Spectrum of dynamical systems arising from Delone sets *

Boris Solomyak †

July 1997

Abstract

We investigate spectral properties of the translation action on the orbit closure of a Delone set. In particular, sufficient conditions for pure discrete spectrum are given, based on the notion of almost periodicity. Connections with diffraction spectrum are discussed.

1 Introduction

A set $\Lambda \subset \mathbf{R}^d$ is a Delone set if there exist positive constants R and r such that every ball of radius R intersects Λ and every ball of radius r contains at most one point of Λ. The collection of all such sets with fixed R and r can be equipped with a metric to form a compact space. The group $\mathbf{R}^d$ acts on this space by translations. We study the spectral properties of this action restricted to some invariant subsets. We begin with a description of eigenvalues (with continuous eigenfunctions) assuming that the restricted action is minimal. Then we consider dynamics with respect to an ergodic invariant measure and obtain sufficient conditions for the system to have pure discrete spectrum. These conditions are based on a variant of almost periodicity for Delone sets.

This work has an application to the theory of quasicrystals, as we explain at the end of Section 2. Roughly speaking, the set Λ models an atomic structure, and pure discrete dynamical spectrum implies pure discrete diffraction spectrum, a property often associated with quasicrystals.

Tiling dynamical systems are closely related to systems arising from Delone sets. A tiling of $\mathbf{R}^d$ is a decomposition of the space into a countable union of compact sets (tiles), having disjoint interiors. The tiles are often assumed to be polygons, each congruent to an element of a finite set of "prototiles". The translation action of $\mathbf{R}^d$ on the space of tilings with a given prototile set (or

---

*1991 *Mathematics Subject Classification.* Primary: 28D15, 52C99, 54H20; Secondary: 42A75, 78A45

†Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195; *E-mail* [solomyak@math.washington.edu]. Supported in part by NSF Grant 9500744.

its restriction on an invariant subspace) is called a tiling dynamical system. Such systems have been intensively studied in recent years, see Radin and Wolff [20] Radin [18, 19] and references therein, E.A. Robinson [22, 23], and Solomyak [24]. There are many ways to go from a tiling to a Delone set and back. If this is done with some care, the tiling and Delone set are going to be "mutually locally derivable", which implies topological conjugacy of the corresponding dynamical systems. In this paper we work with Delone sets but all results can be easily rephrased in terms of tilings.

The paper is organized as follows. In Section 2 the results on Delone sets are stated and connections with diffraction, projection method, and Meyer sets are discussed. These results naturally fit into a more general framework of abstract dynamical systems on compact metric spaces developed in Section 3. The proofs are given in Sections 4 and 5.

2 Delone sets

Denote by $B_R(y)$ the closed ball of radius R centered at $y \in \mathbf{R}^d$. A set $\Lambda \subset \mathbf{R}^d$ is called *relatively dense* if there exists $R = R(\Lambda) > 0$ such that $\Lambda \cap B_R(y) \neq \emptyset$ for all $y \in \mathbf{R}^d$. The set Λ is said to be *uniformly discrete* if there exists $r = r(\Lambda) > 0$ such that $\#[\Lambda \cap B_r(y)] \leq 1$ for all $y \in \mathbf{R}^d$. The set Λ is called a *Delone set* if it is relatively dense and uniformly discrete. The collection of all Delone sets Λ with these properties for given R and r will be denoted $\mathcal{D}(R, r)$. The space $\mathcal{D}(R, r)$ is equipped with a metric:

$$\rho(\Lambda_1, \Lambda_2) = \min\{2^{-1/2},\ \widetilde{\rho}(\Lambda_1, \Lambda_2)\}, \quad \text{where}$$

$$\widetilde{\rho}(\Lambda_1, \Lambda_2) = \inf\{\epsilon > 0 :\ d_H[\Lambda_1 \cap B_{1/\epsilon}(0)\, ,\, \Lambda_2 \cap B_{1/\epsilon}(0)] \leq \epsilon\}\,. \tag{1}$$

Here

$$d_H[A_1, A_2] = \inf\{\delta > 0 :\ A_1 \subseteq A_2 + B_\delta(0) \ \&\ A_2 \subseteq A_1 + B_\delta(0)\}$$

is the Hausdorff distance.

