APERIODIC ORDER – LECTURE 10 SUMMARY

1. Pure discrete spectrum for self-similar tiling systems (see [4]).

We continue the study of tiling dynamical systems. Suppose that ω is a tile-substitution in $\mathbb{R}^d$, with a prototile set $\mathcal{A}$ and expansion similarity map ϕ, which is primitive, and suppose that $\mathcal{T}$ is a fixed point: $\omega(\mathcal{T}) = \mathcal{T}$. We further assume that $\mathcal{T}$ is repetitive and has FLC. We call such tilings *self-similar*. We then consider the associated tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$, which is just the translation action on the orbit closure of $\mathcal{T}$. Recall that such a system is *uniquely ergodic*, so we have a unique invariant measure μ on $X_{\mathcal{T}}$. Moreover, the measure of certain sets can be computed more or less explicitly.

Let P be an admissible patch of $\mathcal{T}$ and let U be a "small nice set" around the origin (such that its translates can be found in every tile interior). Consider

$$X_{P,U} := \{\mathcal{S} \in X_{\mathcal{T}} :\ \exists\, g \in U,\ P + g \subset \mathcal{S}\}.$$

Such sets play the role of "cylinder sets" for the tiling space. It can be shown that such sets generate the Borel σ-algebra on $X_{\mathcal{T}}$. For any $\mathcal{T}$-patch P there exists *uniform frequency*

$$\mathrm{freq}(P, \mathcal{T}) = \lim_{R\to\infty} \frac{\#\{g \in \mathbb{R}^d :\ P + g \subset \mathcal{T} \cap (B_R + \mathbf{z})\}}{\mathcal{L}^d(B_R)}\,.$$

where $B_R = \{\mathbf{x} :\ \|\mathbf{x}\| \le R\}$, and the limit is uniform in $\mathbf{z} \in \mathbb{R}^d$.

Lemma 1.1. *Let $X_{P,U}$ be as above. Then*

$$\mu(X_{P,U}) = \mathcal{L}^d(U) \cdot \mathrm{freq}(P, \mathcal{T}).$$

Now let us discuss the *spectrum* of the measure-preserving system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$. This is analogous to the case of a single transformation, but we need to take the full $\mathbb{R}^d$-action into account. Instead of one Koopman operator we get a group of unitary operators $U_{\mathbf{g}} :\ f \mapsto f \circ T_{\mathbf{g}}$, for $\mathbf{g} \in \mathbb{R}^d$, acting on $L^2(X_{\mathcal{T}}, \mu)$.

Suppose $f \in L^2(X_{\mathcal{T}}, \mu)$. Then there exists a Borel measure σ_f on $\mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d} e^{2\pi i \langle \mathbf{x}, \mathbf{g}\rangle}\, d\sigma_f(\mathbf{g}) = \langle U_{\mathbf{x}} f, f\rangle = \int_{X_{\mathcal{T}}} f(\mathcal{S} - \mathbf{x}) \cdot \overline{f(\mathcal{S})}\, d\mu(\mathcal{S}), \tag{1.1}$$

called the *spectral measure* of f.

Recall the theorem on *eigenvalues* that we discussed in the previous lecture. We define the set of "return vectors" for the tiling $\mathcal{T}$ as follows:

$$\Xi(\mathcal{T}) = \{\mathbf{z} \in \mathbb{R}^d :\ \exists\, T, T' \in \mathcal{T},\ T' = T + \mathbf{z}\}.$$

Theorem 1.2. *Let $\mathcal{T}$ be a self-similar aperiodic tiling with expansion map ϕ, and let μ the unique invariant Borel probability measure for the translation action $(X_{\mathcal{T}}, \mathbb{R}^d)$. The following are equivalent for $\mathbf{g} \in \mathbb{R}^d$:*

(i) $\mathbf{g}$ *is a "continuous" eigenvalue, that is, there exists $f_{\mathbf{g}} \in C(X_{\mathcal{T}})$ such that*

$$f_{\mathbf{g}}(\mathcal{S} - \mathbf{x}) = e^{2\pi i \langle \mathbf{x}, \mathbf{g} \rangle} f_{\mathbf{g}}(\mathcal{S}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d \text{ and } \mathcal{S} \in X_{\mathcal{T}}; \tag{1.2}$$

