APERIODIC ORDER – ASSIGNMENT 4

Due on Thursday, June 25

1. Constant length substitutions.

1.1. **Height.** Let ζ be a (symbolic) substitution of constant length $q \geq 2$ on an alphabet $\mathcal{A}$ of cardinality m. We suppose that the substitution is primitive. Suppose that there is a (one-sided) fixed point for the substitution: $\zeta(u) = u = u_0u_1u_2\dots$

Definition 1.1. Let $S_k := \{j \geq 1 : \ u_{j+k} = u_k\}$ and $g_k = GCD(S_k)$. The *height* of the substitution ζ is defined by

$$h = h(\zeta) = \max\{n \geq 1 : \ (n, q) = 1, \ n|g_0\}.$$

Exercise 1. (i) Prove that $1 \leq h \leq m = \mathrm{card}\mathcal{A}$.
(ii) Prove that if $h = m$, then u is periodic.
(iii) Prove that $h = \max\{n \geq 1 : \ (n, q) = 1, \ n|g_k\}$ for all $k \geq 1$.

Exercise 2. Consider the substitution $0 \to 0123, \ 1 \to 1234, \ 2 \to 2345, \ 3 \to 3450, \ 4 \to 4501, \ 5 \to 5012$. Prove that it is aperiodic and has height $h = 3$.

1.2. **Pure discrete spectrum.**

Definition 1.2. A constant length substitution ζ satisfies the *coincidence condition* if there exists k and n such that $\zeta^k(j)$ has the same n-th letter for every $j \in \mathcal{A}$.

We suppose that the substitution is primitive and shift-aperiodic. The substitution dynamical system is denoted (X_ζ, T, μ), where $X_\zeta \subset \mathcal{A}^{\mathbb{Z}}$, the map T is the left shift, and μ is the unique T-invariant probability measure.

Theorem 1.3 (Dekking 1978)**.** *Suppose that ζ is a primitive substitution of constant length and height $h = 1$. Then the measure-preserving system (X_ζ, T, μ) has pure discrete spectrum if and only if ζ satisfies the coincidence condition.*

Exercise 3. Determine which of the following substitutions have pure discrete spectrum:

$$\text{(a)}\ \ 0 \to 010,\ 1 \to 112,\ 2 \to 220;$$

$$\text{(b)}\ \ 0 \to 011,\ 1 \to 120,\ 2 \to 202;$$

$$\text{(c)}\ \ 0 \to 011,\ 1 \to 020,\ 2 \to 202.$$

1.3. **Rudin-Shapiro sequence and substitution.**

Definition 1.4. The Rudin-Shapiro sequence is defined inductively by $v_0 = 1,\ v_1 = -1,\ v_{2k} = v_k,\ v_{2k+1} = (-1)^k v_k$. The Rudin-Shapiro substitution ζ is defined on the alphabet $\mathcal{A} = \{1, 2, \overline{1}, \overline{2}\}$ by

$$1 \to 12,\ 2 \to 1\overline{2},\ \overline{1} \to \overline{1}\,\overline{2},\ \overline{2} \to \overline{1}2.$$

Exercise 4. What is the height of the Rudin-Shapiro substitution? Prove that the corresponding measure-preserving system is not pure discrete.

Exercise 5. Let $u = u_0u_1u_2u_3\ldots = 121\overline{2}\ldots$ be the fixed point of the Rudin-Shapiro substitution. Prove that the Rudin-Shapiro sequence can be obtained by $v_k = \phi(u_k)$, where $\phi(1) = \phi(2) = 1,\ \phi(\overline{1}) = \phi(\overline{2}) = -1$.

2. Nonconstant length substitutions

Consider the following substitution ζ depending on two parameters:

$$0 \to 01^k0^\ell, \quad 1 \to 0,$$

where $k, \ell \geq 1$. It is clearly primitive, so there is a unique invariant probability measure μ for the substitution dynamical system.

Exercise 6. (i) Compute measures of the cylinder sets: $\mu([0])$ and $\mu([1])$.

(ii) Show that iterating ζ, starting with 0.0, we get a two-sided infinite fixed point for ζ.

(iii) There is a natural self-similar tiling $\mathcal{T}$ of $\mathbb{R}$ associated with the sequence from part (ii). Compute its expansion constant (map) and the lengths of the tiles associated to 0 and 1 (they are defined up to a multiplicative constant).

(iv) Determine for which k and ℓ the tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R})$ has non-trivial discrete spectrum.