

APERIODIC ORDER – LECTURE 3 SUMMARY

1. PERRON-FROBENIUS THEOREM

Theorem 1.1 (Frobenius-Perron). *Let M be a primitive $d \times d$ matrix. Then*

- (i) *there exists a dominant eigenvalue $\theta > 0$, which is simple and strictly greater than any other eigenvalue in absolute value;*
- (ii) *the eigenvector corresponding to θ is strictly positive (component-wise): $\mathbf{x} > \mathbf{0}$;*
- (iii) *for every non-negative vector $\mathbf{y} \geq \mathbf{0}$, the sequence of vectors $M^n \mathbf{y}$ approaches \mathbf{x} in direction, more precisely,*

$$\lim_{n \rightarrow \infty} \frac{M^n \mathbf{y}}{\|M^n \mathbf{y}\|} \rightarrow \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

There are many proofs; see e.g. [1].

Idea of the proof. Without loss of generality we can assume that M is strictly positive. Consider the action of M on the cone of non-negative vectors \mathbb{R}_+^d . It is clear that

$$M(\mathbb{R}_+^d) \subset \mathbb{R}_+^d,$$

and moreover, it is mapped “strictly inside”. It is intuitively clear (but requires proof, which we omit) that the action of M on the set of directions (namely, $\mathbf{y} \mapsto M\mathbf{y}/\|M\mathbf{y}\|$) is strictly contracting, and so the nested sequence of cones $M^n(\mathbb{R}_+^d)$ converges to a single ray:

$$\bigcap_{n=1}^{\infty} M^n(\mathbb{R}_+^d) = \mathbb{R}_+ \mathbf{x} = \{c\mathbf{x} : c \geq 0\}$$

for some $\mathbf{x} > \mathbf{0}$. We claim that \mathbf{x} is an eigenvector for M . Indeed,

$$M(\mathbb{R}_+ \mathbf{x}) = M \left(\bigcap_{n=1}^{\infty} M^n(\mathbb{R}_+^d) \right) = \bigcap_{n=2}^{\infty} M^n(\mathbb{R}_+^d) = \mathbb{R}_+ \mathbf{x},$$

hence $M\mathbf{x} = \theta\mathbf{x}$ for some $\theta > 0$. This implies all the claims of the theorem. □

TERMINOLOGY. The dominant eigenvalue of M is called the Perron-Frobenius (PF) eigenvalue of M , and the corresponding eigenvector \mathbf{x} is called the PF eigenvector. It is usually normalized so that $\|\mathbf{x}\|_1 = \sum_{j=1}^d x_j = 1$.

Now, suppose that M is a $d \times d$ primitive **integer** matrix, as is the case for substitution matrices. Then its characteristic polynomial is an integer monic polynomial, which can be factored over \mathbb{C} :

$$\chi_M(t) = \det(tI - M) = (-1)^d(t - \theta_1)(t - \theta_2) \cdots (t - \theta_d).$$

Its roots θ_j are *algebraic integers*, by definition. In particular, if $\theta_1 = \theta$ is the dominant root, or PF eigenvalue of M , then $\theta_1 > 1$. (This follows from the fact that the product of all roots of the minimal polynomial of an algebraic integer is the absolute value of its lowest coefficient, hence ≥ 1 .)

Definition 1.2. An algebraic integer θ whose *Galois conjugates*, i.e. the other roots of the minimal polynomial, are strictly less than θ in absolute value, is called a **Perron number**.

Thus, Perron-Frobenius Theorem implies that the dominant eigenvalue of a primitive integer matrix is a Perron number. It is interesting that the converse is also true! This is important in symbolic dynamics.

Theorem 1.3 (D. Lind, 1986). *For any Perron number θ there exists a primitive matrix M such that θ is the PF eigenvalue of M .*

2. LETTER AND WORD FREQUENCIES. APERIODICITY.

Definition 2.1. Let W be a word, $|W| = k$, and u a sequence in a finite alphabet $\mathcal{A} = \{1, \dots, m\}$. The frequency of W in u is defined by

$$\text{freq}_u(W) = \lim_{N \rightarrow \infty} \frac{\#\{i \leq N - k : u[i, i + k - 1] = W\}}{N},$$

assuming the limit exists.

The numerator above is the number of occurrences of W in $u[0, N - 1] := u_0 u_1 \dots u_{N-1}$. Note that these occurrences may be overlapping. We say that the frequency of W in u exists *uniformly* if

$$\text{freq}_u(W) = \lim_{N \rightarrow \infty} \frac{\#\{i \in [\ell, \ell + N - k - 1] : u[i, i + k - 1] = W\}}{N},$$

uniformly in $\ell \in \mathbb{N}$.

