1. Aperiodic Order – Introduction

A tiling (or tesselation) of $\mathbb{R}^d$ is a collection of sets, called tiles, which have nonempty disjoint interiors and whose union is the entire $\mathbb{R}^d$.

Tiles are often assumed to be polygons (polyhedra), or at least topological balls, but for us they are just compact sets that are closures of their interiors. To get a meaningful theory, it is usually assumed that there are finitely many "prototiles" up to a group of transformations acting on the space. The two natural choices in $\mathbb{R}^d$ are the group of translations and the group of all Euclidean isometries.

Usually there are additional constraints, such as "face-to-face" for polyhedral tilings, or "matching rules" which specify how the tiles can fit together.

We start with a few historical remarks, mostly taken from the book chapter by E. A. Robinson, Jr. [2].

Question (Tiling Problem). Is there an algorithm that, upon being given a set of prototiles, with matching rules, decides whether a tiling of the entire space exists?

Hao Wang (1961) considered this problem for squares with colored edges, which became known as "Wang tilings." If we ignore the coloring, this is just the periodic tiling of $\mathbb{R}^2$ by the square tiles in a grid. The edges are colored in finitely many colors, and if two tiles touch each other, the colors of their common edge should match.

When $d = 1$, the "Wang tiles" are just intervals with colored endpoints, and there is an easy algorithm to answer the Tiling Problem. Draw a graph whose vertices are prototiles and directed edges indicate which pairs are allowed. A tiling of $\mathbb{R}$ exists if and only if there is an infinite path in this graph, which is equivalent to existence of a cycle.

Definition 1.1. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is called a *periodic tiling* if its *translation group* $\Gamma_{\mathcal{T}} = \{\mathbf{t} \in \mathbb{R}^d : \mathcal{T} - \mathbf{t} = \mathcal{T}\}$ is a *lattice*, that is, a subgroup of $\mathbb{R}^d$ with d linearly independent generators. A tiling is called *aperiodic* if $\Gamma_{\mathcal{T}} = \{0\}$.

From the discussion above it follows that if a tiling of $\mathbb{R}$ with a given prototile set exists, then there is a periodic tiling. Wang conjectured that the same holds for $d > 1$. More precisely, he conjectured that (1) there is an algorithm that decides the Tiling Problem; (2) if a tiling exists, then there exists a periodic tiling. Wang proved that (2) implies (1). However, the conjecture turned out to be false! Wang's student, Robert Berger (1966) proved that the Tiling Problem is undecidable and constructed an "aperiodic tiling system," that is, a prototile set which tile the plane but only aperiodically. Berger's

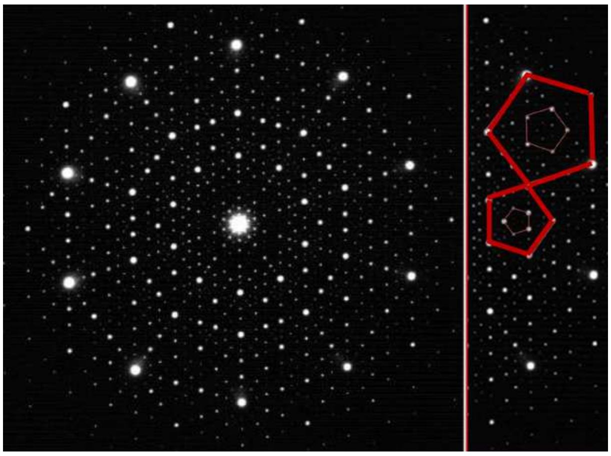

FIGURE 1. Quasicrystal Diffraction Pattern

prototile set was very large; it had more than 20,000 prototiles. Later, Raphael Robinson (1971) found a simpler example with 32 prototiles.

One of the most interesting aperiodic sets is the set of *Penrose tiles*, discovered by Roger Penrose [1]. Penrose tilings play a central role in the theory because they can be generated by any of the three main methods: local matching rules, tiling substitutions, and the projection method. The Penrose tiling has two prototiles up to isometries; below they are defined precisely.

The discovery of quasicrystals in 1982-84 had a profound influence on this subject. A quasicrystal is a solid (usually, metallic alloy) which, like a crystal, has a sharp X-ray diffraction pattern, but unlike a crystal, has an aperiodic atomic structure. Aperiodicity was inferred from a "forbidden" 5-fold symmetry of the diffraction picture. Since the Penrose tilings have this symmetry (not literally, but in an appropriate sense—statistically or for the tiling space), they became a focus of many investigations, both by physicists and by mathematicians. See [3] for an introduction to the mathematics of quasicrystals addressed to a general audience.

Penrose tilings come in several different versions. The simplest to describe has two rhombs—a thick and a thin one—as prototiles, shown in Figure 2. Their smaller angles are $2\pi/5$ and $2\pi/10$ respectively. The "markings" of the boundary define the matching rules. A part of a Penrose tiling is shown in Figure 3. The tiles appear in 10 orientations, so there are 20 prototiles up to translation.