The "cut-off" at $2^{-1/2}$ in the definition of ρ is needed to fulfill the axioms of metric. Although the metric looks complicated, its meaning is simple: two Delone sets are close if they almost coincide on a large neighborhood of the origin. (An equivalent metric was considered by Dworkin in [2]; a similar metric for tiling spaces is used by Robinson in [23].)

A standard diagonalization argument shows that $(\mathcal{D}(R, r), \rho)$ is compact. For all $y \in \mathbf{R}^d$ the translation $\Gamma_y :\ \Lambda \mapsto \Lambda - y$ is a homeomorphism on $\mathcal{D}(R, r)$. Given a Delone set Λ, we consider the *orbit closure* $\mathcal{X}_\Lambda = \text{Clos}\,\{\Lambda - y :\ y \in \mathbf{R}^d\}$ in the metric ρ. The space $\mathcal{X}_\Lambda$, together with the translation action Γ_y, is the topological dynamical system arising from Λ.

A Delone set Λ is often assumed to have the property of *(translational) local finiteness*:

$$\#[(\Lambda - \Lambda) \cap B_t(0)] < \infty, \quad \text{for all} \ \ t > 0. \tag{2}$$

This means that Λ has finitely many "local patterns" up to translation. Lagarias [12] has investigated this property in great detail; he called such Λ a "Delone set of finite type".

When considering locally finite Delone sets, it is more convenient to use another metric ρ' which is defined as the minimum of $2^{-1/2}$ and

$$\widetilde{\rho}'(\Lambda_1, \Lambda_2) = \inf\{\epsilon > 0 : \ \exists\, u, v \in B_\epsilon(0),\ (\Lambda_1 - u) \cap B_{1/\epsilon}(0) = (\Lambda_2 - v) \cap B_{1/\epsilon}(0)\}\,.$$

If Λ is locally finite, ρ and ρ' define the same topology (an easy exercise), and the space $\mathcal{X}_\Lambda$ can be interpreted as the collection of Delone sets all of whose local patterns occur in Λ, up to translation.

Definition. Let us say that $y \in \mathbf{R}^d$ is a *topological δ-almost-period for* Λ if

$$\Lambda \cap B_{1/\delta}(0) = (\Lambda - y) \cap B_{1/\delta}(0). \tag{3}$$

Denote by $\Psi_\delta(\Lambda)$ the set of topological δ-almost-periods.

A locally finite set Λ is called *repetitive* if $\Psi_\delta(\Lambda)$ is relatively dense for all $\delta > 0$. In plain language, this means that all local patterns of Λ occur relatively dense in space. It is well-known that Λ is repetitive if and only if $\mathcal{X}_\Lambda = \mathcal{X}_{\Lambda'}$ for every $\Lambda' \in \mathcal{X}_\Lambda$, or equivalently, $\mathcal{X}_\Lambda$ is the local isomorphism class of Λ. The corresponding dynamical systems $(\mathcal{X}_\Lambda, \Gamma_y)$ are called *minimal* and sometimes "(topologically) almost periodic."

Our first result is a description of eigenvalues for the dynamical system $(\mathcal{X}_\Lambda, \Gamma_y)$. Here we formulate the result in the case when the Delone set is locally finite. The non-locally finite case is contained in Theorem 3.1 given in the next section. Denote by $\langle x, y\rangle$ the standard scalar product in $\mathbf{R}^d$. Recall that $\alpha \in \mathbf{R}^d$ is an eigenvalue for the translation action if there exists a non-zero function $f_\alpha \in C(\mathcal{X}_\Lambda)$ satisfying

$$f_\alpha(\xi - y) = e^{2\pi i \langle y, \alpha\rangle} f_\alpha(\xi) \quad \text{for all} \ \ \xi \in \mathcal{X}_\Lambda \ \ \text{and} \ \ y \in \mathbf{R}^d.$$

