APERIODIC ORDER – LECTURE 5 SUMMARY

1. Intro to topological dynamics (cont.)

Definition 1.1. A topological dynamical system (X, T) is called *uniquely ergodic* (UE) if it has only one invariant probability measure.

Example 1.2.
(i) Irrational circle rotation is minimal and uniquely ergodic. (We prefer to use additive notation for rotations: $R_\alpha : x \mapsto x + \alpha \pmod{\mathbb{Z}}$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.)
(ii) More generally, consider a translation on the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$T_\alpha : \mathbf{x} \mapsto \mathbf{x} + \alpha \mod \mathbb{Z}^d,$$

where $\mathbf{x} = (x_1, \ldots, x_d)$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$.

Theorem 1.3 (Kronecker)**.** *The toral translation T_α is minimal if and only if $\{1, \alpha_1, \ldots, \alpha_d\}$ are linearly independent over the rationals; that is: $\sum_{j=1}^{d} k_j\alpha_j \in \mathbb{Z}$ implies $k_j = 0$, $j \le d$.*

Theorem 1.4 (Weyl)**.** *The toral translation T_α is minimal if and only if it is uniquely ergodic. (The unique invariant probability measure is the Lebesgue=Haar measure on the torus.)*

2. Measure-preserving dynamical systems (see [1, 1.4])

Definition 2.1. Let $(X, \mathcal{B}, \mu)$ be a probability measure space. A measurable transformation $T : X \to X$ is measure-preserving if $\mu(T^{-1}E) = \mu(E)$ for all $E \in \mathcal{B}$. We will abbreviate by writing that $(X, \mu, \mathcal{B}, T)$ is a m.-p. s. It is called *invertible* if T^{-1} exists μ-a.e. and is measurable.

Definition 2.2. Let $(X, \mathcal{B}_X, \mu, T)$ and $(Y, \mathcal{B}_Y, \nu, S)$ be measure-preserving systems on probability spaces.

(i) *Factor:* We say that $(Y, \mathcal{B}_Y, \nu, S)$ is a factor of $(X, \mathcal{B}_X, \mu, T)$ if there are sets $X' \subset X$ and $Y' \subset Y$ of full measure such that $TX' \subset X'$, $SY' \subset Y'$, and a measure-preserving map $\phi : X' \to Y'$ such that

$$\phi \circ T(x) = S \circ \phi(x) \text{ for all } x \in X'.$$

(i) *Isomorphism:* We say that the systems are isomorphic if, in addition, ϕ is invertible.

In measure theory it is natural to ignore sets of zero measure. For metric spaces we will assume that the σ-algebra is Borel σ-algebra and will often drop $\mathcal{B}$ from notation. Also, all maps are assumed to be measurable, even if not stated explicitly.

Example 2.3. Consider $\mathcal{A} = \{0, 1\}$ and the one-sided full shift space $\mathcal{A}^{\mathbb{N}}$ with the infinite product (Bernoulli) measure $\mu = \prod_{\mathbb{N}}(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)$ and the left shift map T. On the other hand, consider $(\mathbb{T}, m, T_2)$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, m is Lebesgue (=Haar) measure, and $T_2(x) = 2x \pmod{\mathbb{Z}}$ is the doubling map. These systems — $(\mathcal{A}^{\mathbb{N}}, \mu, T)$ and $(\mathbb{T}, m, T_2)$ — are measure-theoretically isomorphic, but not topologically conjugate.

Definition 2.4. An m.-p. s. $(X, \mathcal{B}, \mu, T)$ is *ergodic* if $T^{-1}B = B,\ B \in \mathcal{B}$, implies $\mu(B) \in \{0, 1\}$.

Lemma 2.5.

(i) *A system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if every T-invariant measurable function f (i.e. $f(Tx) = f(x)$ for a.e. x) is constant (a.e.).*

(ii) *A uniquely ergodic system is ergodic with respect to the unique invariant measure.*

Part (i) will be an Exercise.

Theorem 2.6 (Von Neumann's Mean Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Then*

$$\frac{1}{N} S_N f(x) := \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \to Pf, \quad \text{as } N \to \infty, \quad \text{for all } f \in L^2(X, \mu),$$

where P is the orthogonal projection onto the subspace of T-invariant L^2-functions, and the convergence is in L^2.