(ii) $\mathbf{g}$ *is a "measurable" eigenvalue, that is, there exists $f_{\mathbf{g}} \in L^2(X_{\mathcal{T}}, \mu)$ (actually, in $L^\infty(X_{\mathcal{T}}, \mu)$) such that (1.2) holds for μ-a.e. $\mathcal{S}$.*

(iii)

$$\lim_{n\to\infty} \langle \phi^n \mathbf{z}, \mathbf{g} \rangle = 0 \ \ (\text{mod } 1), \quad \text{for all } \ \mathbf{z} \in \Xi(\mathcal{T}). \tag{1.3}$$

Using the vectors from $\Xi(\mathcal{T})$ we can also formulate a criterion for pure discrete spectrum. For $\mathbf{z} \in \Xi(\mathcal{T})$ let

$$\mathcal{D}_{\mathbf{z}} := \mathcal{T} \cap (\mathcal{T} + \mathbf{z}),$$

that is, the collection of those tiles of $\mathcal{T}$ for which the translate by $\mathbf{z}$ also belongs to $\mathcal{T}$. Observe that this set has a well-defined density:

$$\text{dens}(\mathcal{D}_{\mathbf{z}}) = \lim_{R\to\infty} \frac{\mathcal{L}^d(\mathcal{D}_{\mathbf{z}} \cap B_R)}{\mathcal{L}^d(B_R)} = \sum_{i=1}^{m} \text{freq}(T_i \cup (\mathbf{z} + T_i), \mathcal{T}) \cdot \mathcal{L}^d(T_i),$$

where $T_1, \ldots, T_m$ are the prototiles.

Theorem 1.3. *Let $\mathcal{T}$ be a self-similar aperiodic tiling with expansion map ϕ, and let μ the unique invariant Borel probability measure for the translation action $(X_{\mathcal{T}}, \mathbb{R}^d)$. Then the following are equivalent:*

(i) *The system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has pure discrete spectrum, that is, there is a basis for $L^2(X_{\mathcal{T}}, \mu)$ consisting of eigenfunctions;*

(ii) $\lim_{n\to\infty} \text{dens}(\mathcal{D}_{\phi^n \mathbf{z}}) = 1$ *for all* $\mathbf{z} \in \Xi(\mathcal{T})$*;*

(iii) $1 - \text{dens}(\mathcal{D}_{\phi^n \mathbf{z}}) \le C(\mathbf{z})\rho^n, \ n \ge 1$, *for some* $\rho \in (0,1)$, *for all* $\mathbf{z} \in \Xi(\mathcal{T})$.

Obviously (iii) implies (ii). The property (ii), and especially (iii), can be viewed as a kind of "almost periodicity" of the tiling $\mathcal{T}$.

2. Substitutions of constant length (see [1, 7.3.1]).

Recall that for a symbolic substitution ζ on the alphabet $\mathcal{A} = \{0, \ldots, m-1\}$ we can consider an associated tile-substitution on the real line $\mathbb{R}$, with prototiles that are line segments whose lengths correspond to the left Perron-Frobenius eigenvector of the substitution matris S_ζ, labeled by the letters. If ζ has constant length q, that is, $|\zeta(j)| = q$ for all $j \in \mathcal{A}$, then the left PF eigenvector is (up to constant multiple) $[1, \ldots, 1]$. As a result, there is very little difference between the tiling (shift) $\mathbb{R}$-action and symbolic (shuft)

$\mathbb{Z}$-action in the constant length case. In particular, the results of the previous section can be applied to the substitution dynamical system (X_ζ, T) in the constant length case almost verbatim.

