (1991, 2005), Lorenz (1993), Puu (2000), and Shone (2002).

complicated rules. This vision had been held by professional economists for a long time. But it has been recently challenged due to the development of nonlinear theory. Nonlinear theory shows how complicated behavior may arise from simple rules. To illustrate this idea, we consider the *Ueda attractor*

$$\ddot{x} + 2\gamma\dot{x} + x^3 = F\cos t\,.$$

This is a simple dynamical system. When $k = 0.025$ and $F = 7.5$, as illustrated in Fig. 1.1.1, its behavior are "chaotic". The model with the specified parameter values does not exhibit any regular or periodic behavioral pattern. Chaos persists for as long as time passes.

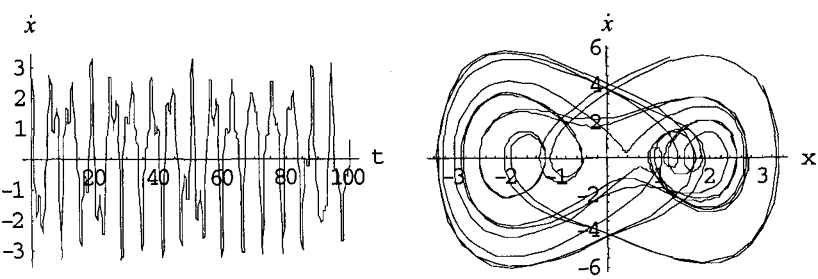

Fig. 1.1.1 Chaos of the Ueda attractor.

Another example is the Lorenz equations. The laws that govern the motion of air molecules and of other physical quantities are well known. The topic of differential equations is some 300 years old, but nobody would have thought it possible that differential equations could behave as chaotically as Edward N. Lorenz found in his experiments. Around 1960, Lorenz constructed models for numerical weather forecasting. He showed that deterministic natural laws do not exclude the possibility of chaos. In other words, determinism and predictability are not equivalent. In fact, recent chaos theory shows that deterministic chaos can be identified in much simpler systems than the Lorenz model.

The system of equations (with the parameter values specified) that Lorenz proposed in 1963 is

$$\dot{x} = 10(-x + y),$$

$$\dot{y} = 28x - y - xz,$$

$$\dot{z} = -\frac{8}{3}z + xy,$$

where x, y, and z are time-dependent variables.[4] We will come back to this system later. If we start with an initial state $(x_0, y_0, z_0) = (6, 6, 6)$, the motion of the system is chaotic, as depicted in Fig. 1.1.2. There are two sheets in which trajectories spiral outwards. When the distance from the center of such a spiral becomes larger than some particular threshold, the motion is ejected from the spiral and is attracted by the other spiral, where it again begins to spiral out, and the process is repeated. The motion is not regular. The number of turns that a trajectory spends in one spiral before it jumps to the other is not specified. It may wind around one spiral twice, and then three times around the other, then ten times around the first and so on.

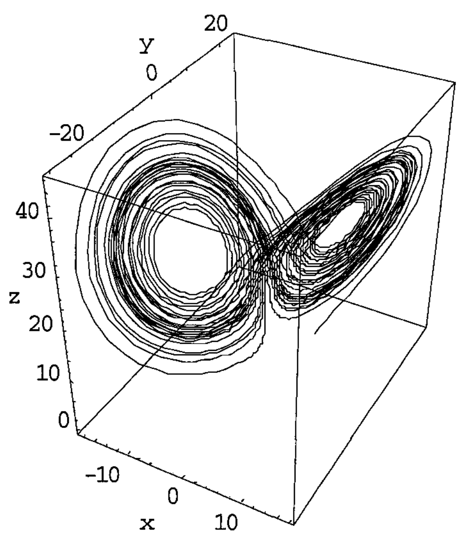

Fig. 1.1.2 The dynamics of the Lorenz equations.

Nonlinear dynamical systems are sufficient to determine the behavior in the sense that solutions of the equations do exist. But it is often

[4] A very thorough treatment of the Lorenz equations is given by Sparrow (1982).

impossible to explicitly write down solutions in algebraic expressions. Nonlinear economics based on nonlinear dynamical theory attempts to provide a new vision of economic dynamics: a vision toward the multiple, the temporal, the unpredictable, and the complex. There is a tendency to replace simplicity with complexity and specialism with generality in economic research. The concepts such as totality, nonlinearity, self-organization, structural changes, order and chaos have found broad and new meanings by the development of this new science. According to this new science, economic dynamics are considered to resemble a turbulent movement of liquid in which varied and relatively stable forms of current and whirlpools constantly change one another. These changes consist of dynamic processes of self-organization along with the spontaneous formation of increasingly subtle and complicated structures. The accidental nature and the presence of structural changes like catastrophes and bifurcations, which are characteristic of nonlinear systems and whose further trajectory is determined by chance, make dynamics irreversible.

