APERIODIC ORDER – LECTURE 11 SUMMARY

1. Substitutions of constant length (cont.)

1.1. **Morse substitution (cont.)** (see [2, 1.5.3, 2.1.3]). Let $\mathcal{A} = \{1, \overline{1}\}$, so that $\zeta(1) = 1\overline{1}$, $\zeta(\overline{1}) = \overline{1}1$ is the Morse substitution. Consider

$$f(x) = \phi(x_0), \quad \text{where } \phi(1) = 1,\ \phi(\overline{1}) = -1. \tag{1.1}$$

We have proved the following

Proposition 1.1. *For the function f given by (1.1), the spectral measure σ_f corresponding to the Morse substitution system, can been expressed as a generalized Riesz product*

$$\sigma_f = \text{weak}^*\text{–}\lim_{n\to\infty} \left(\prod_{k=0}^{n-1} \Big(1 - \cos(2\pi \cdot 2^k t)\Big)\, dt \right) \tag{1.2}$$

Theorem 1.2 (Kakutani). *The spectral measure σ_f is singular with respect to Lebesgue.*

Proof. Note that (1.2) defines a 1-periodic measure on $\mathbb{R}$, since the right-hand side is 1-periodic for all n. Denote this measure by ν and consider $\nu(t/2)$, the push-forward of ν under the map $t \mapsto t/2$ on $\mathbb{R}$. Since the push-forward commutes with taking the weak* limit, we have

$$\begin{aligned} \nu(t/2) &= \text{weak}^*\text{–}\lim_{n\to\infty} \left(\prod_{k=0}^{n-1} \Big(1 - \cos(2\pi \cdot 2^k (t/2))\Big)\, d(t/2) \right) \\ &= \frac{1}{2}\big(1 - \cos(\pi t)\big) \cdot \text{weak}^*\text{–}\lim_{n\to\infty} \left(\prod_{k=0}^{n-2} \Big(1 - \cos(2\pi \cdot 2^k t)\Big)\, dt \right) \\ &= \frac{1}{2}\big(1 - \cos(\pi t)\big) \cdot \nu. \end{aligned} \tag{1.3}$$

We can take absolutely continuous parts of both sides in the formula (1.3), because the push-forward by $t/2$ and multiplication by $1 - \cos(\pi t)$ commute with the decomposition into a.c. and singular parts of the measure. Thus, denoting ν_{ac} by η we have

$$\eta(t/2) = \frac{1}{2}\big(1 - \cos(\pi t)\big) \cdot \eta. \tag{1.4}$$

Observe that η, restricted to $[0, 1]$, is invariant under the map $S_2(t) = 2t \pmod 1$. In fact, such invariance is equivalent to

$$\eta = \eta \circ S_2^{-1} = \eta(t/2) + \eta((t + 1)/2). \tag{1.5}$$

Since η, as ν, is 1-periodic, we have from (1.4) that

$$\eta((t+1)/2) = \frac{1}{2}(1-\cos(\pi(t+1))\eta = \frac{1}{2}(1+\cos(\pi t)),$$

which immediately implies (1.5). It remains to observe that the Lebesgue measure is the only invariant a.c. measure for S_2 (e.g., by the Riemann-Lebesgue Lemma), but Lebesgue does not satisfy (1.4). So η is zero, as claimed. □

1.2. **Rudin-Shapiro sequence and substitution** (see [2, 5.3.2])**.**

Definition 1.3. The Rudin-Shapiro sequence is defined inductively by $v_0 = 1,\ v_1 = -1$, $v_{2k} = v_k$, $v_{2k+1} = (-1)^k v_k$. The Rudin-Shapiro substitution ζ is defined on the alphabet $\mathcal{A} = \{1, 2, \overline{1}, \overline{2}\}$ by

$$1 \to 12,\ 2 \to 1\overline{2},\ \overline{1} \to \overline{1}\,\overline{2},\ \overline{2} \to \overline{1}2.$$

Exercise. Let $u = u_0u_1u_2u_3\ldots = 121\overline{2}\ldots$ be the fixed point of the Rudin-Shapiro substitution. Prove that the Rudin-Shapiro sequence can be obtained by $v_k = \phi(u_k)$, where $\phi(1) = \phi(2) = 1$, $\phi(\overline{1}) = \phi(\overline{2}) = -1$.

It is not hard to see that the RS-substitution has height one and is not pure discrete. We would like to determine the spectral type of $f(x) = \phi(x_0)$.

Theorem 1.4. *The spectral measure σ_f is absolutely continuous with respect to Lebesgue.*

This is really surprising, since substitution dynamical systems are very "non-chaotic", whereas Lebesgue spectral is usually associated with "chaotic" behavior. Of course, the maximal spectral type of the RS substitution is not pure Lebesgue, since it includes the discrete singular part, concentrated on the set of eigenvalues $\exp(2\pi i \cdot \mathbb{Z}[1/2])$. In general, it is known that substitutions *must* have singular part.

