ON THE SPECTRAL THEORY OF ADIC TRANSFORMATIONS[1]

by Boris Solomyak

Department of Mathematics, University of Washington
Seattle, Washington, 98195

In this paper the spectral properties of stationary adic transformations are studied. First sufficient conditions for existence and nonexistence of nonconstant eigenvectors are given, supplementing those obtained by A.N.Livshitz [5] and B.Host [3]. Then we investigate when the spectrum is purely discrete. A sufficient condition is obtained, which is applied to some examples generalizing the well-known "golden mean" (or "Fibonacci") case. Using the results of A.N.Livshitz on the equivalence of substitutional flows and adic transformations [5], we can give the following, equivalent formulation of our main Theorem 4.1 :

the minimal flow generated by the substitution on $\{1, 2, \ldots, m\}$: $1 \to 123\ldots m,\ 2 \to 1,\ 3 \to 2, \ldots, m \to m-1,$ *is metrically isomorphic to a translation on the* $(m-1)$*-dimensional torus.*

This result appeared in [13] without detailed proof.

The author is grateful to A.M.Vershik and A.N.Livshitz for many helpful discussions and valuable suggestions.

1. The notion of adic transformation on a Markov compactum was introduced by A.M.Vershik [14, 15]. Here we give the definition for the special case of stationary compacta.

Let M be an $m \times m$ matrix, having entries 0 or 1. Consider a directed graded graph Γ with levels indexed $0, 1, 2, \ldots,$ having m vertices in each level. The vertices in each level are ordered and indexed $1, 2, \ldots, m$. The edges of Γ connect vertices of i-th level with vertices of $(i+1)$-st level according to the adjacency matrix M.

Consider the space $\mathcal{X}_M$ of all infinite paths in this graph. More precisely,

$$\mathcal{X}_M = \{t = \{t(i)\}_{i=0}^{\infty} \mid 1 \le t(i) \le m,\ M_{t(i)t(i+1)} = 1\}.$$

[1]To appear in Advances of Soviet Mathematics, editor: A.M.Vershik.

In this representation $t(i)$ is the vertex of the path t on i-th level. $\mathcal{X}_M$ is a compact totally disconnected set with topology induced from the product space $\{\overline{1,m}\}^{\mathbf{N}}$ and is called the *Markov compactum* corresponding to the matrix M.

The one-sided shift of finite type, or Markov shift $\sigma : \mathcal{X}_M \to \mathcal{X}_M$ is defined by

$$\sigma t(i) = t(i+1), \quad i \geq 0.$$

We shall be concerned with another transformation which is in a sense transversal to σ.

Introduce a partial ordering on $\mathcal{X}_M$:

$$t \prec t', \quad \text{if for some } n \in \mathbf{N}, \quad \begin{array}{l} t(i) = t'(i), \ i > n; \\ t(n) < t'(n). \end{array}$$

The adic transformation T of $t \in \mathcal{X}_M$ is defined to be the minimal element $t' \in \mathcal{X}_M$ such that $t \prec t'$.

Throughout the paper it is assumed that the matrix M is *primitive,* i.e. that for some $d > 0$ all entries of M^d are strictly positive. In this case the adic transformation is well-defined on $\mathcal{X}_M \setminus \mathcal{K}$, where $\mathcal{K}$ is a finite set of maximal elements with respect to the introduced ordering. It is known (see [16]) that for primitive M the adic transformation is minimal (i.e. all its orbits are dense), and uniquely ergodic. Below we describe the unique invariant measure for T, also called *the central measure.*

By the Perron-Frobenius Theorem, a primitive matrix M has a simple positive eigenvalue θ, such that $\theta > |\lambda|$ for any other eigenvalue λ of M, and there exists a strictly positive eigenvector corresponding to θ.

Denote by ξ_n the partition of $\mathcal{X}_M$ into subsets having the same initial part $i_0, i_1, i_2, \ldots i_n$. The elements of ξ_n will be sometimes identified with "finite paths of length n" (we count the edges of the path), and for $C \in \xi_n$ we shall write $C = C(0)C(1)\ldots C(n)$, where $C(k) = i_k$. We shall also write $t \sim s \,(\mathrm{mod}\ \xi_n)$ if $t(i) = s(i), \ i \leq n$.

Central measure. A measure on $\mathcal{X}_M$ is called central if it is invariant under the adic transformation. It is easy to see that a measure μ is central if and only if $\mu(C)$ depends on n and $C(n)$ only. We have the following description of the unique central measure (see [16], and for substitutions [7], [9, p.100]).

Lemma 1.1 *Let* T *be the adic transformation on a Markov compactum* $\mathcal{X}_M$ *with primitive* $M = \{M_{ij}\}$. *Then the central measure* μ *is a Markov measure with initial distribution* $\mu_l = \mu(\{t(0) = l\})$, $l = 1, 2, \ldots, m$, *and stationary transition probabilities* $[p_{ij}]_{i,j=1}^m$, *where* $[\mu_l]_{l=1}^m$ *is the normalized eigenvector corresponding to* θ, *and* $p_{ij} = M_{ij}\mu_j/(\theta\mu_i)$.