Theorem 2.2. *Let u be a fixed point of a primitive substitution ζ on the alphabet $\mathcal{A} = \{1, \dots, m\}$. Then*

- (i) *for any $W \in \mathcal{L}(u)$, the frequency $\text{freq}_u(W)$ exists uniformly;*

(ii) Let $x_i = \text{freq}_u(i)$ (that is, the frequency of a single letter i). Then $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ is

the PF eigenvector of S_ζ , normalized so that $\|\mathbf{x}\|_1 = 1$.

Proof of (ii). Recall the population vector of a word W : $\ell(W) = \begin{bmatrix} \ell_1(W) \\ \vdots \\ \ell_m(W) \end{bmatrix}$, where

$\ell_i(W)$ is the number of letters i in the word W .

Step 1. We will prove that for any $i, j \in \mathcal{A}$,

$$\lim_{n \rightarrow \infty} \frac{\ell_i(\zeta^n(j))}{|\zeta^n(j)|} = x_i,$$

geometrically fast. Denote by \mathbf{e}_j the j -th unit vector. We have

$$\ell(\zeta^n(j)) = S_\zeta^n(\ell(j)) = S_\zeta^n \mathbf{e}_j.$$

Therefore,

$$\ell_i(\zeta^n(j)) = \langle S_\zeta^n \mathbf{e}_j, \mathbf{e}_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^m .

Now let E be the invariant subspace for S_ζ corresponding to the eigenvectors and root vectors of eigenvalues other than $\theta = \theta_1$, the PF eigenvalue of S_ζ (if S_ζ is diagonalizable, then E is the linear span of the other eigenvectors). Then the restriction of S_ζ to E , denoted by $S_\zeta|_E$, is a linear transformation with maximal eigenvalue θ_2 , such that $|\theta_2| < \theta_1$, by the Perron-Frobenius Theorem. We will need the following standard

Lemma 2.3. *Given a linear transformation A with spectral radius ρ_A , for any $\delta > 0$ there exists $C > 0$ such that*

$$(2.1) \quad \|A^n\| \leq C(\rho_A + \delta)^n, \quad \text{for any } n \in \mathbb{N}.$$

In fact, $\|A^n\| \leq Cn^j \rho_A^n$, where j is the size of the maximal Jordan block of A , but we will not need it.

Now represent \mathbf{e}_j as follows:

$$\mathbf{e}_j = a\mathbf{x} + P_E \mathbf{e}_j,$$

where P_E is the projection onto the subspace E , parallel to \mathbf{x} . We have

$$(2.2) \quad S_\zeta^n \mathbf{e}_j = a\theta_1^n \mathbf{x} + S_\zeta^n (P_E \mathbf{e}_j).$$

Observe that by Lemma 2.3,

$$(2.3) \quad \|\mathbf{S}_\zeta^n(P_E \mathbf{e}_j)\| \leq C(|\theta_2| + \delta)^n \leq C(\theta_1 - \delta)^n,$$

for $\delta > 0$ sufficiently small. Therefore, the first term in (2.2) is dominant (we note that $a > 0$ by part (iii) of Perron-Frobenius Theorem). We obtain from (2.2) and (2.3):

$$\ell_i(\zeta^n(j)) = \langle \mathbf{S}_\zeta^n \mathbf{e}_j, \mathbf{e}_i \rangle = a\theta_1^n \langle \mathbf{x}, \mathbf{e}_i \rangle + O((\theta_1 - \delta)^n) = ax_i\theta_1^n + O((\theta_1 - \delta)^n).$$

On the other hand, using that $\|\mathbf{x}\|_1 = \sum_{i=1}^m x_i = 1$, we have

$$|\zeta^n(j)| = \sum_{i=1}^m \ell_i(\zeta^n(j)) = a\theta_1^n + O((\theta_1 - \delta)^n).$$

It follows that

$$\left| \frac{\ell_i(\zeta^n(j))}{|\zeta^n(j)|} - x_i \right| \leq C \left(\frac{\theta_1 - \delta}{\theta_1} \right)^n,$$

concluding the proof of Step 1.

Step 2. We need a preliminary result. For a word W , its initial part (beginning) of any length is called its *prefix*, and its final part (ending) of any length is called its *suffix*.

Lemma 2.4. *Let ζ be a primitive substitution on an alphabet \mathcal{A} which has a fixed point $u = \zeta(u) \in \mathcal{A}^{\mathbb{N}}$. We consider the language $\mathcal{L}(u)$ consisting of all words which occur in u . Show that every word $W \in \mathcal{L}(u)$ can be expressed as follows:*

$$(2.4) \quad W = U_0 \zeta(U_1) \zeta^2(U_2) \dots \zeta^k(U_k) \zeta^k(V_k) \dots \zeta^2(V_2) \zeta(V_1) V_0,$$

where $k \in \mathbb{N}$, U_0, \dots, U_k are suffixes (possibly empty!) of the words $\zeta(a)$, $a \in \mathcal{A}$ and V_0, \dots, V_k are prefixes (possibly empty!) of the words $\zeta(a)$, $a \in \mathcal{A}$, but at least one of U_k, V_k is nonempty.