Proof sketch. We will use Proposition 2.6 from Lecture 10. Denote the beginning of the RS sequence of length 2^n by

$$A_n = \phi(\zeta^n(1)).$$

Further, let $B_n = \phi(\zeta^n(2))$. Since $\phi(\zeta^n(\overline{2})) = -\phi(\zeta^n(2))$, we have

$$A_{n+1} = A_nB_n, \quad B_{n+1} = A_n(-B_n). \tag{1.6}$$

Now consider the trigonometric polynomial

$$g_n(t) := \sum_{j=0}^{2^n-1} v_j e^{-2\pi i j t} = \Phi(A_n, t),$$

using the notation from Lecture 10, p.6. Similarly, consider

$$h_n(t) := \Phi(B_n, t) = \sum_{j=0}^{2^n-1} w_{n,j} e^{-2\pi i j t},$$

where $B_n = w_{n,0} \dots w_{n,2^n-1}$. Then (1.6) yields

$$\begin{aligned} g_{n+1}(t) &= g_n(t) + e^{-2\pi i \cdot 2^n t} h_n(t); \\ h_{n+1}(t) &= g_n(t) - e^{-2\pi i \cdot 2^n t} h_n(t), \end{aligned}$$

which immediately imply

$$|g_{n+1}^2(t)| + |h_{n+1}^2(t)| = 2\big(|g_n^2(t)| + |h_n^2(t)|\big).$$

Since $g_0(t) \equiv 1$ and $h_0(t) \equiv -1$, we obtain

$$|g_n^2(t)| + |h_n^2(t)| \equiv 2^{n+1} \quad \text{for all } t. \tag{1.7}$$

Recall Proposition 2.6 from Lecture 10, which says in our case:

$$\sigma_f = \text{weak}^*\text{–} \lim_{n\to\infty} 2^{-n} |g_n^2(t)|\, dt.$$

However, it follows from (1.7) that $2^{-n}|g_n^2(t)| \le 2$; in particular, all these densities are uniformly bounded in L^2. Since weak*-convergence is equivalent to convergence of Fourier coefficients, we obtain that the limiting measure σ_f has square-summable Fourier coefficients, hence it is actually absolutely continuous with a density in L^2. The theorem is proved. (An additional argument shows that the spectral measure σ_f is actually itself Lebesgue.) □

2. Projection method (see [3, 2.6.1] and [1, 7.2]).

We start with some terminology.

- A finitely-generated free Abelian subgroup of $\mathbb{R}^d$ is called a *$\mathbb{Z}$-module*. It has the form

$$\Omega = \{\mathbf{x} = n_1\mathbf{b}_1 + \cdots + n_k\mathbf{b}_k :\ n_1, \dots, n_k \in \mathbb{Z}\}, \tag{2.1}$$

  where $\mathbf{b}_1, \dots, \mathbf{b}_k$, for $k \le d$, are vectors in $\mathbb{R}^d$. These vectors do not have to be linearly independent. However, we can choose them *integrally independent*; in this case k is the *rank* of the $\mathbb{Z}$-module.
- The set $\Omega \subset \mathbb{R}^d$ is discrete iff the vectors $\mathbf{b}_1, \dots, \mathbf{b}_k$ are linearly independent. If in addition, $k = d$, then Ω is a *lattice* in $\mathbb{R}^d$.
- Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d$. The *dual lattice* $\mathcal{L}^*$ of $\mathcal{L}$ is defined by

$$\mathcal{L}^* = \{\mathbf{y} \in \mathbb{R}^d :\ \langle \mathbf{x}, \mathbf{y}\rangle \in \mathbb{Z} \text{ for all } \mathbf{x} \in \mathcal{L}\}.$$

- The lattice $\mathcal{L}$ is *integral* if $\mathcal{L} \subset \mathcal{L}^*$.

- Equivalent condition: a lattice $\mathcal{L}$ is integral iff for every $\mathbf{x} \in \mathcal{L}$ we have $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \in \mathbb{N}$.
- If $\mathcal{L}$ is an integral lattice, then there is $n \in \mathbb{N}$ such that $\mathcal{L}^* \subset \mathcal{L}$. This follows from the fact $\mathcal{L}^*/\mathcal{L}$ is a finite Abelian group.
- A linear subspace $\mathcal{E} \subset \mathbb{R}^d$ is *totally irrational* if $\mathcal{E} \cap \mathbb{Z}^d = \{\mathbf{0}\}$.
- The *orthogonal complement* of $\mathcal{E}$ in $\mathbb{R}^d$ is denoted $\mathcal{E}^\perp$.