T and T^{-1} are defined μ-almost everywhere. Our goal is to investigate the spectral properties of T as an automorphism of $(\mathcal{X}_M, \mu)$. By U_T we denote the unitary operator $U_T f(t) = f(Tt)$ on $L^2(\mathcal{X}_M, \mu)$.

Substitutions. Let $D = \{1, 2, \ldots, m\}$ and consider the function $\omega : D \to D^m$ which assigns to every $i \le m$ the increasing sequence of numbers j such that $M_{ji} = 1$ (M is the same as above). This function is called the *substitution of alphabet* D. Substitutions generate minimal flows on certain sequence spaces, whose properties were studied in many works (see the bibliography in [9]). It was discovered by A.N.Livshitz [5] that essentially all stationary adic transformations are metrically isomorphic to corresponding substitutional flows. On the other hand, "generalized adic transformations" were introduced (see [16]), which correspond (essentially) to all substitutional flows. Thus the two theories are parallel, and some methods and ideas turned out to be quite similar. For a long time the emphasis was on *substitutions of constant length*, which means in our case that all the column sums are equal. In the case of *nonconstant length* the spectral theory is far from being complete. We are going to deal with adic transformations but indicate the links with theory of substitutions.

2. Eigenvalues of the adic transformation. The set of eigenvalues of the unitary operator U_T will be denoted by $\sigma_{disc}(U_T)$. Since U_T arises from an automorphism of a measure space, $\sigma_{disc}(U_T)$ is a subgroup of the unit circle $\mathbf{T}$. We shall also write $\sigma_{disc}(U_T) = \exp(2\pi i G)$, where G is a subgroup of $\mathbf{R}$.

Consider the vectors $P^{(n)} = [P_i^{(n)}]_{i=1}^m \in \mathbf{Z}^m$, $n \ge 0$, where $P_i^{(n)} = \#\{C \in \xi_n \mid C(n) = i\}$. Clearly

$$P_i^{(n+1)} = \sum_{j:M_{ji}=1} P_j^{(n)} ,$$

hence $P^{(n)} = M^T P^{(n-1)} = \ldots = (M^T)^n P^{(0)}$, $P^{(0)} = [1, 1, \ldots, 1]^T$. Thus all the components of $P^{(n)}$ satisfy the recurrence relation

$$x_{n+m} = -a_{m-1}x_{n+m-1} - a_{m-2}x_{n+m-2} - \cdots - a_0 x_n, \tag{1}$$

where $p(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ is the characteristic polynomial of the matrix M.

Lemma 2.1 (i) *If* $\zeta \in \sigma_{disc}(U_T)$, *then*

$$\lim_{n\to\infty} \zeta^{L_n} = 1, \tag{2}$$

for some positive sequence $L_n \in \mathbf{Z}$ *satisfying the recurrence relation* (1).

(ii) *If*

$$\lim_{n\to\infty} \zeta^{P_i^{(n)}} = 1, \quad i = 1, 2, \ldots, m, \tag{3}$$

then $\zeta \in \sigma_{disc}(U_T)$.

Remark. (i) was proved by A.Livshitz in [5], where the sequences $\{L_n\}$, appearing in (2), are described. For substitutions the result corresponding to this lemma (even more precise) was obtained by B.Host [3, Theorem (1.4)].

Proof of Lemma 2.1(ii). Our construction resembles that of [12], see also [5].

Consider the partition ξ_k. For each $t \in \mathcal{X}_M$ let $C_t^{(k)} \in \xi_k$ be the set containing t. A total ordering on ξ_k : is now defined by

$$\begin{aligned} C \prec D, \quad &\text{if } C(k) < D(k), \text{ or if for some } l < k, \\ &C(l) < D(l),\ C(i) = D(i),\ l < i \le k. \end{aligned}$$

This ordering is not to be confused with the partial ordering on $\mathcal{X}_M$, though if $t \prec s$, then $C_t^{(k)} \prec C_s^{(k)}$ for k sufficiently large.

Now enumerate all elements of ξ_k starting from zero with respect to this ordering and let $N_k(t)$ be the index of $C_t^{(k)}$. For ζ satisfying (3) let

$$f_\zeta(t) = \lim_{k\to\infty} \zeta^{N_k(t)}. \tag{4}$$

It is not hard to see that if $\lim \zeta^{L_n} = 1$ for a sequence $\{L_n\}$ satisfying a recurrence relation, then the convergence is exponential. Thus (3) implies

$$\sum_n |\zeta^{P_i^{(n)}} - 1| < \infty,\ i = 1, \ldots, m, \tag{5}$$

Now the existence of the limit is implied by the following simple assertion:

$$N_{k+1}(t) - N_k(t) \text{ is a linear combination of } P_i^{(k)}, P_i^{(k+1)},\ i = 1, \ldots, m, \text{ with coefficients } 0, \pm 1.$$

It remains to note that if Tt is defined, then $N_k(Tt) = N_k(t) + 1$ for k large enough, so $f_\zeta(Tt) = \zeta f_\zeta(t)$. The function f_ζ is continuous, since the convergence in (4) is uniform, and thus $\zeta \in \sigma_{disc}(U_T)$.

Now we will discuss how algebraic properties of the characteristic polynomial of the matrix M affect the (non)existence of eigenvalues for the adic transformation. Different methods work for the case of eigenvalues $\exp(2\pi i\alpha)$ with rational and irrational α. We shall assume that the characteristic polynomial is irreducible. Some analogs can be proved for the general case, but the situation becomes more complicated.

