APERIODIC ORDER – LECTURE 6 SUMMARY

1. Elements of Spectral Theory (see [1, 1.5])

At the beginning of the class we had a quick review of the section on spectral theory, which was first covered on April 30. The only new material in this section was Subsection 1.1.

Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. The operator $U_T : f \mapsto f \circ T$ on $L^2(X, \mu)$ is called the *Koopman operator* associated with T.

Lemma 1.1. *The Koopman operator is an isometry. If* T *is invertible, then* U_T *is unitary and* $U_{T^{-1}} = U_T^{-1}$.

Spectral theory of the measure-preserving transformation is the spectral theory of the Koopman operator U_T.

Definition 1.2. A complex number λ is an eigenvalue of U_T if there exists $f \in L^2(X, \mu)$, called eigenfunction, such that $U_T f = \lambda f$. Equivalently: $f(Tx) = \lambda f(x)$ a.e. Note that $\lambda = 1$ is always an eigenvalue, corresponding to the "trivial" constant eigenfunction.

Lemma 1.3.

(i) *A measure-preserving system is ergodic if and only if* $\lambda = 1$ *is a simple eigenvalue.*

(ii) *If* T *is ergodic, then every eigenvalue is simple, and all eigenfunctions have constant modulus.*

(iii) *Eigenvalues of ergodic* T *form a group (a subgroup of the circle* $\mathbb{T}$*).*

Definition 1.4. The spectrum of T is said to be (pure) discrete if there is a Hilbert space basis for $L^2(X, \mu)$ consisting of eigenfunctions. The spectrum of T is said to be continuous if $\lambda = 1$ is the only eigenvalue (which is simple). (Such T are also called *weakly mixing.*)

Example 1.5.

(i) The circle rotation R_α has discrete spectrum.

(ii) The doubling map T_2 has continuous spectrum.

Lemma 1.6. *Let* $\alpha \notin \mathbb{Q}$. *The rotation* $(\mathbb{T}, R_\alpha)$ *is a measure-theoretic factor of a m.-p. s.* $(X, \mathcal{B}, \mu, T)$ *if and only if* $e^{2\pi i \alpha}$ *is an eigenvalue of* T.

Example 1.7. (Full details not provided.) Let G be a compact Abelian group, with the Haar measure m_G, and let $g \in G$. The translation $T_g : \ x \mapsto x + g$ on G has discrete spectrum. The eigenfunctions are the characters (elements of the Pontryagin dual G^*).

Theorem 1.8 (Halmos-Von Neumann)**.**

(i) *Two invertible and ergodic m.-p. s. with identical discrete spectrum are measure-theoretically isomorphic.*

(ii) *Every m.-p. s. with discrete spectrum is measure-theoretically isomorphic to a translation on a compact Abelian group, with the Haar measure.*

Definition 1.9. Let (X, T) be a topological dynamical system. A function $f \in C(X)$ is called a continuous eigenfunction, with eigenvalue λ, if $f(Tx) = \lambda f(x)$ for all x. The system is said to have topological discrete spectrum if the eigenfunctions span a dense subset of $C(X)$.

1.1. **Toral endomorphisms and automorphisms.** Let A be an integer $d \times d$ matrix. Consider the transformation

$$T_A(\mathbf{x}) = A\mathbf{x} \mod \mathbb{Z}^d, \ \ \mathbf{x} \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d.$$

It is well-defined (strictly speaking T_A acts not on $\mathbf{x}$, but on the equivalence class of $\mathbf{x}$). When A has non-zero determinant, T_A is a toral endomorphism, and when $\det(A) = \pm 1$, it is a toral automorphism. Note that $(T_A)^{-1} = T_{A^{-1}}$ in the latter case. Toral endomorphisms preserve the Haar (=Lebesgue) measure m_d. This is especially easy to see for the automorphisms. (T_A acts by matrix multiplication, which preserves the volume when $|\det(A)| = 1$, followed by piecewise translation, which also preserves the volume. For endomorphisms we need to check that the volume is preserved under the **inverse image**, which is not very hard either.)

Theorem 1.10. *Let A be an integer matrix, with $|\det(A)| = 1$. Consider the measure-preserving system $(\mathbb{T}^d, T_A, m_d)$. Then*

(i) *the system is ergodic if and only if none of the eigenvalues of A is a root of unity;*

(ii) *if the system is ergodic, then it has continuous spectrum.*

This was proved using the Fourier series on $\mathbb{R}^d$. Take $f \in L^2(\mathbb{T}^d, m_d)$ and expand it in terms of the orthonormal basis $\phi_{\mathbf{n}}(\mathbf{x}) = e^{2\pi i \langle \mathbf{x}, \mathbf{n} \rangle}$, $\mathbf{n} \in \mathbb{Z}^d$. Then consider the equation $f(T_A \mathbf{x}) = f(\mathbf{x})$, for ergodicity; and the equation $f(T_A \mathbf{x}) = \lambda f(\mathbf{x})$ for the eigenfunctions.

1.2. **Spectral type.** Let μ be a Borel probability measure on $\mathbb{T}$. The Fourier coefficients $(\widehat{\mu}(n))_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ are defined by

$$\widehat{\mu}(n) = \int_{\mathbb{T}} \varepsilon^{2\pi i n t} \, d\mu(t), \ \ \mathbf{n} \in \mathbb{Z}.$$

A sequence $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is *positive definite* if for any complex sequence $(z_j)_{j\geq 1}$, we have

$$\forall\ n \geq 1, \qquad \sum_{1\leq i,j\leq n} z_i\overline{z_j}a_{i-j} \geq 0.$$

EXERCISE. Show that $(\widehat{\mu}(n))_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is positive definite.