How do we know that a Penrose tiling exists? In other words, why is it possible to tile the whole plane? This is not obvious; in fact, when one starts playing with the Penrose

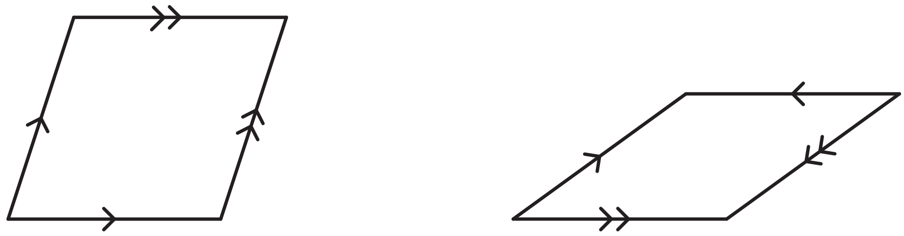

FIGURE 2. The Penrose tiles (rhombs)

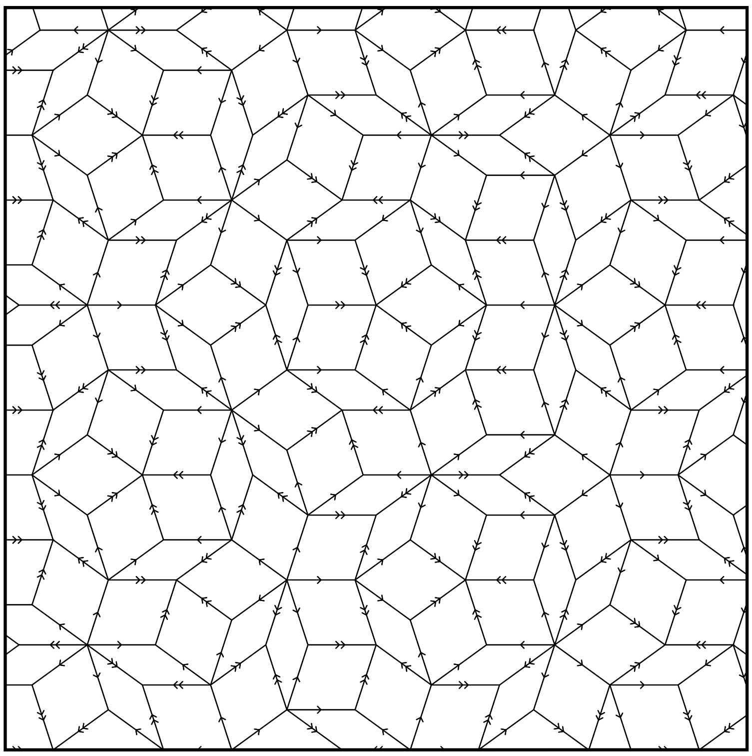

FIGURE 3. A part of a Penrose tiling

tiles as with a "jigsaw puzzle," it becomes clear that there are many non-extendable configurations. Penrose proved the existence of a tiling using *inflation.* It is easiest to explain this using *triangular Penrose tiles* introduced by R. Robinson, see Figure 4. They are obtained by cutting each rhomb into two triangles, and the markings are chosen in such a way that any tiling by triangles can be converted into a Penrose tiling by combining adjacent triangles. Now we can do the following: inflate one of the triangles by a factor of $(1+\sqrt{5})/2$ (the golden ratio) and subdivide it according to the rule indicated in the figure,

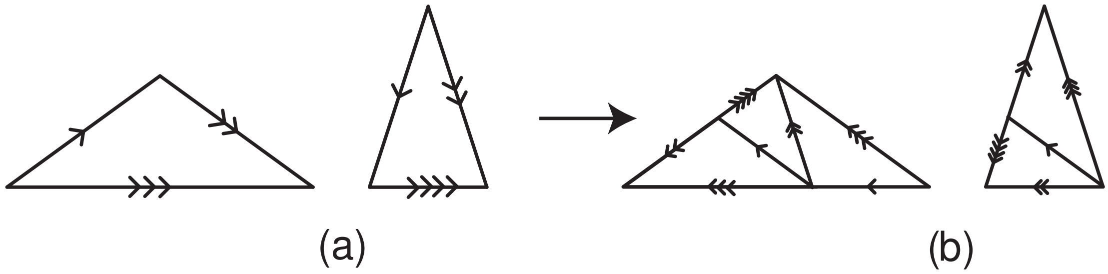

FIGURE 4. The *triangular Penrose tiles.*

then repeat this with the entire patch. Note that some of the triangles in the subdivision are obtained using a reflection; the subdivision rule respects this.

Exercise. Verify that when we inflate and subdivide repeatedly, adjacent triangles in the patch satisfy the matching rules on the boundary.

Thus we can iterate this procedure producing larger and larger patches. In the limit (appropriately defined) we get a tiling of the entire plane. This inflation-subdivision procedure is a powerful mechanism to create hierarchical structures, and it will be one of the main topics of the lectures.

The use of **dynamical systems** has been a major ingredient in the study of aperiodic tilings. Given a tiling in $\mathbb{R}^d$, one associates with it a space which is the closure of its $\mathbb{R}^d$-translation orbit, the closure being in the "local" topology, which compares tilings for more or less exact match in regions around the origin. We call it the *tiling space*; in the literature on mathematical quasicrystals it is often called the *dynamical hull.* Tiling spaces provide a new point of view; many properties of tilings are really properties of the tiling space. Moreover, many of these properties can be interpreted in dynamical terms. Explaining this will be one of the goals of this course.

REFERENCES

[1] R. Penrose, The role of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. Appl.* **10** (1974), 266-271. Pentaplexity: a class of nonperiodic tilings of the plane, *Math. Intelligencer* **2** (1979/80), no. 1, 32–37.

[2] E. A. Robinson, Jr., Symbolic dynamics and tilings of $\mathbb{R}^d$, in *Symbolic dynamics and its applications*, Proc. Sympos. Appl. Math., Vol. 60, Amer. Math. Soc., Providence, RI, 2004, pp. 81–119.

[3] M. Senechal, *Quasicrystals and geometry*, Cambridge University Press, 1995.