This is sometimes called the *prefix-suffix decomposition* of the word W , or its “accordion form”, and it will be used repeatedly. The proof of the lemma is left as an **EXERCISE**, see *Assignment 1*.

Using Step 1 and the lemma, it is not hard to conclude the proof of the claim that $\text{freq}_u(i)$ exist uniformly. \square

Corollary 2.5. *Let u be a fixed point of a primitive substitution ζ , and suppose that the PF eigenvalue θ_1 of \mathbf{S}_ζ is irrational (this holds, e.g. when the characteristic polynomial $\chi_\zeta(t)$ is irreducible). Then u is non-periodic.*

Proof. If u is eventually periodic, ending with V^∞ for some finite word V , then clearly,

$$\text{freq}_u(i) = \frac{\ell_i(V)}{|V|} \in \mathbb{Q}.$$

On the other hand, by Theorem 2.2, we have that $\text{freq}_u(i) = x_i$ are the components of the PF eigenvector for S_ζ . Since $S_\zeta \mathbf{x} = \theta_1 \mathbf{x}$, we immediately see that θ_1 has to be rational, contradicting the assumption. \square

The last corollary doesn't apply to substitutions of constant length, which have $\theta = \ell \in \mathbb{N}$ (the common length of $\zeta(j)$, $j \in \mathcal{A}$). Fortunately, there is an easy criterion for nonperiodicity.

Lemma 2.6 (Pansiot [2]). *Let ζ be a substitution of constant length, which is 1-to-1 on the alphabet \mathcal{A} . Let u be a fixed point of ζ . It is non-periodic if and only if there is a letter $\alpha \in \mathcal{A}$ which appears in u with at least two distinct right extensions $\alpha\beta, \alpha\gamma$ (note that one of β, γ may equal α , we just require that $\beta \neq \gamma$).*

Proof. One direction is obvious: if every letter, which appears in u , has a unique right continuation, then u is periodic. Conversely, suppose that $\alpha\beta$ and $\alpha\gamma$ both appear in u (we write this as $\alpha\beta, \alpha\gamma \in \mathcal{L}(u)$), and $\beta \neq \gamma$. We can write

$$\zeta(\alpha\beta) = \zeta(\alpha)W_1\beta_1U_1, \quad \zeta(\alpha\gamma) = \zeta(\alpha)W_1\gamma_1V_1, \quad \text{with } \beta_1 \neq \gamma_1.$$

Here W_1 is the common prefix of $\zeta(\beta), \zeta(\gamma)$, maybe empty. Since ζ has constant length and is 1-to-1 on \mathcal{A} , we have $\beta_1 \neq \gamma_1$. Thus, $U_1 := \zeta(\alpha)W_1 \in \mathcal{L}(u)$ and has at least two right extensions: β_1 and γ_1 . Now we iterate the procedure inductively and obtain a sequence of words $U_n \in \mathcal{L}(u)$,

$$U_n = \zeta^n(\alpha)\zeta^{n-1}(W_1) \dots \zeta(W_{n-1})W_n,$$

with at least two right extensions β_n and γ_n . Here W_j are all proper prefixes of words $\zeta(i)$, $i \in \mathcal{A}$ hence $|W_j| < \ell$. We have $|U_n| \geq |\zeta^n(\alpha)| = \ell^n$, and

$$|U_n| = |\zeta^n(\alpha)| + \sum_{k=0}^{n-1} |\zeta^k(W_k)| \leq \ell^n + (\ell - 1) \sum_{k=0}^{n-1} \ell^k < 2\ell^n.$$

Since $\ell \geq 2$, it follows that $|U_n| > |U_{n-1}|$ for all n , and therefore all U_n are distinct. Thus, there are arbitrarily long words in $\mathcal{L}(u)$ with a non-unique right extension, and hence the complexity $p_u(n)$ is unbounded. By a theorem from Lecture 1 we obtain that u is non-periodic. \square

3. SYMBOLIC DYNAMICAL SYSTEMS (SEE CHAPTER 1 IN [3])

Let \mathcal{A} be a finite alphabet. We consider the space of sequences $\mathcal{A}^{\mathbb{N}}$ (one-sided) as an infinite products of finite sets with discrete topology. It is equipped with a metric

$$d(u, v) = 2^{-\min\{n \in \mathbb{N}: u_n \neq v_n\}}$$

This way, $\mathcal{A}^{\mathbb{N}}$ becomes a compact metric space, which is topologically a Cantor set (compact, totally disconnected, without isolated points). We consider the *one-sided left shift map*

$$\sigma : \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}, \quad \sigma((u_n)_{n \geq 0}) = (u_{n+1})_{n \geq 0}.$$

Note that σ is continuous, onto, but not 1-to-1.

