APERIODIC ORDER – LECTURE 4 SUMMARY

1. Symbolic dynamical systems (see [1, Section 1.1.3])

We consider sequence spaces $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$, with a finite alphabet $\mathcal{A} = \{0, \dots, m-1\}$ (or $\mathcal{A} = \{1, \dots, m\}$, or $\mathcal{A} = \{a, b\}$, or $\dots$). The metric is defined by

$$\rho(x, y) = 2^{-k}, \quad \text{where } k = \min\{|n| : \ x_n \neq y_n\}.$$

In other words, two sequences x and y are within a distance of at most 2^{-k} if and only if they coincide up to index k (or coincide from $-k$ to $+k$ in the 2-sided case).

We (almost) proved that these spaces are compact.

We then consider the left shift transformation T defined by

$$(Tx)_n = x_{n+1} \quad \text{for all } n.$$

Note that T is continuous, it is invertible on $\mathcal{A}^{\mathbb{Z}}$ and surjective, but non-injective on $\mathcal{A}^{\mathbb{N}}$.

A **symbolic dynamical system**, by definition, is (X, T), where $X \subset \mathcal{A}^{\mathbb{N}}$ (or $\subset \mathcal{A}^{\mathbb{Z}}$) is closed and T-invariant, with T the left shift.

Lemma 1.1. *Let $u \in \mathcal{A}^{\mathbb{N}}$. Consider $X_u = \mathrm{clos}\{T^n u : n \geq 0\}$, the orbit closure of u. Then $x \in X_u$ if and only if $\mathcal{L}(x) \subset \mathcal{L}(u)$, where $\mathcal{L}(x)$ denotes the language of x (the set of all finite words which appear in x).*

Recall that u is called *uniformly recurrent* if every word in u appears in it with bounded gaps. A top. dyn. system is *minimal* if every orbit is dense. In view of the last lemma, (X_u, T) is minimal whenever $\mathcal{L}(x) \subset \mathcal{L}(u)$ implies $\mathcal{L}(x) = \mathcal{L}(u)$.

Theorem 1.2. *Let $u \in \mathcal{A}^{\mathbb{N}}$. The system (X_u, T) is minimal if and only if u is uniformly recurrent.*

Proof sketch. Suppose first that X_u is minimal, but u is not uniformly recurrent. Then some word in u, say W, does not appear in it with bounded gaps. This means we can find arbitrary long subwords of u without W. Shifting u in such a way that these subwords start at index 0 and passing to a convergent subsequence by compactness, we will obtain an element $x \in X_u$ which does not contain W. This means that the language of x is a proper subset of $\mathcal{L}(u)$ contradicting minimality.

Conversely, suppose that u is uniformly recurrent and take $x \in X_u$. We want to show that $\mathcal{L}(x) = \mathcal{L}(u)$, that is, every word of u appears in x. Let W be such a word, $W \in \mathcal{L}(u)$. By the bounded gap property, there is $N \in \mathbb{N}$ such that every subword of u of length N contains W (it could be very large, say, N could be 10^6 whereas $|W| = 10$, but it exists). Then take any word in x of length N. Since every word of x appears in u, and every subword of u of length N contains W, it follows that x contains W, as desired. □

Corollary 1.3. *Let u be a fixed point of a primitive substitution. Then (X_u, T) is minimal.*

Example 1.4. There exist non-primitive substitutions with such a property: let $\zeta(0) = 0010$, $\zeta(1) = 0$. This is a well-known *Chacon substitution*. It is easy to see that the fixed point of ζ is uniformly recurrent.

Exercise. If ζ is a *growing substitution*, that is, $|\zeta^n(\alpha)| \to \infty$ for every letter $\alpha \in \mathcal{A}$, then minimality of X_u implies primitivity.

2. Topological dynamical systems (see [1, Section 1.4.1])

A **topological dynamical system** is a pair (X, T), where X is a compact metric space and $T : X \to X$ is continuous. It is called invertible if T is a homeomorphism.

Let $\mathcal{M}_X$ be the set of Borel probability measures on X. A measure $\mu \in \mathcal{M}_X$ is said to be *invariant* for T, or T-invariant if

$$\mu(T^{-1}B) = \mu(B) \quad \text{for all Borel sets } B \subset X.$$

Theorem 2.1 (Krylov and Bogoliubov)**.** *For every topological dynamical system there is at least one invariant measure.*

Sketch of the proof. The proof relies on some facts from Functional Analysis.

Consider $C(X)$, the Banach space of continuous (complex-valued) functions on the compact set X. Recall *Riesz Representation Theorem*, which asserts that the space complex-valued Borel measures on X can be identified with the space of continuous functionals on $C(X)$. Now, note that $\mathcal{M}_X$ is a closed subset of the unit ball in the space of Borel measures (a measure is in $\mathcal{M}_X$ iff it is positive and $\int 1\, d\mu = \mu(X) = 1$). By *Banach-Alaoglu Theorem*, the unit ball of the dual space is compact in the weak* topology. This means, in our case, that for any sequence $\mu_n \in \mathcal{M}_X$ there is a subsequence μ_{n_k} such that

$$\int f\, d\mu_{n_k} \to \int f\, d\mu \quad \text{for some } \mu \in \mathcal{M}_X.$$

Now we start with the proof of the existence of invariant measure: for any $a \in X$ let δ_a denote the point mass at a. Clearly, $\delta_a \in \mathcal{M}_X$ and $\int f(x)\,d\delta_a(x) = f(a)$. Pick any $x \in X$ and define

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x}.$$

We have $\mu_n \in \mathcal{M}_X$, so there is a subsequence which converges weak* to some measure $\mu \in \mathcal{M}_X$. We claim that μ is T-invariant. Observe that T-invariance of μ means, by definition, the equality $\mu \circ T^{-1} = \mu$. By Riesz Representation Theorem, a measure is uniquely determined by its action on $C(X)$, hence $\mu \circ T^{-1} = \mu$ iff for every continuous f on X,

$$\int f(z)\,d\mu(z) = \int f(z)\,d(\mu \circ T^{-1})(z) = \int f(Ty)\,d\mu(y).$$

(In the last step we used the change of variable formula for integrals.) Now observe that

$$\int f(z)\,d\mu_n(z) = \frac{1}{n}\sum_{i=0}^{n} f(T^i x)$$

and

$$\int f(Ty)\,d\mu_n(y) = \frac{1}{n}\sum_{i=1}^{n+1} f(T^i x).$$

It follows that

$$\left|\int f(z)\,d\mu_n(z) - \int f(Ty)\,d\mu_n(y)\right| = \frac{|f(x) - f(T^n x)|}{n} \to 0, \quad \text{as } n \to \infty,$$

because f is bounded (it is a continuous function on a compact set). This implies that for $\mu =$ the weak*-cluster point of μ_n we have $\int f(z)\,d\mu(z) = \int f(Ty)\,d\mu(y)$, and so μ is T-invariant. □

References

[1] Pytheas N. Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Edited by V. Berthé, S. Ferenczi, C. Mauduit, and A. Siegel. Lecture Notes in Math., 1794, Springer-Verlag, Berlin, 2002.