Theorem 2.2. *Suppose that $\mathcal{X}_M$ is a Markov compactum, where M is a primitive matrix with a characteristic polynomial $p(x)$ irreducible over* $\mathbf{Q}$. *Then the following are equivalent:*

(i) *there is a polynomial $g(x) \in \mathbf{Z}[x]$ constant and irrational on the set* $\{x_i | \, p(x_i) = 0,\ |x_i| \geq 1\}$;

(ii) *the adic transformation on $\mathcal{X}_M$ has an eigenvalue $\exp(2\pi i\alpha)$ with irrational α.*

Moreover, if $g(x) \in \mathbf{Z}[x]$, $g(x_i) = \alpha$ for $|x_i| \geq 1,\ p(x_i) = 0$, then $\exp(2\pi i\alpha^k)$ is an eigenvalue for all $k > 0$.

Remark. There are conditions (see [5]) which guarantee that there are no eigenvalues $\exp(2\pi i\alpha)$ with rational nonintegral α. For example, this is the case if $\det M = \pm 1$, and $M_{11} = 1$. Combining this with Theorem 2.2 and corollaries below, one can obtain some conditions for the adic transformation to have continuous spectrum, or equivalently, to be weakly mixing.

Proof of Theorem 2.2. The implication (ii)$\Rightarrow$(i) is a special case of a theorem due to A.Livshitz [6, Theorem 2].

(i)$\Rightarrow$(ii). Suppose that $g(x) \in \mathbf{Z}[x]$ is such that $g(x_i) = \alpha \notin \mathbf{Q}$, for each root x_i of $p(x)$ with $|x_i| \geq 1$. Let $\zeta = \exp(2\pi i\alpha)$. We shall prove that $\zeta^{L_n} \to 1$, for *every* sequence $L_n \in \mathbf{N}$ satisfying (1), and then make use of Lemma 2.1 (ii) to assert that $\zeta \in \sigma_{disc}(U_T)$.

Since $p(x)$ is irreducible, it has m distinct roots x_i $(m = \deg p)$, and any sequence L_n satisfying the recurrence relation (1) can be written as

$$L_n = \sum_{i=1}^{m} c_i x_i^n$$

(see [1]). We have

$$\alpha L_n = \alpha \sum_{i:\,|x_i|\geq 1} c_i x_i^n + \alpha \sum_{i:\,|x_i|<1} c_i x_i^n. \tag{6}$$

Let $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots + g_l x^l$. Then

$$K_n \stackrel{\text{def}}{=} \sum_{j=0}^{l} g_j L_{n+j} = \sum_{j=0}^{l} g_j \sum_{i=1}^{m} c_i x_i^{n+j} =$$

$$= \sum_{i=1}^{m} c_i x_i^n g(x_i) = \alpha \sum_{i:\,|x_i|\geq 1} c_i x_i^n + \sum_{i:\,|x_i|<1} c_i x_i^n g(x_i).$$

Comparing this with (6) we see that $\alpha L_n - K_n \to 0$. Since $K_n \in \mathbf{Z}$, we have

$$\zeta^{L_n} = \exp(2\pi i(\alpha L_n - K_n)) \to 1.$$

One can take $L_n = P_i^{(n)}$, and by Lemma 2.1(ii) $\zeta \in \sigma_{disc}(U_T)$.

To prove that $\exp(2\pi i \alpha^k)$ is an eigenvalue, write $g^k(x) = p(x)Q(x) + R(x)$, where $R(x) \in \mathbf{Z}[x]$, $\deg R < m = \deg p$. Then $R(x_i) = \alpha^k$ for $|x_i| \geq 1$, and α^k is irrational since $p(x)$ is the minimal polynomial for x_1, the maximal in modulus root. Thus, the same argument works if we replace $g(x)$ with $R(x)$, and the proof is complete.

Corollary 2.3. *Under the conditions of Theorem 2.2 let s be the smallest prime divisor of $m = \deg p$. If*

$$\#\{x_i \mid p(x_i) = 0,\ |x_i| \geq 1\} > m/s,$$

then $G \subset \mathbf{Q}$.

Remark. Related results were obtained by B.Host [3] and A.N.Livshitz [5]. These authors do not assume that $p(x)$ is irreducible. In [5] it is proved that if all zeroes of $p(x)$ have modulus greater or equal than one, then $G \subset \mathbf{Q}$. After the translation from the language of substitutions, the statement of [3, (6.5)] runs as follows: Suppose that $M_{11} = 1$ and more than half of the zeroes of the minimal polynomial for the Perron eigenvalue θ have modulus greater or equal than one. Then $G \subset \mathbf{Q}$. This result can be proved similarly to Corollary 2.3.

Definition. An algebraic integer is called a *Pisot number*, if all its conjugates have modulus less than one.

A special case of Corollary 2.3 is the following

Corollary 2.4. *Under the conditions of Theorem* 2.2, *if* $m = \deg p$ *is prime, then eigenvalues with irrational* α *exist if and only if* θ *is a Pisot number.*

In [11, 12] it is proved that if θ is a Pisot number, then $G \setminus \mathbf{Q}$ is rather large: it contains a free Abelian group of rank $m-1$. Theorem 2.2 contains this result. Below we give an example when θ is not a Pisot number, but still eigenvalues $\exp(2\pi i\alpha)$ with irrational α exist.