Theorem 2.1 *Let Λ be a locally finite, repetitive Delone set. Then α is an eigenvalue for $(\mathcal{X}_\Lambda, \Gamma_y)$ if and only if*

$$\lim_{\delta \to 0} \sup_{y \in \Psi_\delta(\Lambda)} |e^{2\pi i \langle y, \alpha\rangle} - 1| = 0. \tag{4}$$

Next we consider measurable dynamics. It is known that there exists a Borel probability measure μ on $\mathcal{X}_\Lambda$ which is translation-invariant and ergodic, that is, any Borel translation-invariant set has measure 0 or 1. (Here we do not consider the question of uniqueness for this measure; it is related to the existence of "uniform patch frequencies", see [13] for some sufficient conditions.) A non-zero μ-measurable function on $\mathcal{X}_\Lambda$ is an eigenfunction for the measure-preserving system $(\mathcal{X}_\Lambda, \mu, \Gamma_y)$ if the eigenfunction equation holds for μ-almost every $\xi \in \mathcal{X}_\Lambda$. We write "continuous eigenfunctions" and "measurable eigenfunctions" to distinguish between the topological and measure-theoretic settings. It is not always true that each measurable eigenfunction coincides with some continuous eigenfunction a.e.; of course, every continuous eigenfunction is measurable.

The system $(\mathcal{X}_\Lambda, \mu, \Gamma_y)$ is said to have *pure discrete spectrum* if measurable eigenfunctions form a basis for $L^2(\mathcal{X}_\Lambda, \mu)$. (Ergodicity implies that all eigenspaces are one-dimensional and Γ_y-invariance of the measure implies that eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal.)

Definition. The lower uniform frequency of a discrete set A is defined by

$$\underline{\mathrm{freq}}(A) = \liminf_{L\to\infty} \sup_{a\in\mathbf{R}^d} L^{-d}\,\#[A \cap (C_L + a)], \tag{5}$$

where $C_L = [-L/2, L/2]^d$. If $\liminf$ can be replaced with $\lim$ in (5), we say that A has uniform frequency $\mathrm{freq}(A)$.

Recall that $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of sets. Let us say that $x \in \mathbf{R}^d$ is a *statistical δ-almost-period* for Λ if

$$\underline{\mathrm{freq}}[\Lambda \triangle (\Lambda - x)] \le \delta. \tag{6}$$

Theorem 2.2 *Let Λ be a Delone set. If the set of statistical δ-almost-periods for Λ is relatively dense for all $\delta > 0$, then the system $(\mathcal{X}_\Lambda, \mu, \Gamma_y)$ has pure discrete spectrum for any ergodic invariant measure μ.*

Remarks. 1. We propose to call a Delone set Λ *statistically almost periodic* if the conditions of Theorem 2.2 are fulfilled.

2. If the system $(\mathcal{X}_\Lambda, \Gamma_y)$ is uniquely ergodic, that is, if μ is unique, then the frequency (rather than lower frequency) exists in (6), and it is always uniform, so that one can let $a = 0$ and drop $\sup_a$ in (5).

3. Notice that in this theorem we do not assume the set Λ to be locally finite.

Diffraction spectrum. Physicists use Delone sets to model infinite atomic structures, e.g. in statistical mechanics and the theory of diffraction. For instance, one can consider the measure $\nu = \sum_{x\in\Lambda} \delta_x$ as "atomic density" (here δ_x is the Dirac's δ at x). If the autocorrelation γ of ν exists, the diffraction is described by the Fourier transform $\widehat{\gamma}$, see papers [6, 8] by Hof. If Λ is locally finite and there exist “pair correlations” (frequencies n_a of $a \in \Lambda - \Lambda$), then the autocorrelation exists and is given by $\gamma = \sum_{a \in \Lambda - \Lambda} n_a \delta_a$. Moreover, $\widehat{\gamma}$ is then a measure as well, as shown by Hof [6], and one can speak about the discrete component of $\widehat{\gamma}$ (“Bragg peaks”), continuous component (“diffuse spectrum”), etc.

The diffraction spectrum is related to the dynamical spectrum which we study. It follows from the work of Dworkin [2], see also Hof [8], that (assuming unique ergodicty) $\widehat{\gamma}$ is pure discrete (resp. pure continuous), if the spectrum of the associated dynamical system is such. Thus, we obtain the following.