Traditional economists were mainly concerned with regular motion of the economic systems. Even when they are concerned with economic dynamics, students are still mostly limited to their investigations of differential or difference equations to regular solutions (which include steady states and periodic solutions). In particular, economists were mainly interested in existence of a unique stable equilibrium. Students trained in traditional economics tend to imagine that the economic reality is uniquely determined and will remain invariant over time under "ideal conditions" of preferences, technology, and institutions. Nevertheless, common experiences reveal more complicated pictures of economic reality. Economic structures change even within a single generation. Economic systems collapse or suddenly grow without any seemingly apparent signs of structural changes.

1.2 Overview

This book presents the mathematical theory in linear and nonlinear differential equations and its applications to many fields of economics.

The book is for economists and scientists of other disciplines who are concerned with modeling and understanding the time evolution of nonlinear dynamic economic systems. It is of potential interest to professionals and graduate students in economics and applied mathematics, as well as researchers in social sciences with an interest in applications of differential equations to economic systems.

The book is basically divided into three parts - Part I concerns with one-dimensional differential equations; Part II concentrates on planar differential equations; Part III studies higher dimensional dynamical systems. Each part consists of three chapters – the first chapter is concentrated on linear systems, the second chapter studies nonlinear systems, and the third chapter applies concepts and techniques from the previous two chapters to economic dynamic systems of different schools.

Part I consists of three chapters. Chapter 2 deals with one-dimensional linear differential equations. Section 2.1 solves one-dimensional linear first-order differential equations. In Sec. 2.2, we examine a few special types of first-order equations, which may not be linear. Using the special structures of the equations, we can explicitly solve them. The types include separable differential equations, exact differential equations, and the Bernoulli equation. This section also examines the most well-known growth model, the Solow model and provides a few examples of applications. Section 2.3 is concerned with second-order differential equations. Section 2.4 gives general solutions to higher-order differential equations with continuous coefficients in time. Section 2.5 gives general solutions to higher-order differential equations with constant coefficients.

Chapter 3 is organized as follows. Section 3.1 introduces some fundamental concepts and theorems, such as equilibrium, trajectory, solution, periodic solution, existence theorems, and stability, about nonlinear differential equations. To avoid repetition in later chapters, the contents of this section are not limited to one dimension; they are valid for any finite dimensions. Section 3.2 states stability conditions of equilibria for scalar autonomous equations. We also apply the theory to two well-known economic models, the Cagan monetary model and the generalized Solow model with poverty traps. Section 3.3 introduces bifurcation theory and fundamental results for one-dimensional nonlinear systems. We examine saddle-node, transcritical, pitchfork, and cusp bifurcations. In Sec. 3.4, we demonstrate periodic solutions of one-dimensional second-order differential equations, using the Van der Pol equation and the Duffing equation as examples. Section 3.5 illustrates the energy balance method for examining periodic solutions. Section 3.6 introduces how to estimate amplitude and frequency of the periodic solutions examined in the previous section.

Chapter 4 applies concepts and theorems from the previous two chapters to analyze different models in economic model. Although the economic relations in these models tend to be complicated, we show that the dynamics of all these models are determined by motion of one-dimensional differential equations. Section 4.1 examines a one-sector growth model. As the economic mechanisms of this model will be applied in some other models in this book, we explain the economic structure in details. This section also applies the Liapunov theorem to guarantee global asymptotical stability of the equilibrium. Section 4.2 depicts the one-sector growth model proposed in Sec. 4.1 with simulation. Section 4.3 examines the one-sector-growth model for general utility functions. Section 4.4 examines a model of urban economic growth with housing production. In Sec. 4.5, we examine a dynamic model to how leisure time and work hours change over time in association with economic growth. Section 4.6 examines dynamics of sexual division of labor and consumption in association of modern economic growth. We illustrate increases of women labor participation as a "consequence" of economic growth as well as changes of labor market conditions. Section 4.7 introduces the Uzawa two-sector model. In Sec. 4.8, we re-examine the Uzawa model with endogenous consumer behavior. The models of this chapter show the essence of economic principles in many fields of economics, such as equilibrium economics (as a stationary state of a dynamic economics), growth theory, urban economics, and gender economics. The basic ideas and conclusions of this chapter require some books to explain, if that is possible. This also proves power of differential equations theory.

Part II consists of three chapters. Chapter 5 studies planar linear differential equations. Section 5.1 gives general solutions to planar linear first-order homogeneous differential equations. We also depict phase portraits of typical orbits of the planar systems. Section 5.2 introduces some concepts, such as positive orbit, negative orbit, orbit, limit set, and invariant set, for qualitative study. Section 5.3 shows how to calculate matrix exponentials and to reduce planar differential equations to the canonical forms. In Sec. 5.4, we introduce the concept of topological equivalence of planar linear systems and classify the planar linear homogeneous differential equations according to the concept. Section 5.5 studies planar linear first-order non-homogeneous differential equations. This section examines dynamic behavior of some typical economic models, such as the competitive equilibrium model, the Cournot duopoly model with constant marginal costs, the Cournot duopoly model with increasing marginal costs, the Cagan model with sluggish wages. Section 5.6 solves some types of constant-coefficient linear equations with time-dependent terms.