Example. Let $m = 4$,

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We have $p(x) = x^4 - 2x^3 - x^2 + 2x - 1 = \psi(g(x))$, where $g(x) = x^2 - x$, $\psi(y) = y^2 - 2y - 1$. The polynomial $p(x)$ is irreducible and it has four zeroes: x_1, x_2, x_3, x_4, where $x_{1,2} = 1/2(1 \pm \sqrt{5 + 4\sqrt{2}})$ have modulus greater than one, and $x_{3,4} = 1/2(1 \pm i\sqrt{4\sqrt{2} - 5})$ have modulus less than one. We have $g(x_1) = g(x_2) = 1 + \sqrt{2}$. So Theorem 2.2 implies that $\exp(2\pi i\sqrt{2})$ is an eigenvalue of the adic transformation on $\mathcal{X}_M$.

Proof of Corollary 2.3. Let $\mathcal{Z}$ be the set of all zeroes of the irreducible polynomial p. If $G \not\subset \mathbf{Q}$, then by Theorem 2.2 there exists $g(x) \in \mathbf{Z}[x]$, such that $g(x_i) = \alpha$ for $x_i \in \mathcal{Z}$, $|x_i| \geq 1$. Let F be the splitting field for the polynomial p (see [17, p.121]), and let $\mathcal{A}_\alpha = \{x_i \in \mathcal{Z} | \, g(x_i) = \alpha\}$. We can assume that $\deg g < \deg p$, replacing $g(x)$ by the remainder in the division by $p(x)$, if necessary. Thus we have $\#\mathcal{A}_\alpha < \#\mathcal{Z}$. Let τ be an automorphism of the field F over $\mathbf{Q}$ which maps $x_1 \in \mathcal{A}_\alpha$ to some $x_j \notin \mathcal{A}_\alpha$ (exists by [17, p.166]). Then $g(\tau x_i) = \tau g(x_i) = \tau\alpha$, so $\tau\mathcal{A}_\alpha \cap \mathcal{A}_\alpha = \emptyset$, $\tau\mathcal{A}_\alpha \subset \mathcal{Z}$, $\#\tau\mathcal{A}_\alpha = \#\mathcal{A}_\alpha$. Proceeding in this fashion we see that the set $\mathcal{Z}$ is partitioned into sets having the same number of elements, on which g is constant. Thus $\#\mathcal{A}_\alpha$ divides $\deg p$, and we get a contradiction with the hypothesis of the corollary.

3. Purely discrete spectrum. Discreteness of the spectrum is established with the help of the following proposition of a general nature.

Proposition 3.1. *Suppose that T is an automorphism of a Lebesgue space $(\mathcal{X}, \mu)$, U_T the corresponding unitary operator on $L^2(\mathcal{X}, \mu)$, and E its projection spectral measure on the circle $\mathbf{T}$. We assume that there exists a sequence of increasing measurable partitions of $\mathcal{X}$ denoted by $\{\xi_n\}$, such that $\xi_n \to \epsilon$ (partition into points) and a sequence $p_n \to \infty$ of integers such that*

$$\text{for all } k\text{, for all } C \in \xi_k, \quad \sum_n \mu(T^{p_n} C \triangle C) < \infty. \tag{7}$$

Then $E(\mathbf{T} \setminus \{\zeta : \zeta^{p_n} \to 1\}) = 0$.

Proof was given in [13] but we repeat it for convenience of the reader.

Let $\chi(C)$ be the indicator function of the set $C \in \xi_k$, $\nu_C(\cdot) = (E(\cdot)\chi(C), \chi(C))$. Then

$$\int |\zeta^{p_n} - 1|^2 \, d\nu_C = \| (U_T^{p_n} - I)\chi(C) \|_2^2 = \| \chi(T^{p_n}C) - \chi(C) \|_2^2 =$$

$$= \mu(T^{p_n} C \triangle C).$$

Now (7) implies that

$$\sum_k \nu_C\{\zeta : |\zeta^{p_n} - 1| > 2^{-l}\} < \infty$$

for all $l > 0$. Therefore $\nu_C\{\mathbf{T} \setminus \{\zeta : \zeta^{p_n} \to 1\}) = 0$. Since the set of indicators for all C is total in $L^2(\mathcal{X}, \mu)$ we conclude that $E(\mathbf{T} \setminus \{\zeta : \zeta^{p_n} \to 1\}) = 0$.

Proposition 3.1 was used by A.N.Livshitz [4, 6] to obtain combinatorial conditions for discreteness of spectrum of adic transformations and substitutions.

Let M be an $m \times m$ matrix such that $M_{1,1} = 1$ and $\mathcal{X}_M$ the corresponding Markov compactum. In this case we have a unique minimal path $t_0 \in \mathcal{X}_M$, $t_0 = 111\dots$. Analogous condition for substitutions (viz. the word assigned to 1 starts with 1) is often used (see [3, 9]).

Let p_n denote $P_1^{(n)}$, the number of paths of length n, leading to the vertex 1.

