APERIODIC ORDER – ASSIGNMENT 1

Due on Thursday, April 16

Do as many exercises as you can, but try to do at least one of the 2-part problems completely.

Exercise 1. Construct an infinite sequence u which is recurrent, but not uniformly recurrent.

Exercise 2. Let ζ be a substitution on a finite alphabet. Suppose that $\lim_{n\to\infty} |\zeta^n(a)| = \infty$. Show that there is a periodic point for ζ.

Exercise 3 (Morse substitution). Let $\zeta(0) = 01,\ \zeta(1) = 10$ be the Morse substitution. Prove the following two properties of the fixed point

$$u = u_0 u_1 u_2 \ldots = \lim_{n\to\infty} \zeta^n(0) = 01101001\ldots :$$

(i) In the Morse sequence u no word of the form UUv occurs, where U is any nonempty word and v is the first letter of U. (This is sometimes expressed by saying that the Morse sequence is *free of powers* $2+\varepsilon$ for any $\varepsilon > 0$.)

(ii) Prove that the n-th term of the Morse sequence can be expressed as follows:

$$u_n = \text{sum of the digits in the binary representation of } n \text{ modulo } 2,$$

e.g. $0 = 0$, $u_0 = 0$; $1 = 1$, $u_1 = 1$; $2 = 10$ (in binary), so $u_2 = 1$; $3 = 11$ (in binary), so $u_3 = 1 + 1 = 0 \bmod 2$, etc.

Exercise 4 (Fibonacci numeration system and substitution). Let F_n be the sequence of Fibonacci numbers: $F_0 = 1, F_1 = 2, F_2 = 3,\ F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

(i) Prove that every non-negative integer n has a unique expansion

$$n = \sum_{i\geq 0} n_i F_i, \tag{0.1}$$

such that $n_i \in \{0, 1\}$ for all i and there are no two consecutive 1's, i.e. if $n_i = 1$, then $n_{i+1} = 0$. (This is also called *Zeckendorff numeration system*; it is used in computer science.) *Hint: induction!*

(ii) Let $u = u_0u_1u_2\ldots = 01001\ldots$ be the fixed point of the Fibonacci substitution $\zeta(0) = 01,\ \zeta(1) = 0$. **Prove that u_n is equal to the last digit n_0 in the representation (0.1).** For example:

$0 = 0 \cdot F_0 \implies u_0 = 0;$

$1 = 1 \cdot F_0 \implies u_1 = 1;$

$2 = 1 \cdot F_1 + 0 \cdot F_0 \implies u_2 = 0;$

$3 = 1 \cdot F_2 + 0 \cdot F_1 + 0 \cdot F_0 \implies u_3 = 0;$

$4 = 1 \cdot F_2 + 0 \cdot F_1 + 1 \cdot F_0 \implies u_4 = 1$, etc.

Exercise 5 (Complexity of substitution sequences). For a word W, its initial part (beginning) of any length is called its *prefix*, and its final part (ending) of any length is called its *suffix*.

(i) Let ζ be a primitive substitution on an alphabet $\mathcal{A}$ which has a fixed point $u = \zeta(u) \in \mathcal{A}^{\mathbb{N}}$. We consider the *language* $\mathcal{L}(u)$ consisting of all words which occur in u. Show that every word $W \in \mathcal{L}(u)$ can be expressed as follows:

$$W = U_0\,\zeta(U_1)\,\zeta^2(U_2)\ldots\zeta^k(U_k)\,\zeta^k(V_k)\ldots\zeta^2(V_2)\,\zeta(V_1)\,V_0, \qquad (0.2)$$

where $k \in \mathbb{N}$, $U_0, \ldots, U_k$ are suffixes (possibly empty!) of the words $\zeta(a)$, $a \in \mathcal{A}$ and $V_0, \ldots, V_k$ are prefixes (possibly empty!) of the words $\zeta(a)$, $a \in \mathcal{A}$, but at least one of U_k, V_k is nonempty. (This is sometimes called the *prefix-suffix decomposition* of the word W, or its "accordion form".)

(ii)* Use part (i) to prove that the substitution sequence has linear complexity:

$$p_u(n) \le Cn \ \text{ for some } C > 0 \ \text{ and all } n \in \mathbb{N}.$$

Hint: you can use the fact that $|\zeta^n(a)| \sim \theta^n$, where θ is dominant eigenvalue of the substitution matrix S_ζ; this is a consequence of the Perron-Frobenius Theorem.