APERIODIC ORDER – LECTURE 7 SUMMARY

1. Delone sets and associated dynamical systems

A Delone set $\Lambda \subset \mathbb{R}^d$ is a uniformly discrete, relatively dense set. This means, by definition, that there exist $0 < r < R < \infty$ such that every ball of radius r contains at most one point of Λ and every ball of radius R contains at least one point of Λ.

We can define a metric on the space of Delone sets, similarly to tilings, and consider resulting spaces and dynamical systems. Instead of "patches" for tilings we will talk about "patterns" for Delone sets (i.e., finite subsets). Sometimes we also consider "colored" Delone sets, or more precisely, Delone multisets. This is more in line with tiling theory.

There are three important properties of Delone sets, which are relevant for us:

- $\Lambda - \Lambda$ is discrete. This means every point is isolated, equivalently, there are finitely many points in every ball. In fact, this is equivalent to finite local complexity (finitely many patterns of diameter $< R$, for any R, up to translation), because two-point patterns determine all patterns.
- $[\Lambda] := \mathbb{Z}[\mathbf{x} : \mathbf{x} \in \Lambda]$, the abelian group (subgroup of $\mathbb{R}^d$) generated by Λ, is finitely generated.
- $\Lambda - \Lambda$ is uniformly discrete. Such Λ are called *Meyer sets.*

Lemma 1.1. *If $\Lambda - \Lambda$ is discrete, then $[\Lambda]$ is finitely generated, but the converse is false.*

1.1. Sliding block codes and local derivability.

Let $\mathcal{A} = \{0, \dots, m-1\}$ be a finite alphabet. Recall that a (2-sided) symbolic dynamical system over $\mathcal{A}$ is (X, T), where $X \subset \mathcal{A}^{\mathbb{Z}}$ is closed and T-invariant, T being the left shift map.

Definition 1.2. Let $\phi : \mathcal{A}^{2n+1} \to \mathcal{A}$. The sliding block code corresponding to ϕ is a map $F_\phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by

$$y = (y_k)_{k \in \mathbb{Z}} := F_\phi(x), \quad y_k = \phi(x_{k-n}, x_{k-n+1}, \dots, x_{k+n-1}, x_{k+n}).$$

Theorem 1.3 (Curtis-Hedlund-Lyndon)**.** *Let (X, T) and (Y, T) be two symbolic dynamical systems over $\mathcal{A}$, and suppose that $F : X \to Y$ is a continuous map wich commutes with the shift: $F \circ T = T \circ F$. Then F is a sliding block code.*

This easily generalizes to the case of symbolic systems over different alphabets.

Exercise. Prove the Curtis-Hedlund-Lyndon theorem.

"Local derivation" is a tiling/Delone set version of sliding block code.

Definition 1.4. Let Λ_1 and Λ_2 be two Delone sets in $\mathbb{R}^d$. We say that Λ_2 is locally derivable (LD) from Λ_1 with a radius $R > 0$ if

$$(\Lambda_1 - x) \cap B_R(0) = (\Lambda_1 - y) \cap B_R(0) \implies (\Lambda_2 - x) \cap \{0\} = (\Lambda_2 - y) \cap \{0\}.$$

We say that Λ_1 and Λ_2 are mutually locally derivable (MLD) if each of the Delone sets is LD from the other one.

These notions extend naturally to the cases "tiling – tiling" and "tiling – Delone set".

Lemma 1.5. *(i) If Λ_2 is LD from Λ_1, then there is an associated topological factor map from $(X_{\Lambda_1}, \mathbb{R}^d)$ to $(X_{\Lambda_2}, \mathbb{R}^d)$.*

(ii) If the Delone sets are MLD, then the corresponding dynamical systems are topologically conjugate.

Proof idea. Define $F(\Lambda_1 - g) := \Lambda_2 - g$ for $g \in \mathbb{R}^d$. This way we have the factor map defined on the orbit of Λ_1 (a dense set in X_{Λ_1}). If we can show that this map is uniformly continuous, we can extend F by continuity to the whole space. The uniform continuity follows from the LD property. □

Remark 1.6. Unlike the symbolic dynamics context, where we have an inverse theorem to the analog of the last lemma (i.e. the Curtis-Hedlund-Lyndon theorem), this is no longer true for tilings/Delone sets. In particular, there are examples where a topological conjugacy is not given by an MLD relation.

Lemma 1.7 (not very precise)**.** *It is always possible to pass from a tiling to a Delone (multi)set, and vice versa, in an MLD manner.*

Proof idea. One direction: choose a specific point in the interior of every prototile, labeled with the label of the prototile, and "spread this points around" by translation.

The other direction: consider, e.g., the Voronoi tesselation, with the cells labeled by the label of the point in a Delone set. □

2. Tiling substitutions and self-similar tilings (after W. Thurston)

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear similarity map, that is, $\phi\mathbf{x} = \lambda\mathcal{O}\mathbf{x}$, where $\lambda > 1$ is an expansion constant and $\mathcal{O}$ is an orthogonal transformation (linear isometry).

Definition 2.1. Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a prototile set in $\mathbb{R}^d$. A map $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is called a **tile-substitution with expansion** ϕ if

$$\operatorname{supp}(\omega(T_j)) = \phi(\operatorname{supp}(T_j)), \quad j \leq m. \tag{2.1}$$

Note that the definition implies that there are finite sets $D_{ij} \subset \mathbb{R}^d$ such that

$$\phi T_j = \bigcup_{i=1}^{m} \bigcup_{g \in D_{ij}} (T_i + g), \quad j \leq m,$$

where all the sets in the union have disjoint interiors. Thus, there are $\# D_{ij}$ translates of T_i in the substitution of T_j. The matrix $\mathsf{S}_\omega(i,j) = \# D_{ij}$ is the *substitution matrix* of ω. It is a direct analog of the substitution matrix for symbolic word substitutions. The tile-substitution ω is *primitive* if the matrix S_ω is primitive.

The substitution ω is extended to translates of tiles by

$$\omega(T_j + g) = \phi g + \omega(T_j),$$

and to patches and tilings by

$$\omega(P) = \cup\{\omega(T) : T \in P\}.$$

This is well-defined by (2.1), and so we can iterate ω. A tiling $\mathcal{T}$ is a *fixed point* of ω if $\omega(\mathcal{T}) = \mathcal{T}$.

Lemma 2.2 (Exercise)**.** *For a primitive tile-substitution ω there exists $n \in \mathbb{N}$ such that ω^n has a fixed point.*

Definition 2.3. A repetitive FLC fixed point of a primitive tile-substitution is called a *self-similar tiling.*

Examples show that the assumptions of FLC and repetitivity cannot be dropped.

Lemma 2.4 (Exercise)**.** *Let $\mathcal{T}$ be an FLC fixed point of a tile substitution ω with substitution matrix S_ω.*

(i) *If $\mathcal{T}$ is repetitive, then S_ω is primitive.*

(ii) *If S_ω is prmitive and $\mathbf{0}$ is in the interior of a $\mathcal{T}$-tile, then $\mathcal{T}$ is repetitive.*

References

[1] E. A. Robinson, *Symbolic Dynamics and Tilings of $\mathbb{R}^d$*, Proceedings of Symposia in Applied Mathematics **60**, AMS, Providence, 2014, available at `http://wms.mat.agh.edu.pl/~sem_ds/abstract/nr_1.pdf`

[2] B. Solomyak, *Tilings and Dynamics*, available at `https://www.math.washington.edu/~solomyak/PREPRINTS/notes6.pdf`