Theorem 2.7 (Birkhoff's Pointwise Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be an* ergodic *measure-preserving system. Then*

$$\frac{1}{N} S_N f(x) \to \int_X f \, d\mu, \quad \text{as } N \to \infty, \quad \text{for all } f \in L^1(X, \mu),$$

where convergence is **almost everywhere** *and in L^1.*

Corollary 2.8. *Suppose that $(X, \mathcal{B}, T)$ is a measurable dynamical system (e.g., topological dynamical system, with Borel σ-algebra $\mathcal{B}$), for which there are two different invariant probability measures μ and ν. Then $\mu \perp \nu$, that is, the measures are mutually singular.*

This will be an Exercise.

Theorem 2.9. *Let $T : X \to X$ be a continuous map on a compact metric space. Then the following are equivalent:*

(i) *T is uniquely ergodic;*
(ii) *for every $f \in C(X)$, we have $\frac{1}{N} S_N f(x) \to C_f$, where the constant C_f is independent of x.*

Moreover, $C_f = \int_X f \, d\mu$, where μ is the unique invariant probability measure, and the convergence in (ii) is **uniform**.

Exercise. Prove the equivalence (i) $\Longleftrightarrow$ (ii).

Corollary 2.10. *Let $u \in \mathcal{A}^{\mathbb{N}}$ for a finite alphabet $\mathcal{A}$, and consider the symbolic dynamical system (X_u, T). Suppose that this system is uniquely ergodic. Then for any word W which occurs in u there exists a uniform frequency $\mathrm{freq}_u(W)$, equal to $\mu([W])$.* (In fact, the converse is also true.)

3. Elements of Spectral Theory (see [1, 1.5])

Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. The operator $U_T : f \mapsto f \circ T$ on $L^2(X, \mu)$ is called the *Koopman operator* associated with T.

Lemma 3.1. *The Koopman operator is an isometry. If T is invertible, then U_T is unitary and $U_{T^{-1}} = U_T^{-1}$.*

Spectral theory of the measure-preserving transformation is the spectral theory of the Koopman operator U_T.

Definition 3.2. A complex number λ is an eigenvalue of U_T if there exists $f \in L^2(X, \mu)$, called eigenfunction, such that $U_T f = \lambda f$. Equivalently: $f(Tx) = \lambda f(x)$ a.e. Note that $\lambda = 1$ is always an eigenvalue, corresponding to the "trivial" constant eigenfunction.

Lemma 3.3.
(i) *A measure-preserving system is ergodic if and only if $\lambda = 1$ is a simple eigenvalue.*
(ii) *If T is ergodic, then every eigenvalue is simple, and all eigenfunctions have constant modulus.*
(iii) *Eigenvalues of ergodic T form a group (a subgroup of the circle $\mathbb{T}$).*

Definition 3.4. The spectrum of T is said to be (pure) discrete if there is a Hilbert space basis for $L^2(X, \mu)$ consisting of eigenfunctions. The spectrum of T is said to be continuous if $\lambda = 1$ is the only eigenvalue (which is simple). (Such T are also called *weakly mixing.*)

Example 3.5.
(i) The circle rotation R_α has discrete spectrum.
(ii) The doubling map T_2 has continuous spectrum.

Lemma 3.6. *Let $\alpha \notin \mathbb{Q}$. The rotation $(\mathbb{T}, R_\alpha)$ is a measure-theoretic factor of a m.-p. s. $(X, \mathcal{B}, \mu, T)$ if and only if $e^{2\pi i \alpha}$ is an eigenvalue of T.*

Example 3.7. (Full details not provided.) Let G be a compact Abelian group, with the Haar measure m_G, and let $g \in G$. The translation $T_g: \ x \mapsto x + g$ on G has discrete spectrum. The eigenfunctions are the characters (elements of the Pontryagin dual G^*).

