APERIODIC ORDER – ASSIGNMENT 2

Due on Thursday, May 21

Do as many exercises as you can; you can skip an exercise and do the following one(s) assuming the one you didn't do.

Exercise 1. Prove that a system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if every T-invariant measurable function f (i.e. $f(Tx) = f(x)$ for a.e. x) is constant (a.e.). *Use only the definition of ergodicity.*

Exercise 2. Suppose that $(X, \mathcal{B}, T)$ is a measurable dynamical system (e.g., topological dynamical system, with Borel σ-algebra $\mathcal{B}$), for which there are two different invariant probability measures μ and ν. Then $\mu \perp \nu$, that is, the measures are mutually singular.

Hint: use Birkhoff's Ergodic Theorem.

Exercise 3. Let $T : X \to X$ be a continuous map on a compact metric space. Then the following are equivalent:

(i) T is uniquely ergodic;

(ii) for every $f \in C(X)$, we have $\frac{1}{N} S_N f(x) \to C_f$, where the constant C_f is independent of x.

Moreover, $C_f = \int_X f \, d\mu$.

Hint: use Birkhoff's Ergodic Theorem and the proof of Krylov-Bogolyubov Theorem.

Exercise 4. Show that $(\widehat{\mu}(n))_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is positive definite, where μ is a positive Borel measure on $\mathbb{T}$.

Exercise 5. Let U_T be a Koopman operator of an invertible m.-p. s. $(X, \mathcal{B}, \mu, T)$. Let $f \in L^2(X, \mu)$. Consider the sequence

$$\langle U_T^n f, f \rangle := \int_X U_T^n f(x) \overline{f(x)} \, d\mu(x), \quad n \in \mathbb{Z}.$$

Prove that this sequence is positive definite.

Exercise 6. Let f be an eigenfunction corresponding to an eigenvalue λ of unit norm: $\|f\|_2 = 1$. Prove that $\varrho_f = \delta_\lambda$.

Exercise 7. Prove that any correlation sequence is positive definite.

Exercise 8. Suppose that (X_u, T) is a uniquely ergodic symbolic dynamical system. Prove that

(i) u has a unique correlation measure ϱ;

(ii) the correlation measure ϱ is the spectral type of the map $\pi_0 : x \mapsto x_0$, defined for $x = (x_j)_{j \in \mathbb{Z}} \in X_u$.