First let us examine the eigenvalues. Suppose that u is a fixed point of the substitution: $\zeta(u) = u$ and let $\mathcal{T}$ be the associated self-similar tiling of $\mathbb{R}_+$. (Even though it is a tiling of the half-line only, we can consider the tiling dynamical system $X_\mathcal{T}$ on $\mathbb{R}$ by specifying the "atlas" of patches being those of $\mathcal{T}$. Or, if we are bothered by this, we can assume that u is a 2-sided fixed point of ζ.) Then

$$\Xi(\mathcal{T}) = \Xi(u) = \{k \in \mathbb{Z} :\ \exists\, j,\ u_{j+k} = u_j\}. \tag{2.1}$$

We have by Theorem 1.2 and the comments above:

Corollary 2.1. *Let ζ be a primitive aperiodic substitution of constant length q. Then the following are equivalent:*
(i) $\alpha \in \mathbb{R}$ *is an eigenvalue for the substitution* $\mathbb{R}$*-action;*
(ii) $e^{2\pi i\alpha}$ *is an eigenvalue for the substitution (shift)* $\mathbb{Z}$*-action;*
(iii) $q^n k\alpha \to 0$ *modulo 1, as* $n \to \infty$*, for all* $k \in \Xi(\mathcal{T})$*;*
(iv)

$$\alpha \in \bigcap_{k\in\Xi(\mathcal{T})} \frac{1}{k}\cdot \mathbb{Z}\left[\frac{1}{q}\right].$$

This naturally leads to the notion of *height* introduced by M. Dekking.

Definition 2.2. Let $S_\ell := \{k \geq 1 :\ u_{\ell+k} = u_\ell\}$ and $g_\ell = GCD(S_\ell)$. The *height* of the substitution ζ is defined by

$$h = h(\zeta) = \max\{n \geq 1 :\ (n, q) = 1,\ n|g_0\}.$$

Properties [Exercise]. (i) We have $1 \leq h \leq m = \text{card}\mathcal{A}$.

(ii) If $h = m$, then u is periodic.

(iii) We have $h = \max\{n \geq 1 :\ (n, q) = 1,\ n|g_\ell\}$ for all $\ell \geq 1$.

Exercise. Consider the substitution $0 \to 0123,\ 1 \to 1234,\ 2 \to 2345,\ 3 \to 3450,\ 4 \to 4501,\ 5 \to 5012$. Prove that it is aperiodic and has height $h = 3$.

Theorem 2.3 (Dekking, 1978)**.** *Let ζ be a non-periodic substitution of constant length q of height h. Then the group of eigenvalues for the substitution dynamical system is* $\exp\big(2\pi i(\mathbb{Z}[1/q] \times \mathbb{Z}/h\mathbb{Z})\big)$.

2.1. **Pure discrete spectrum.**

Definition 2.4. A constant length substitution ζ satisfies the *coincidence condition* if there exists ℓ and p such that $\zeta^\ell(j)$ has the same p-th letter for every $j \in \mathcal{A}$.

We suppose that the substitution is primitive and shift-aperiodic. The substitution dynamical system is denoted (X_ζ, T, μ), where $X_\zeta \subset \mathcal{A}^{\mathbb{Z}}$, the map T is the left shift, and μ is the unique T-invariant probability measure.

Theorem 2.5 (Dekking 1978)**.** *Suppose that ζ is a primitive substitution of constant length and height $h = 1$. Then the measure-preserving system (X_ζ, T, μ) has pure discrete spectrum if and only if ζ satisfies the coincidence condition.*

Sketch of the proof. We will reduce this theorem to Theorem 1.3. We can assume, without loss of generality, that $\ell = 1$ (from the definition of coincidences), by passing to a power of the substitution if necessary. Recalling (2.1), we have $\Xi(u) = \Xi(\mathcal{T}) \subset \mathbb{Z}$. Suppose that $x \in X_\zeta$ is a two-sided fixed point for ζ: $x = \zeta(x)$. We will estimate the frequency of "coincidences" between x and $T^{q^n k}x$ for $k \in \Xi(u)$. In fact, pick *any* $k \in \mathbb{N}$ and consider the pair of sequences as a sequence of pairs:

$$(x, T^k x) = \{(x_j, x_{j+k})\}_{j\in\mathbb{Z}}.$$

We can view it as an element of $X_\zeta \times X_\zeta$. The left shift T and substitution ζ naturally extend to $X_\zeta \times X_\zeta$. Observe that

$$\zeta^n(x, T^k x) = (\zeta^n(x), T^{q^n k}(\zeta^n(x)) = (x, T^{q^n k}x), \quad n \geq 1.$$