Corollary 2.3 *If a Delone set is statistically almost periodic and the corresponding dynamical system is uniquely ergodic, then the diffraction spectrum is pure discrete.*

The discrete part of $\widehat{\gamma}$ is concentrated on the discrete part of the dynamical spectrum, however, they need not coincide. Gähler and Klitzing in [4] investigated the discrete component of the diffraction spectrum for self-similar tilings; some of their results are parallel to our results from [24] on dynamical spectrum. The work of Robinson [21] implies that there is a connection between continuous eigenfunctions and the standard procedure used by physicists to determine the “Bragg peaks”.

Although the statistical almost periodicity condition is quite strong, it can often be verified. In the special case of self-similar tilings, an analog of Theorem 2.2 was used to develop a concrete algorithm (the “overlap algorithm”) for checking pure discrete spectrum. Pure discrete spectrum was confirmed for the “chair” and “sphinx” tilings in our paper [24]; as well as for the three-dimensional version of the “chair” tiling (in preparation). It is likely that Theorem 2.2 can also be applied to hierarchical tilings which are not strictly self-similar.

Projection Method. As an illustration, let us give an alternative proof of a result by Hof [8] that the dynamical system associated with a set obtained by projection method has pure discrete spectrum. (Earlier, Robinson in [22] has shown that the dynamical system associated with the Penrose tiling is an almost 1:1 extension of a translation action on the torus $\mathbf{T}^4$, and hence has pure discrete spectrum.)

Let $\mathcal{T}$ be a lattice in $\mathbf{R}^m$ and E a d-dimensional subspace such that the dual lattice $\mathcal{T}^*$ satisfies $\mathcal{T}^* \cap E = \{0\}$. Further, let $K \subset E^\perp$ be a compact set in the orthogonal complement of E. As in [8], we assume that K has non-empty interior and ∂K has zero $(m-d)$-dimensional Lebesgue measure $\mathcal{L}_{m-d}$. The set K is called a “window”. Let π denote the orthogonal projection onto E. The set Λ is defined by

$$\Lambda = \pi[\mathcal{T} \cap (E \times K)].$$

It is well-known that Λ is a Delone set in E. Let us check that the hypotheses of Theorem 2.2 are satisfied. We shall use the fact that for such a set there exists uniform frequency proportional to $\mathcal{L}_{m-d}(K)$ (see Hof (1996a) and references therein). For each $\delta > 0$, replacing K with the ball $B_\delta(0)$ in $E^\perp$, we obtain that the set

$$A(\delta) = \pi[\mathcal{T} \cap (E \times B_\delta(0))]$$

is relatively dense in E. Let $x \in A(\delta)$. Since $\mathcal{T}$ is a lattice, it is immediate that

$$\Lambda \bigtriangleup (\Lambda + x) \subset \pi[\mathcal{T} \cap (E \times (\partial K + B_\delta(0)))].$$

The set in the right-hand side has uniform frequency proportional to the measure of $\partial K + B_\delta(0)$, which tends to zero as $\delta \to 0$. It follows that x satisfies (6) for δ sufficiently small, so Theorem 2.2 yields that $(\mathcal{X}_\Lambda, \mu, \Gamma_y)$ has pure discrete spectrum.

Meyer sets. Recently the notion of a *Meyer set* was introduced: A Delone set Λ is Meyer if $\Lambda - \Lambda \subset \Lambda + F$ for some finite set F. There are many equivalent characterizations, see Moody [16]. Lagarias in [11] has shown that a Delone set Λ is Meyer if and only if $\Lambda - \Lambda$ is Delone. Meyer in [14, 15] developed a duality theory for such sets. A question arises if there is any connection between the spectral properties of the system $(\mathcal{X}_\Lambda, \Gamma_y)$ and the Meyer theory.