The following theorem is standard; we will not prove it, see [3, p.52] for details.

Theorem 2.1. *Let $\phi : \mathbb{R}^k \to \mathbb{R}^d$ be a surjective linear mapping, where $k > d$. Then there are (possibly trivial) subspaces V and W of $\mathbb{R}^d$ such that $\mathbb{R}^d = V \oplus W$ and*
(i) $\phi(\mathbb{Z}^k) = \phi(\mathbb{Z}^k) \cap W + \phi(\mathbb{Z}^k) \cap V$*;*
(ii) $\phi(\mathbb{Z}^k) \cap W$ *is a (discrete) point lattice in* W*;*
(iii) $\phi(\mathbb{Z}^k) \cap V$ *is a dense subgroup of* V*.*

Projection is a powerful technique for constructing nonperiodic Delone sets and tilings. Let $\mathcal{E}$ be a linear subspace of $\mathbb{R}^n$. Denote by $\Pi = \Pi_{\mathcal{E}}$ the orthogonal projection to $\mathcal{E}$ and by $\Pi^\perp$ the orthogonal projection to $\mathcal{E}^\perp$.

Proposition 2.2. *Let $\mathcal{L}$ be an integral lattice in $\mathbb{R}^n$ and $\mathcal{E} \subset \mathbb{R}^n$ a linear subspace. Then the following are equivalent:*
(i) $\Pi(\mathcal{L})$ *is dense in* $\mathcal{E}$*;*
(ii) $\mathcal{E} \cap \mathcal{L} = \{0\}$*;*
(iii) $\Pi^\perp|_{\mathcal{L}}$ *is 1-to-1.*

Next we define the *cut and project scheme* (CPS). We restrict ourselves to Euclidean "internal space" for simplicity, but note that more general schemes (with LCAG internal spaces) are considered. For the rest of the lecture we follow [1, Section 7.2] very closely.

Definition 2.3. A *cut and project scheme* (CPS) is a triple $(\mathbb{R}^d, H, \mathcal{L})$, with the *internal space* $H \approx \mathbb{R}^\ell$ and $\mathcal{L} \subset \mathbb{R}^d \times H$ a lattice, with the two natural projections $\pi : \mathbb{R}^d \times H \to \mathbb{R}^d$ and $\pi_{\text{int}} : \mathbb{R}^d \times H \to H$ subject to the conditions that $\pi|_{\mathcal{L}}$ is injective and $\pi_{\text{int}}(\mathcal{L})$ is dense in H.

Definition 2.4. A subset $W \subset H$ is called a *window* or *acceptance domain* if W is bounded and has nonempty interior. It is called *regular* if the boundary ∂W has zero Lebesgue measure.

Definition 2.5. For $W \subset H$ consider

$$\Lambda(W) := \{\pi(\mathbf{x}) : \ \mathbf{x} \in \mathcal{L}, \ \pi_{\text{int}}(\mathbf{x}) \in W\}.$$

It is called a *model set* if W is a window. The model set is *generic* if $\pi_{\text{int}}(\mathcal{L}) \cap \partial W = \emptyset$, otherwise it is called *singular*.

Theorem 2.6. *Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS and $W \subset H$.*

(i) *If W is bounded, then $\Lambda(W)$ has finite local complexity (FLC) and hence it is uniformly discrete;*

(ii) *if W has nonempty interior, then $\Lambda(W)$ is relatively dense in $\mathbb{R}^d$;*

(iii) *if $\Lambda(W)$ is a model set (i.e. W is bounded and $W^\circ \neq \emptyset$), then $\Lambda(W)$ is a Meyer set.*

Proof sketch. (i) Consider a ball $B_R(\mathbf{y}) \subset \mathbb{R}^d$. The pattern $\Lambda(W) \cap B_R(\mathbf{y})$ is obtained by projecting $\mathcal{L} \cap (B_R(\mathbf{y}) \times W)$ to $B_R(\mathbf{y})$. However, $\mathcal{L}$, being a lattice, has FLC, hence there are finitely many possible patterns $\mathcal{L} \cap (B_R(\mathbf{y}) \times W)$ up to translation. This property is obviously preserved after translation.