Theorem 3.8 (Halmos-Von Neumann)**.**
(i) *Two invertible and ergodic m.-p. s. with identical discrete spectrum are measure-theoretically isomorphic.*
(ii) *Every m.-p. s. with discrete spectrum is measure-theoretically isomorphic to a translation on a compact Abelian group, with the Haar measure.*

Definition 3.9. Let (X, T) be a topological dynamical system. A function $f \in C(X)$ is called a continuous eigenfunction, with eigenvalue λ, if $f(Tx) = \lambda f(x)$ for all x. The system is said to have topological discrete spectrum if the eigenfunctions span a dense subset of $C(X)$.

3.1. **Spectral type.** Let μ be a Borel probability measure on $\mathbb{T}$. The Fourier coefficients $(\widehat{\mu}(n))_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ are defined by

$$\widehat{\mu}(n) = \int_{\mathbb{T}} \varepsilon^{2\pi i n t} \, d\mu(t), \quad n \in \mathbb{Z}.$$

A sequence $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is *positive definite* if for any complex sequence $(z_j)_{j\geq 1}$, we have

$$\forall\, n \geq 1, \qquad \sum_{1\leq i,j\leq n} z_i \overline{z_j} a_{i-j} \geq 0.$$

Exercise. Show that $(\widehat{\mu}(n))_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is positive definite.

Theorem 3.10 (Bochner-Herglotz)**.** *Any positive definite sequence is the sequence of Fourier coefficients of a positive finite Borel measure.*

Exercise. Let U_T be a Koopman operator of an invertible m.-p. s. $(X, \mathcal{B}, \mu, T)$. Let $f \in L^2(X, \mu)$. Consider the sequence

$$\langle U_T^n f, f \rangle := \int_X U_T^n f(x) \overline{f(x)} \, d\mu(x), \quad n \in \mathbb{Z}.$$

Prove that this sequence is positive definite.

Definition 3.11. Let $f \in L^2(X, \mu)$. The spectral type ϱ_f of f is the finite Borel measure on the circle $\mathbb{T}$ such that $\widehat{\varrho}(n) = \langle U_T^n f, f \rangle$ for $n \in \mathbb{Z}$.

Exercise. Let f be an eigenfunction corresponding to an eigenvalue λ of unit norm: $\|f\|_2 = 1$. Prove that $\varrho_f = \delta_\lambda$.

From general spectral theory it follows that there exists a unique, up to mutual absolute continuity, measure ϱ such that $\varrho = \varrho_f$ for some $f \in L^2(X, \mu)$ and $\varrho_g << \varrho$ for any

$g \in L^2(X, \mu)$. (Here $<<$ denotes the relation of absolute continuity.) Such ϱ is called the *maximal spectral type* of U_T (or T).

3.2. **Correlation measures of a sequence.** Let $u \in \mathcal{A}^{\mathbb{N}}$ be a sequence, for a finite alphabet $\mathcal{A}$. Consider the sequence of uniformly bounded sequences γ_N, for $N \in \mathbb{N}$:

$$\gamma_N(k) = \frac{1}{N} \sum_{n<N} u_{n+k}\overline{u_n}.$$

By compactness, there exists at least one cluster point $\gamma = (\gamma_k)_{k \geq 0}$, that is,

$$\gamma(k) = \lim_{j \to \infty} \frac{1}{N_j} \sum_{n<N_j} u_{n+k}\overline{u_n}.$$

We extend it to negative k by defining $\gamma(-k) = \overline{\gamma(k)}$. Any such sequence is called a *correlation sequence* for u.

Exercise. Prove that any correlation sequence is positive definite.

Definition 3.12. A *correlation measure* of u is the measure whose sequence of Fourier coefficients is a correlation sequence of u.

Exercise. Suppose that (X_u, T) is a uniquely ergodic symbolic dynamical system. Prove that

(i) u has a unique correlation measure ϱ;
(ii) the correlation measure ϱ is the spectral type of the map $\pi_0 : x \mapsto x_0$, defined for $x = (x_j)_{j \in \mathbb{Z}} \in X_u$.

References

[1] Pytheas N. Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Edited by V. Berthé, S. Ferenczi, C. Mauduit, and A. Siegel. Lecture Notes in Math., 1794, Springer-Verlag, Berlin, 2002.