We will also consider $\mathcal{A}^{\mathbb{Z}}$, the set of two-sided infinite sequences, on which the shift map σ is a homeomorphism. Now the metric is defined by

$$d(u, v) = 2^{-\min\{n \in \mathbb{N}: u_{|n|} \neq v_{|n|}\}}$$

We will sometimes work with $\mathcal{A}^{\mathbb{N}}$ and sometimes with $\mathcal{A}^{\mathbb{Z}}$. For simplicity, in this section below we consider $\mathcal{A}^{\mathbb{N}}$, but everything is essentially unchanged for $\mathcal{A}^{\mathbb{Z}}$.

Definition 3.1. Denote by

$$\mathcal{O}(u) = \{\sigma^n(u) : n \geq 0\}$$

the orbit of u under σ , and consider $X_u := \overline{\mathcal{O}(u)}$, the orbit closure of u . Then X_u is a compact subset of $\mathcal{A}^{\mathbb{N}}$ invariant under σ . We call (X_u, σ) the *symbolic dynamical system associated with u* .

Lemma 3.2. *The following are equivalent for u and v in $\mathcal{A}^{\mathbb{N}}$:*

- (i) $w \in X_u$;
- (ii) *there exists a sequence $k_n \rightarrow \infty$ such that $w_0 \dots w_n = u_{k_n} \dots u_{k_n+n}$;*
- (iii) $\mathcal{L}_n(w) \subset \mathcal{L}_n(u)$ for all n .

Definition 3.3. A sequence u is said to be *uniformly recurrent* (or *minimal*) if every word that occurs in u occurs with bounded gaps. Equivalently: for every n there exists K , such that $u_0 \dots u_n$ occurs in every subword of u of length K .

A topological dynamical system (i.e. a continuous self-map of a compact metric space) is called *minimal* if every orbit is dense, or equivalently, there are no nontrivial closed invariant subsets.

Theorem 3.4. *A sequence u is uniformly recurrent if and only the associated symbolic dynamical system is minimal.*

This will be proved in the next lecture.

Lemma 3.5. *Let u be a fixed point of a primitive substitution. Then u is uniformly recurrent.*

Proof. Let $u = \zeta(u) = u_0u_1\dots$, where ζ is a primitive substitution on \mathcal{A} . First we show that every letter $\alpha \in \mathcal{A}$ occurs with bounded gaps. Let $k \in \mathbb{N}$ be such that $\zeta^k(\beta)$ contains all letters of \mathcal{A} , for every $\beta \in \mathcal{A}$ (exists by primitivity). We have

$$u = \zeta(u) = \zeta^k(u) = \zeta^k(u_0)\zeta^k(u_1)\dots\zeta^k(u_n)\dots$$

Every word $\zeta^k(u_j)$ contains α , hence the gap between consecutive occurrences of α is not greater than $\max_{\beta \in \mathcal{A}} |\zeta^k(\beta)|$.

Now let W be an arbitrary word from $\mathcal{L}(u)$ (the language of u). Then W occurs in $\zeta^n(\alpha)$ for some $n \in \mathbb{N}$ and α (actually, any α by primitivity). Writing

$$u = \zeta^{kn}(u) = \zeta^n(\zeta^k(u_0))\zeta^n(\zeta^k(u_1))\dots,$$

we see that every $\zeta^n(\zeta^k(u_j))$ contains $\zeta^n(\alpha)$, hence W . It follows that the gap between consecutive occurrences of W is at most $\max_{\beta \in \mathcal{A}} |\zeta^{nk}(\beta)|$. \square

Exercise. Modify the proof to show that u is *linearly recurrent*, that is, there exists $C > 0$ such that for any word W in u , the gap between two consecutive occurrences of W is not greater than $C|W|$.

REFERENCES

- [1] M. Boyle, *NOTES ON THE PERRON-FROBENIUS THEORY OF NONNEGATIVE MATRICES*, available at <http://www.math.umd.edu/~mboyle/courses/475sp05/spec.pdf>
- [2] J.-J. Pansiot, Decidability of periodicity for infinite words. *RAIRO Inform. Théor. Appl.* **20** (1986), no. 1, 43–46.
- [3] Pytheas N. Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Edited by V. Berthé, S. Ferenczi, C. Mauduit, and A. Siegel. Lecture Notes in Math., 1794, Springer-Verlag, Berlin, 2002.