Lemma 3.2. *Suppose that* $M_{1,1} = 1$, *and set*

$$D_{n,k} = \{t \in \mathcal{X}_M : t \not\sim T^{p_n} t (\mathrm{mod}\ \xi_k)\}.$$

a) *If the adic transformation* T *has purely discrete spectrum, then* $\forall k,\ \mu(D_{n,k}) \to 0,\ n \to \infty$.

b) *If* $\forall k,\ \sum_n \mu(D_{n,k}) < \infty$, *then* T *has purely discrete spectrum.*

Remark. A similar result for substitutions on $\{0, 1\}$ was obtained by B.Host (see [9, VI.27])

Proof of Lemma 3.2. It follows from [5] that if $M_{1,1} = 1$, then in Lemma 2.1 (i) one can set $p_n = L_n$. Thus if ζ is an eigenvalue of U_T, then $\lim_{n\to\infty} \zeta^{p_n} = 1$. If eigenvalues of U_T form a total set, then $(U_T^{p_n} - I)\chi(C) \to 0$, for each $C \in \xi_k$. This implies that $\mu(T^{p_n} C \triangle C) \to 0$. This proves a) since

$$D_{n,k} = \cup_{C\in\xi_k}(T^{p_n} C \triangle C).$$

Assertion b) follows from Proposition 3.1 and Lemma 3.3 below.

Lemma 3.3. *If* $\{p_n\}$ *is any sequence satisfying the recurrence relation* (1) *with* $a_k \in \mathbf{Z}$, *then the set* $\{\zeta \in \mathbf{T} \mid \zeta^{p_n} \to 1\}$ is *countable.*

Proof of Lemma 3.3. Let $\zeta = \exp(2\pi i\alpha)$. Then $\zeta^{p_n} \to 1$ if and only if $\parallel \alpha p_n \parallel \to 0$, where $\parallel b \parallel$ is the distance of b to the nearest integer. We can write

$$\alpha p_n = R_n + \varepsilon_n; \quad R_n \in \mathbf{Z},\ \varepsilon_n \to 0.$$

Since $\{\alpha p_n\}_{n\geq 1}$ satisfies (1), the sequence $\{R_n\}$ satisfies the same recurrence relation (1) starting from some $n = n(\alpha)$. We have $\alpha = \lim_{n\to\infty} p_n^{-1} R_n$. Since the set of integral sequences satisfying a given recurrence relation is countable, we are done.

4. In this section we consider matrices of a special form. Let M be the following $m \times m$ matrix:

$$M = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

We shall consider the Markov compacta $\mathcal{X}_M$ and $\mathcal{X}_{M^T}$ where M^T is the transpose of M. The characteristic polynomial of M is

$$p(x) = x^m - x^{m-1} - \cdots - x - 1.$$

Let θ be the Perron-Frobenius eigenvalue, $1 < \theta < 2$. It is not hard to see that all other zeroes of $p(x)$ have modulus less than one.

Let T be the adic transformation. Using Lemma 1.1, we find the "initial probabilities" on $\mathcal{X}_M, \mathcal{X}_{M^T}$, denoted by μ_l, μ_l^* correspondingly:

$$\mu_l = \frac{\theta - 1}{m - 1}(\theta^{l-1} - \theta^{l-2} - \cdots - 1), \quad \mu_l^* = \theta^{-l}, \quad l = 1, \dots, m.$$

Theorem 4.1. *The adic transformation on* $\mathcal{X}_M$ $(\mathcal{X}_{M^T})$ *has purely discrete spectrum and is metrically isomorphic to a translation on the* $(m-1)$*-dimensional torus:*

$$\mathbf{R}^{m-1}/\mathbf{Z}^{m-1} \ni \vec{x} \mapsto \vec{x} + \vec{\alpha}\ , \quad \text{where } \vec{\alpha} = [\mu_l]_{l=1}^{m-1}\ ([\mu_l^*]_{l=1}^{m-1}).$$

Remarks. 1) Special cases of the theorem are: $m = 2$ ("Fibonacci" or "golden mean" case), and $m = 3$, studied by G.Rauzy [10] in the language of substistutions.

2) Theorem 4.1 appeared in [13] without detailed proof. Here the formulation is slightly modified, using the observation of B.Host [3] that if $M_{1,1} = 1$, $\det M = 1$, then the generators for the group of eigenvalues can be taken equal to μ_l.

Proof of Theorem 4.1. The Markov compacta $\mathcal{X}_M$ and $\mathcal{X}_{M^T}$ are studied similarly, so we confine ourselves to the adic transformation T on $\mathcal{X}_M$. By the theorem of J. von Neumann (see [2, p.46]) two automorphisms with the same purely discrete spectrum are metrically isomorphic. So it is sufficient to prove that U_T has purely discrete spectrum and the group of eigenvalues is generated by $\exp(2\pi i\mu_l)$, $l = 1, \dots, m-1$. The description of eigenvalues follows from Lemma 2.1, which in our case implies that

$$\zeta \in \sigma_{disc}(U_T) \Leftrightarrow \zeta^{p_n} \to 1.$$

It follows also from the results of B.Host [3] on substitutions, so we will not give details of this description here.

Thereby it remains to prove that U_T has purely discrete spectrum. We are going to make use of Lemma 3.2, but first some preparation is needed.