Theorem 1.11 (Bochner-Herglotz)**.** *Any positive definite sequence is the sequence of Fourier coefficients of a positive finite Borel measure.*

EXERCISE. Let U_T be a Koopman operator of an invertible m.-p. s. $(X, \mathcal{B}, \mu, T)$. Let $f \in L^2(X, \mu)$. Consider the sequence

$$\langle U_T^n f, f\rangle := \int_X U_T^n f(x)\overline{f(x)}\, d\mu(x), \quad n \in \mathbb{Z}.$$

Prove that this sequence is positive definite.

Definition 1.12. Let $f \in L^2(X, \mu)$. The spectral type ϱ_f of f is the finite Borel measure on the circle $\mathbb{T}$ such that $\widehat{\varrho}(n) = \langle U_T^n f, f\rangle$ for $n \in \mathbb{Z}$.

EXERCISE. Let f be an eigenfunction corresponding to an eigenvalue λ of unit norm: $\|f\|_2 = 1$. Prove that $\varrho_f = \delta_\lambda$.

From general spectral theory it follows that there exists a unique, up to mutual absolute continuity, measure ϱ such that $\varrho = \varrho_f$ for some $f \in L^2(X, \mu)$ and $\varrho_g << \varrho$ for any $g \in L^2(X, \mu)$. (Here $<<$ denotes the relation of absolute continuity.) Such ϱ is called the *maximal spectral type* of U_T (or T).

2. TILING AND DELONE SET DYNAMICAL SYSTEMS I

(SEE SECTION 2 OF [3] AND SECTIONS 2.2, 2.3 OF [2])

Here are I am going to be very brief; see the Notes for details.

We consider tilings of the Euclidean space $\mathbb{R}^d$.

- **Tile** is a compact set, which is a closure of its interior. (More precisely, it is a pair (A, i), where A is a compact set and i is its label. This is needed in order to distinguish between geometrically identical tiles. However, we are often sloppy about this, and consider tiles to be simply sets.)
- **Prototile set** is a finite set of tiles $\mathcal{A} = \{T_1, \ldots, T_m\}$; this is our "alphabet".
- **Patch** is a finite set of tiles which have disjoint interiors. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches whose every tile is a translate of one of the prototiles.
- **Tiling** is a set of tiles with disjoint interiors, whose union is $\mathbb{R}^d$. We usually assume that all tiles are translates of one of the prototiles.

- A tiling $\mathcal{T}$ has **finite local complexity** (FLC) if there are finitely many $\mathcal{T}$-patches of any given "size" up to translation. (Note that $\mathcal{T}$-patch is a finite subset of $\mathcal{T}$, and its "size" can be measured by the diameter of its support. The support is the union of the (supports) of the tiles.)
- A tiling is **repetitive** if for every patch P there exists $R = R(P)$ such that there is a translate of P in every ball of radius R. (This is sometimes called *uniform recurrence*, like for sequences.)
- **Tiling metric** is defined, roughly, as follows: two tilings are ε-close, for a small $\varepsilon > 0$ whenever they agree on a ball $B_{1/\varepsilon}(0)$, up to a translation of size $\leq \varepsilon$. Denote the distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ by $d(\mathcal{T}_1, \mathcal{T}_2)$.
- **Tiling space** is a closed, translation invariant set of tilings. One can consider tiling spaces $X_{\mathcal{A}}$ = the set of all tilings with tiles that are translates of the prototiles in $\mathcal{A}$. More often, given a tiling $\mathcal{T}$, we consider the associated tiling space which is its orbit closure $X_{\mathcal{T}} = \mathrm{clos}\{\mathcal{T} - g : g \in \mathbb{R}^d\}$.

Theorem 2.1. *The tiling metric is complete. If $\mathcal{T}$ has FLC, then the tiling space $X_{\mathcal{T}}$ is compact.*

- The group $\mathbb{R}^d$ acts on $X_{\mathcal{T}}$ by translations $T_g : \mathcal{S} \mapsto \mathcal{S} - g$. This is a continuous action, and we call the resulting system $(X_{\mathcal{T}}, T_g)_{g \in \mathbb{R}^d}$ the **(topological) tiling dynamical system** associated with $\mathcal{T}$. We usually simply write $(X_{\mathcal{T}}, \mathbb{R}^d)$.
- A tiling $\mathcal{T}$ is **aperiodic** if $\mathcal{T} - g = \mathcal{T}$ only for $g = \mathbf{0}$.

Lemma 2.2. *Let $\mathcal{T}$ be an aperiodic repetitive FLC tiling. Then a small neighborhood of $X_{\mathcal{T}}$ is homeomorphic to $C \times U$, where C is a Cantor set (totally disconnected perfect compact set) and U is a small open subset of $\mathbb{R}^d$.*

References

[1] Pytheas N. Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Edited by V. Berthé, S. Ferenczi, C. Mauduit, and A. Siegel. Lecture Notes in Math., 1794, Springer-Verlag, Berlin, 2002.

[2] E. A. Robinson, *Symbolic Dynamics and Tilings of $\mathbb{R}^d$*, Proceedings of Symposia in Applied Mathematics **60**, AMS, Providence, 2014, available at `http://wms.mat.agh.edu.pl/~sem_ds/abstract/nr_1.pdf`

[3] B. Solomyak, *Tilings and Dynamics*, available at `https://www.math.washington.edu/~solomyak/PREPRINTS/notes6.pdf`