Since ζ has constant length and satisfies the coincidence condition with $\ell = 1$, applying the substitution to any pair of letters (α, β), we get a sequence of pairs of length q among which there is at least one coincidence (γ, γ). On the other hand, applying ζ to a coincidence pair will only yield coincidence pairs. Therefore,

$$1 - \mathrm{dens}(\mathcal{D}_{\phi^{n+1}k}) \leq \frac{q-1}{q} \cdot \big(1 - \mathrm{dens}(\mathcal{D}_{\phi^n k})\big).$$

This implies that the spectrum is pure discrete, by Theorem 1.3. On the other hand, one can show that if the coincidence condition does not hold, then the frequency of "noincoincidences" will stay bounded away from zero, as $n \to \infty$. □

2.2. **Continuous spectrum (see** [3]**).** When the system (X_ζ, T, μ) does not have pure discrete spectrum, we would like to understand the spectral type of test functions which are orthogonal to the subspace spanned by the eigenfunctions. Usually it is enough to consider the test functions which only depend on the single coordinate x_0, or equivalently, linear combinations of characteristic (indicator) functions of cylinder sets $1\!\!1_{[\alpha]}$, $\alpha \in \mathcal{A}$.(For instance, in the constant length-q case, the maximal spectral type is

$$\sum_{\alpha\in\mathcal{A}} \sum_{n=0}^{\infty} \sigma_a * \omega_{q^n}$$

where σ_a is the spectral measure of $\mathbb{1}_{[\alpha]}$, ω_{q^n} is the discrete uniform measure on the finite set of roots of unity of order q^n, and $*$ denotes the convolution of measures.

In order to investigate continuous spectrum, the following proposition is often useful.

Proposition 2.6. *Let $\phi : \mathcal{A} \to \mathbb{C}$ be any function, and let $f(x) = \phi(x_0)$ for $x \in X_\zeta$. In other words,*

$$f = \sum_{\alpha \in \mathcal{A}} \phi(\alpha) \mathbb{1}_{[\alpha]}.$$

Then the spectral measure of f can be represented as

$$\sigma_f = \text{weak}^*\text{-} \lim_{N\to\infty} \left(\frac{1}{N} \Big| \sum_{j=0}^{N-1} \phi(x_j) e^{-2\pi i j t} \Big|^2 dt \right) \tag{2.2}$$

The right-hand side of (2.2) is a weak* limit of absolutely contiunuous measures on $[0,1] \sim \mathbb{T}$; it exists uniformly in (and is independent of) $x \in X_\zeta$.

Proof. Weak* convergence of measures is equivalent to convergence of their Fourier coefficients. We have for any ergodic invertible measure-preserving system (X, T, μ), any $k \in \mathbb{Z}$, and any $f \in L^2(X, \mu)$, by definition:

$$\widehat{\sigma}_f(-k) = \int_0^1 e^{2\pi i k t}\, d\sigma_f(t) = \langle U_T^k f, f \rangle = \int_X f(T^k x) \overline{f(x)}\, d\mu(x).$$

Now, if the system is uniquely ergodic and f is continuous (as in our case), we can rewrite the last integral as a limit of Birkhoff sums to obtain:

$$\widehat{\sigma}_f(-k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{n+k} x) \overline{f(T^n x)}.$$

Further specializing to our case, we have $f(T^n x) = \phi(x_n)$, hence

$$\widehat{\sigma}_f(-k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi(x_{n+k}) \overline{\phi(x_n)}. \tag{2.3}$$

Now let us consider the right-hand side of (2.2), before taking the limit. We have

$$\frac{1}{N} \cdot \sum_{j=0}^{N-1} \phi(x_j) e^{-2\pi i j t} \cdot \sum_{n=0}^{N-1} \overline{\phi(x_n)} e^{2\pi i n t} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} \phi(x_j) \overline{\phi(x_n)} e^{2\pi i (n-j) t}.$$

Writing $j = n + k$, we see obtain that the $(-k)$-Fourier coefficient of the last expression is

$$\frac{1}{N} \sum_{n=\max(0,k)}^{\min(N-1, N-k-1)} \phi(x_{n+k}) \overline{\phi(x_n)}.$$