It is convenient to add negative levels indexed $-m+1, -m+2, \dots, -1$ to the graded graph Γ, and to extend all the paths $t \in \mathcal{X}_M$ to negative levels so that

$$t \sim \{t(i)\}_{i=-m+1}^{\infty}, \quad t(-m+1) = 1,$$

and if $t(r) = 1$, $r \leq 0$, then $t(i) = 1$, $i \leq r$. Such extension is unique because only one edge leads to each of the vertices $2, 3, \dots, m$.

Recall that p_n is the number of paths $\{t(i)\}_{i=0}^{n}$ for which $t(n) = 1$. It is clear from the form of the graph that the number of paths of length n leading to the l-th vertex is equal to p_{n-l+1}. Set $p_i = 1$, $-m+1 \leq i \leq 0$. The sequence p_i satisfies the recurrence relation

$$p_{k+m} = p_{k+m-1} + \dots + p_{k+1}, \quad k \geq -m+1.$$

We shall say that a path $t \in \mathcal{X}$ is *eventually straight* if for some $k > 0$, we have $t(i) = 1$, $i \geq k$. The set of eventually straight paths is linearly ordered by the partial ordering in $\mathcal{X}_M$. Now enumerate all eventually straight paths, starting from zero, and let $N(t)$ be the index of t. The path indexed zero is $t_0 \sim \{t_0(i) = 1\}_{i=-m+1}^{\infty}$. Evidently $t = T^{N(t)} t_0$.

Instead of working with the sequence $\{t(i)\}$, it is more convenient to deal with another representation of paths.

Symbolic representation. We shall assign to each element $t \in \mathcal{X}_M$, a sequence $\{\varepsilon_i(t)\}_{i=-m+1}^{\infty}$:

$$\varepsilon_i(t) = \begin{cases} 0, \text{ if } t(i) = 1; \\ 1, \text{ otherwise.} \end{cases}$$

Lemma 4.2. *If* $t \in \mathcal{X}_M$ *is eventually straight , then*

$$N(t) = \sum_{i=-m+1}^{\infty} \varepsilon_i(t) p_i. \tag{8}$$

Remark. G.Rauzy [10] has a similar representation for $m = 3$.

Proof. Straightforward.

Admissible sequences. A sequence $\{\varepsilon_i\}_{i=-m+1}^{\infty}$ is said to be admissible if it arises from some $t \in \mathcal{X}_M$. It is easily seen that $\{\varepsilon_i\}_{-m+1}^{\infty} \subset \{0,1\}^{\mathbf{N}}$ is admissible if and only if

(a) there is no segment of m successive "ones" ;
(b) $\varepsilon_{-m+1} = 0$, ε_i are monotone for $i \le 0$.

We shall also say that a sequence $\{\varepsilon_i\}_{i=\kappa_1}^{\kappa_2}$ is admissible, if it corresponds to a part of the path from κ_1 to κ_2.

Note that condition (a) describes the sequences arising in β-expansions ($\beta = \theta$) of real numbers, see [8].

Proposition 4.3. *If* $t \in \mathcal{X}_M$, $\varepsilon_{k+1}(t) = \varepsilon_{k+2}(t) = \ldots = \varepsilon_{k+3m}(t) = 0$, *then* $T^{p_n} t \sim t (\mathrm{mod}\ \xi_k)$ *for any* $n > k$.

Before we give the proof of the proposition let us deduce Theorem 4.1.

Set $D_{n,k} = \{t \in \mathcal{X}_M | \, T^{p_n} t \not\sim t\, (\mathrm{mod}\ \xi_k)\}$. It follows from Proposition 4.3 that if $t \in D_{n,k}$, then the part of the path from k-th to n-th level does not contain a segment of $3m$ successive "ones". Recall that the conditional probability for the path to go from i-th vertex to j-th vertex is equal to p_{ij} and does not depend on the level (see Lemma 1.1). Thus for any $l \ge 0$ and any $C \in \xi_l$,

$$\mu(\{t \in C | \, t(l{+}1) = t(l{+}2) = \ldots = t(l{+}3m) = 1\}) = \mu(C) p_{C(l)1} p_{11}^{3m-1} > \delta \mu(C),$$

where δ does not depend on l or $C(l)$. Hence

$$\mu(\{t \in \mathcal{X}_M | \; t(k+1)t(k+2)\ldots t(k+3m) \ne 11\ldots1,$$
$$t(k+3m+1)t(k+3m+2)\ldots t(k+6m) \ne 11\ldots1, \ldots,$$
$$t(k+3md+1)t(k+3md+2)\ldots t(k+3m(d+1)) \ne 11\ldots1\}) < (1-\delta)^{d+1}$$

Therefore
$$\mu(D_{n,k}) \le (1-\delta)^{[(n-k)/3m]}$$
Since k and m are fixed, $\sum_n \mu(D_{n,k}) < \infty$, and by Lemma 3.2 (ii) U_T has purely discrete spectrum.

It remains to prove Proposition 4.3.