(ii) We need the following

Lemma 2.7. *Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS and $U \subset H$ a nonempty open set. Then there exists a compact set $K \subset \mathbb{R}^d$ such that*

$$\mathbb{R}^d \times H = \mathcal{L} + (K \times U).$$

Proof. Since $\mathcal{L}$ is a lattice in $\mathbb{R}^d \times H$, we can find a compact set $C \subset \mathbb{R}^d \times H$ such that $\mathbb{R}^d \times H = \mathcal{L} + C$. Let $K_1 = \pi(C) \subset \mathbb{R}^d$ and $K_2 = \pi_{\text{int}}(C) \subset H$, then

$$\mathbb{R}^d \times H = \mathcal{L} + (K_1 \times K_2). \tag{2.2}$$

Now recall that $\pi_{\text{int}}(\mathcal{L})$ is dense in H, which implies that

$$\bigcup_{p \in \mathcal{L}} (\pi_{\text{int}}(p) + U) = H \supset K_2.$$

By compactness of K_2, this open cover contains a finite subcover, that is, for some finite $F \subset \mathcal{L}$,

$$\bigcup_{p \in F} (\pi_{\text{int}}(p) + U) = H \supset K_2. \tag{2.3}$$

Fix an arbitrary $z \in \mathbb{R}^d \times H$. Then there exists $p \in \mathcal{L}$ such that $z - p \in K_1 \times K_2$ by (2.2). Then $\pi_{\text{int}}(z-p) \in K_2$, and by (2.3) we have $\pi_{\text{int}}(z-p) \subset \pi_{\text{int}}(q) + U$ for some $q \in F$; equivalently,

$$\pi_{\text{int}}(z - p - q) \in U.$$

Note that

$$\pi(z-p-q) = \pi(z-p) - \pi(q) \in K_1 - \pi(F) =: K \subset \mathbb{R}^d,$$

with K compact. It follows that

$$z = p + q + (z - p - q) \in \mathcal{L} + (K \times U),$$

and since z was arbitrary, the lemma is proved. □

Now, to prove Claim (ii) of the theorem, we assume that the window $W \subset H$ has nonempty interior and pick an open $U \subset -W$. We apply the lemma to find a compact set $K \subset \mathbb{R}^d$ such that $\mathbb{R}^d \times H = \mathcal{L} + (K \times U)$. Pick any $x \in \mathbb{R}^d$. We have

$$(x, 0) = p + (k, u)$$

for some $p \in \mathcal{L}$, $k \in K$, and $u \in U$. Then

$$0 = \pi_{\text{int}}(p) + u \implies \pi_{\text{int}}(p) = -u \in -U \subset W.$$

Thus $\pi(p)\Lambda(W)$, by definition, and we have

$$x = \pi(p) + k \in \Lambda(W) + K.$$

We have shown that $\mathbb{R}^d = \Lambda(W) + K$, which means $\Lambda(W)$ is relatively dense.

(iii) If $\Lambda(W)$ is a model set, then it is Delone by parts (i) and (ii). Moreover,

$$\Lambda(W) - \Lambda(W) \subset \Lambda(W - W),$$

which is uniformly discrete by part (i). This means that $\Lambda(W)$ is Meyer, as desired. □

Lemma 2.8. *Let* $(\mathbb{R}^d, H, \mathcal{L})$ *be a CPS and* W *a bounded window with nonempty interior. If* $\pi_{\text{int}}|_{\mathcal{L}}$ *is 1-to-1, then* $\Lambda(W)$ *is non-periodic.*

Remark 2.9. Recall that in the definition of CPS we required that $\pi|_{\mathcal{L}}$ be 1-to-1. Assuming that $\mathcal{L}$ is an integral lattice, we have, in view of Proposition 2.2:

$$\pi_{\text{int}}|_{\mathcal{L}} \text{ is 1-to-1} \iff (\mathbb{R}^d \times \{0\}) \cap \mathcal{L} = \{\mathbf{0}\};$$

$$\pi|_{\mathcal{L}} \text{ is 1-to-1} \iff (\{0\} \times H) \cap \mathcal{L} = \{\mathbf{0}\}.$$

Proof. Suppose that $\Lambda(W) + z = \Lambda(W)$ for some $z \neq 0$. Then $z = \pi(u)$ for some $u \in \mathcal{L}$, $u \neq 0$. A direct verification shows that

$$\Lambda(W) + z = \Lambda(W) + \pi(u) = \Lambda(W + \pi_{\text{int}}(u)).$$

Note that $g := \pi_{\text{int}}(u) \neq 0$ by assumption. It remains to observe that $\Lambda(W) \neq \Lambda(W + g)$ for any $g \neq 0$. Indeed, $U := W^\circ \setminus (W + g)$ is nonempty (otherwise, the interior of W would be invariant under the translation by g, which is impossible for bounded sets). By Theorem 2.1, $\Lambda(U)$ is relatively dense, hence nonempty. Now, $\Lambda(U) \subset \Lambda(W) \setminus \Lambda(W + g)$, which shows theses model sets are not the same. □

References

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[2] Pytheas N. Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Edited by V. Berthé, S. Ferenczi, C. Mauduit, and A. Siegel. Lecture Notes in Math., 1794, Springer-Verlag, Berlin, 2002.

[3] M. Senechal, *Quasicrystals and Geometry*, Cambridge, 1995.