Comparing this with (2.3), we see that the difference is at most $2k\|\phi\|_\infty / N$, which tends to zero as $N \to \infty$, and the desired claim follows. $\square$

Corollary 2.7. *In the setting of the previous proposition, we have for any $\alpha \in \mathcal{A}$:*

$$\sigma_f = \text{weak}^*\text{-}\lim_{n\to\infty}\left(\frac{1}{|\zeta^n(\alpha)|}\cdot\left|\sum_{j=0}^{|\zeta^n(\alpha)|-1}\phi\left(\zeta^n(\alpha)_j\right)\, e^{-2\pi ijt}\right|^2 dt\right) \tag{2.4}$$

2.3. **Morse substitution.** Now let's apply all this theory to the Morse substitution:

$$\zeta : 0 \to 01,\ 1 \to 10.$$

In fact, it is more convenient to use the alphabet $\mathcal{A} = \{1, \overline{1}\}$, so that $\zeta(1) = 1\overline{1},\ \zeta(\overline{1}) = \overline{1}1$. The "$\overline{\cdot}$"-operation extends to an involution on the set of words and infinite sequences over $\mathcal{A}$, using the convention $\overline{\overline{1}} = 1$. Observe that the Morse substitution has the property:

$$\zeta(\overline{V}) = \overline{\zeta(V)} \text{ for any } V \in \mathcal{A}^+.$$

It is clear that the height is equal to 1. By Theorem 1.2, the group of eigenvalues is $\exp(2\pi i \cdot \mathbb{Z}[1/2])$. The coincidence condition obviously fails, so the spectrum is not pure discrete. One can show that the closed linear span of the eigenfunctions consists of those $f \in L^2(X_\zeta, \mu)$ which are *even* with respect to this involution: $f(\overline{x}) = f(x)$. (This is obviously true for the trivial, constant eigenfunction.) Thus, we are interested in the spectral type of *odd* functions. The natural choice is

$$f(x) = \phi(x_0), \quad \text{where } \phi(1) = 1,\ \phi(\overline{1}) = -1. \tag{2.5}$$

Now we apply Corollary 2.7. For a word V in the alphabet $\mathcal{A}$ let

$$\Phi(V,t) = \sum_{j=0}^{|V|-1}\phi(V_j)e^{-2\pi ijt} \quad \text{and} \quad \Phi_n(t) := \Phi(\zeta^n(1), t),$$

so that we have by (2.4):

$$\sigma_f = \text{weak}^*\text{-}\lim_{n\to\infty}\left(2^{-n}|\Phi_n(t)|^2\, dt\right).$$

Note that $\Phi(\overline{V}, t) = -\Phi(V, t)$ for all $V \in \mathcal{A}^+$. Observe that $\Phi_0(t) = 1$, and for $n \geq 1$:

$$\begin{aligned}\Phi_n(t) = \Phi(\zeta^n(1), t) &= \Phi(\zeta^{n-1}(1)\,\zeta^{n-1}(\overline{1}), t)\\ &= \Phi(\zeta^{n-1}(1), t) + e^{-2\pi i\cdot 2^{n-1}t}\cdot\Phi(\zeta^{n-1}(\overline{1}), t)\\ &= \Phi(\zeta^{n-1}(1), t)(1 - e^{-2\pi i\cdot 2^{n-1}t}).\end{aligned}$$

Thus,

$$\Phi_n(t) = \prod_{k=0}^{n-1}\left(1 - \lambda^{2^k}\right), \quad \text{where } \lambda = e^{-2\pi it}.$$

Next,

$$|1-\lambda|^2 = 2 - 2\Re(\lambda) = 2(1 - \cos(2\pi t)),$$

hence

$$2^{-n}|\Phi_n(t)|^2 = \prod_{k=0}^{n-1}\Big(1-\cos(2\pi\cdot 2^k t)\Big)\,.$$

We have proved the following

Proposition 2.8. *For the function f given by (2.5), the spectral measure σ_f corresponding to the Morse substitution system, can been expressed as a generalized Riesz product*

$$\sigma_f = \text{weak}^*\text{–}\lim_{n\to\infty}\left(\prod_{k=0}^{n-1}\Big(1-\cos(2\pi\cdot 2^k t)\Big)\,dt\right)$$

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