Proof of Proposition 4.3. It is assumed that $T^{p_n}t$ is defined. This means that $T^{p_n}t \sim t\,(\mathrm{mod}\ \xi_K)$ for some K for which $t(K)=1$. Clearly the claim of the proposition for t is equivalent to that for the path t' such that $t'(i)=t(i),\ i\le K;\ t(i)=1,\ i>K$. Thus it is sufficient to prove the proposition for eventually straight paths.

We can define the *addition* of eventually straight paths by the formula:
$$N(t+s) = N(t) + N(s).$$
Let e_n be the eventually straight path such that $\varepsilon_i(e_n)=\delta_{ni}$ (Kronecker's symbol). Then by Lemma 4.2, $T^{p_n}t = t + e_n$.

Let $\{d_i\}_{i=-m+1}^{\infty}$, $\{d_i'\}_{i=-m+1}^{\infty}$ be two integral sequences having finitely many nonzero terms. We shall call them *equivalent*, if
$$\sum_{i=-m+1}^{\infty} d_i p_i \;=\; \sum_{i=-m+1}^{\infty} d_i' p_i$$
Our goal is to transform the sequence $\{\varepsilon_i(t)+\delta_{ni}\}$ to obtain an equivalent admissible sequence. Recall that a sequence of zeroes and ones is said to be admissible if it does not contain a segment of m ones, and its terms with nonpositive indices (if any) are nondecreasing.

Definition 4.4. A sequence $\{d_i\}_{i=-m+1}^{\infty}$ will be called *an admissible sequence with perturbation* if

(a) $0 \le d_i \le 2$.

(b) There exists $\nu > 0$, such that $d_{\nu+1}=0$, and both $\{d_i\}_{i=-m+1}^{\nu-m}$ and $\{d_i\}_{i=\nu+1}^{\infty}$ are admissible.

The integer ν will be called the *final point of the perturbation.*

First we note that the sequence $\{\varepsilon_i(t)+\delta_{ni}\}$ is admissible with perturbation, since gaps between two successive zeroes in $\{\varepsilon_i(t)\}$ are not greater than $m-1$. Roughly speaking, the idea of what follows is that the perturbation "propagates" and is "damped" in the straight segment of the path t, leaving its initial part untouched.

We shall use two types of operations leaving sequences equivalent.

Operation A. Suppose that for a sequence $\{b_i\}$ having finitely many nonzero terms, we have

$$b_{r+1} = b_{r+2} = \ldots = b_{r+m} = 1,\ b_{r+m+1} = 0.$$

Them we can put

$$b'_i = \begin{cases} 0, & \text{if } \ r+1 \le i \le r+m; \\ 1, & \text{if } \ i = r+m+1; \\ b_i, & \text{otherwise.} \end{cases}$$

Since $p_{r+m+1} = p_{r+1} + p_{r+2} + \cdots + p_{r+m}$ by the recurrence relation, $\{b'_i\}$ is equivalent to $\{b_i\}$. The following is readily seen.

Assertion 1. If $\{b_i\}_{i=-m+1}^{\infty}$ is a sequence with finitely many nonzero terms, such that $b_i \le 1$ for $i \ge l$, and $b_l = 0$, then using Operation A several times one can obtain an admissible sequence $\{b'_i\}_{i=-m+1}^{\infty}$ equivalent to the original one, and such that $b'_i = b_i,\ i \le l$.

Returning to $T^{p_n}t$, we see that if $\varepsilon_n(t) = 0$, then $d_i = \varepsilon_i(t) + \delta_{ni} \le 1$. Assertion 1 shows that $T^{p_n}t \sim t \,(\mathrm{mod}\ \xi_k)$.

If $\varepsilon_n(t) = 1$, we have to apply another operation.

Operation B. Suppose that $\{b_i\}_{i=-m+1}^{\infty}$ is an admissible sequence with perturbation where ν is the final point of the perturbation. Let

$$r = \max\{i \,|\, b_i = 2\} - 1.$$

So $b_{r+1} = 2$, and we can assume that (if $r+1 < \nu$) $b_{r+2} = b_{r+3} = \ldots = b_\nu = 1$, or otherwise we could take a smaller final point of the perturbation. Now we use that

$$p_{r+1} = p_r + p_{r-1} + \cdots + p_{r-m+1}, \quad p_{\nu-m+1} + p_{\nu-m+2} + \cdots + p_\nu = p_{\nu+1}$$

and put

$$b'_i = \begin{cases} b_i + 1, & \text{if } \ i = \nu+1; \\ 0, & \text{if } \ r+1 \le i \le \nu; \\ b_i + 1, & \text{if } \ r-m+1 \le i \le \nu - m; \\ b_i, & \text{otherwise.} \end{cases}$$

The sequences $\{b_i\}$ $\{b'_i\}$ are equivalent. It is possible that $\{b'_i\}_{i=\nu}^{\infty}$ is inadmissible. This will happen if $b_{\nu+2} = b_{\nu+3} = \cdots = b_{\nu+m} = 1$. But in this case we can apply Assertion 1 to obtain a sequence $\{b''_i\}_{i=-m+1}^{\infty}$ equivalent to $\{b_i\}_{i=-m+1}^{\infty}$, such that $b''_i = b'_i$ for $i \leq \nu$, and $\{b''_i\}_{i=\nu}^{\infty}$ is admissible.