Chapter 6 deals with nonlinear planar differential equations. Section 6.1 carries out local analysis and provides conditions for validity of linearization. We also provide relations between linear systems and almost linear systems with regard to dynamic qualitative properties. This section examines dynamic properties of some frequently-applied economic models, such as the competitive equilibrium model, the Walrasian-Marshallian adjustment process, the Tobin-Blanchard model, and the Ramsey model. Section 6.2 introduces the Liapunov methods for stability analysis. In Sec. 6.3, we study some typical types of bifurcations of planar differential equations. Section 6.4 demonstrates motion of periodic solutions of some nonlinear planar systems. Section 6.5 introduces the Poincaré-Bendixon Theorem and applies the theorem to the Kaldor model to identify the existence of business cycles. Section 6.6 states Lienard's Theorem, which provides conditions for the existence and uniqueness of limit cycle in the Lienard system. Section 6.7 studies one of most frequently applied theorems in nonlinear economics, the Andronov-Hopf Bifurcation Theorem and its applications in the study of business cycles.

Chapter 7 applies the concepts and theorems related to two-dimensional differential equations to various economic issues. Section 7.1 introduces the IS-LM model, one of the basic models in contemporary macroeconomics and examines its dynamic properties.

Section 7.2 examines an optimal foreign debt model, maximizing the life-time utility with borrowing. In Sec. 7.3, we consider a dynamic economic system whose construction is influenced by Keynes' General Theory. Applying the Hopf bifurcation theorem, we demonstrate the existence of limit cycles in a simplified version of the Keynesian business model. Section 7.4 examines dynamics of unemployment within the framework of growth theory. In particular, we simulate the model to demonstrate how unemployment is affected by work amenity and unemployment policy. In Sec. 7.5, we establish a two-regional growth model with endogenous time distribution. We examine some dynamic properties of the dynamic systems. Section 7.6 models international trade with endogenous urban model formation. We show how spatial structures evolve in association of global growth and trade. In Sec. 7.7, we introduce a short-run dynamic macro model, which combines the conventional IS-LM model and Phillips curve. We also illustrate dynamics of the model under different financial policies. Section 7.8 introduces a growth model with public inputs. The public sector is treated as an endogenous part of the economic system. The system exhibits different dynamic properties examined in the previous two chapters.

Chapter 8 studies higher-dimensional differential equations. Section 8.1 provides general solutions to systems of linear differential equations. Section 8.2 examines homogeneous linear systems with constant coefficients. Section 8.3 solves higher-order homogeneous linear differential equations. Section 8.4 introduces diagonalization and introduces concepts of stable and unstable subspaces of the linear systems. Section 8.5 studies the Fundamental Theorem for linear systems and provides a general procedure of solving linear equations.

Chapter 9 deals with higher dimensional nonlinear differential equations. Section 9.1 studies local stability and validity of linearization. Section 9.2 introduces the Liapunov methods and studies Hamiltonian systems. In Sec. 9.3, we examine differences between conservative and dissipative systems. We examine the Goodwin model in detail. Section 9.4 defines the Poincaré maps. In Sec. 9.5, we introduce center manifold theorems. Section 9.6 applies the center manifold theorem and Liapunov theorem to a simple planar system. In Sec. 9.7, we introduce the Hopf bifurcation theorem in higher dimensional cases and apply it to a predator-prey model. Section 9.8 simulates the Loren equations, demonstrating chaotic motion of deterministic dynamical systems.

Chapter 10 applies the mathematical concepts and theorems of higher differential equations introduced in the previous two chapters to differential economic models. Section 10.1 examines some tâtonnement price adjustment processes, mainly applying the Liapunov methods. Section 10.2 studies a three-country international trade model with endogenous global economic growth. Section 10.3 extends the trade model of the previous section by examining impacts of global economic group on different groups of people not only among countries but also within countries. We provide insights into complexity of international trade upon different people. Section 10.4 examines an two-region growth model with endogenous capital and knowledge. Different from the trade model where international migration is not allowed, people freely move among regions within the interregional modeling framework. Section 10.5 introduces money into the growth model. We demonstrate the existence of business cycles in the model, applying the Hopf bifurcation theorem. Section 10.6 guarantees the existence of limit cycles and aperiodic behavior in the traditional multi-sector optimal growth model, an extension of the Ramsey growth model. Section 10.7 proposes a dynamic model with interactions among economic growth, human capital accumulation, and opening policy to provide insights into the historical processes of Chinese modernization. Analysis of behavior of this model requires almost all techniques introduced in this book.

As concluding remarks to this book, we address two important issues which have been rarely studied in depth in economic dynamical analysis, changeable speeds and economic structures. The understanding of these two issues are essential for appreciating validity and limitations of different economic models in the literature, but should also play a guarding role in developing general economic theories. We also include an appendix. App. A.1 introduces matrix theory. App. A.2 shows how to solve linear equations, based on matrix theory. App. A.3 defines some basic concepts in study of functions and states the Implicit Function Theorem. App. A.4 gives a general expression of the Taylor Expansion.

App. A.5 briefly mentions a few concepts related to structural stability. App. A.6 introduces optimal control theory.