Let us prove that $\{b''_i\}_{i=-m+1}^{\infty}$ is almost admissible. First, $\{b''_i\}_{i=r+1}^{\infty}$ is admissible as $b''_{r+1} = b''_{r+2} = \cdots = b''_{\nu} = 0$ (and there is at least one zero since $r+1 \leq \nu$).

Since $\{b_i\}_{i=-m+1}^{\nu-m}$ is admissible, we have that $\{b'_i\}_{i=-m+1}^{r-m-1} = \{b_i\}_{i=-m+1}^{r-m-1}$ is admissible and $b'_i \leq b_i + 1 \leq 2$ for $i \leq \nu - m$. For $\nu - m + 1 \leq i \leq r$ we have $b'_i = b_i \leq 2$. Thus $\{b''_i\}$ is admissible with perturbation, and the final point of the perturbation is $r < \nu$.

We have proved the following

Assertion 2. If $\{b_i\}_{i=-m+1}^{\infty}$ is an admissible sequence with perturbation, ν is the final point of perturbation, then using Operations B and A it is possible to obtain an equivalent admissible sequence with perturbation $\{b''_i\}_{i=-m+1}^{\infty}$ with the final point of the perturbation r, $\nu - m \leq r < \nu$, and such that $b''_i = b_i$ for $i \leq r - m$.

Let us return to the proof of Proposition 4.3 and put $d_i = \varepsilon_i + \delta_{ni}$. As was mentioned already, the sequence $\{d_i\}_{i=-m+1}^{\infty}$ is admissible with perturbation. Using Assertion 2 repeatedly we can reduce it to an equivalent admissible sequence with perturbation $\{d'_i\}_{i=-m+1}^{\infty}$ with the final point of the perturbation ν', $k+3m+1 \leq \nu' \leq k+4m$, and such that $d'_i = d_i = \varepsilon_i(t)$ for $i < \nu' - m$. (Of course it is possible that at some point the sequence becomes admissible, but then we are done.) From the assumption of Proposition 4.3 that $\varepsilon_i(t) = 0$ for $k+1 \leq i \leq 2m$ it follows that

$$d'_{\nu'-m} = d'_{\nu'-m-1} = \cdots = d'_{\nu'-2m+1} = 0.$$

We shall continue to perform Operations B and A. But now "adding ones" at the places $i \leq \nu' - m$ will not produce new "twos" in the sequence (take into account that now for every operation B the new interval of adding does not intersect the previous one). Thus with each application of Operation B, the number of terms of the sequence equal to two, decreases. On the other hand, the final point of the perturbation remains greater or equal than $\nu' - m + 1$, so the property of having a straight segment (all zeroes) of length m preceding the perturbation, preserves. So eventually, after no more than m steps, we come to a sequence $\{d''_i\}$ such that $d''_i = d_i = \varepsilon_i(t)$ for $i \leq k$, and $d''_i \le 1$. Now applying Operation A, we obtain an admissible sequence corresponding to $T^{p_n} t$. The proof of the proposition, and therefore the proof of Theorem 4.1 is now complete.

References

[1] A.O.Guelfond, Calcul des différences finies. Dunod, Paris, 1963.

[2] P.R.Halmos, Lectures on Ergodic Theory. The Mathematical Soc. of Japan, Tokyo, 1956.

[3] B.Host, Valeurs propres de systèmes dynamiques définis par de substitutions de longueur variable. Ergodic Theory and Dynamical Systems 6 (1986), 529-540.

[4] A.N.Livshitz, On the spectra of adic transformations of Markov compacta. Russian Math. Surveys 42:3 (1987), 222-223.

[5] A.N.Livshitz, A sufficient condition for weak mixing of substitutions and stationary adic transformations. Math. Notes 44:6 (1988), 920-925.

[6] A.N.Livshitz, Some examples of adic transformations and substitutions. To appear in Selecta Mathematica Sovietica.

[7] P.Michel, Stricte ergodicité d'ensembles minimaux de substitutions. C.R.Acad.Sc. Paris 278 (1974), 811-813.

[8] W.Parry, On the β-expansions of real numbers. Acta Math. Acad. Sci. Hungar. 11 (1960), 401-416.

[9] M.Queffélec, Substitution dynamical systems - spectral analysis. Lecture Notes in Math. 1294 (1987).

[10] G.Rauzy, Nombres algébraiques et substitutions. Bull. Soc. math. France 1982 (110), 147-178.

[11] M.Solomyak, Master Thesis. Leningrad University, 1982.

[12] M.Solomyak, The simultaneous action of adic transformation and Markov shift on the torus, to appear in Advances of Soviet Mathematics, 1991.

[13] B.Solomyak, On a dynamical system with discrete spectrum. Russian Math Surveys 41:2 (1986), 219-220.

[14] A.M.Vershik, Uniform algebraic approximation of shift and multiplication operators. Soviet Math. Dokl. 24:1 (1981), 97-100.

[15] A.M.Vershik, A theorem on periodic Markov approximation in ergodic theory. Journal of Soviet Math. 28 (1985), 667-674.

[16] A.M.Vershik and A.N.Livshitz, Adic models of ergodic transformations, spectral theory, substitutions, and related topics (an overview), present volume

[17] B.L.van der Waerden, Algebra, Volume 1. Frederick Ungar Publ.Co., New York, 1970.