We note that being a repetitive Meyer set does not imply pure discrete spectrum or even the presence of non-trivial discrete spectral component. Indeed, any relatively dense subset of a Meyer set is Meyer, so any self-similar tiling with vertices in a lattice gives rise to a Meyer set. In [24] we gave an example of such a tiling with mixed spectrum (the "domino" tiling). To get a Meyer set without discrete spectrum one can take a weakly mixing substitution ζ on two symbols 0 and 1, and let Λ be the set of integers corresponding to 0 in some bi-infinite sequence arising from ζ (see [17] for the definitions). On the other hand, one can find a Delone set with non-trivial discrete spectrum which is not Meyer: take the product of a one-dimensional set with discrete spectrum and a non-Meyer set. We do not know if a repetitive Delone set with finitely many local patterns can have pure discrete spectrum but be non-Meyer.

3 Abstract Dynamical Systems

In order to prove the results stated in the previous section, it is convenient to put them into the more general framework of $\mathbf{R}^d$-actions. Let $(\mathcal{X}, \rho)$ be a compact metric space. A topological $\mathbf{R}^d$-action is a continuous homomorphism $y \mapsto F_y$ from $\mathbf{R}^d$ as a topological group, into the group of homeomorphisms of $\mathcal{X}$. Recall that a dynamical system is said to be *minimal* if it has no non-trivial closed invariant subsets, or equivalently, if its every orbit is dense. For $\delta > 0$ and $\xi \in \mathcal{X}$ let

$$\Phi_\delta(\xi) = \{y \in \mathbf{R}^d : \ \rho(F_y\xi, \xi) \le \delta\}. \tag{7}$$

By a well-known argument (see Chapter 1 in Furstenberg's book [3]),

$$(\mathcal{X}, F_y) \ \text{ is minimal } \iff \Phi_\delta(\xi) \ \text{ is relatively dense } \ \forall \xi \in \mathcal{X}, \ \forall \delta > 0. \tag{8}$$

Recall that $\alpha \in \mathbf{R}^d$ is an eigenvalue for $(\mathcal{X}, F_y)$ if there exists a non-zero function $f_\alpha \in C(\mathcal{X})$ satisfying

$$f_\alpha(F_y\xi) = e^{2\pi i \langle y, \alpha \rangle} f_\alpha(\xi) \quad \text{for all } \ \xi \in \mathcal{X} \ \text{ and } \ y \in \mathbf{R}^d. \tag{9}$$

Theorem 3.1 *Let $(\mathcal{X}, F_y)$ be a minimal $\mathbf{R}^d$-action on a compact metric space. Then α is an eigenvalue if and only if*

$$\lim_{\delta \to 0} \sup_{y \in \Phi_\delta(\xi)} |e^{2\pi i \langle y, \alpha \rangle} - 1| = 0, \quad \textit{for } \ \xi \in \mathcal{X}. \tag{10}$$

If (10) is satisfied for a single $\xi \in \mathcal{X}$, then it holds for all $\xi \in \mathcal{X}$.

Let μ be an ergodic F_y-invariant Borel probability measure on $\mathcal{X}$. A non-zero μ-measurable function on $\mathcal{X}$ is an eigenfunction for the measure-preserving system $(\mathcal{X}, \mu, F_y)$ if the equality in (9) holds for μ-a.e. $\xi \in \mathcal{X}$ and all $y \in \mathbf{R}^d$. The measure-preserving system $(\mathcal{X}, \mu, F_y)$ is said to have pure discrete spectrum if μ-measurable eigenfunctions form a basis for $L^2(\mathcal{X}, \mu)$.

Let us say that $x \in \mathbf{R}^d$ is a *measure-theoretic ϵ-almost-period* for the $\mathbf{R}^d$-action $(\mathcal{X}, \mu, F_y)$ if

$$\mu\{\xi \in \mathcal{X} : \ \rho(F_x\xi, \xi) > \epsilon\} < \epsilon. \tag{11}$$

Theorem 3.2 *Let $(\mathcal{X}, \mu, F_y)$ be an ergodic $\mathbf{R}^d$-action. If the set of measure-theoretic ϵ-almost-periods is relatively dense for all $\epsilon > 0$, then the system has pure discrete spectrum.*

Although we are not aware of such a result in the literature, perhaps, it is known. The proof is rather simple, using that a finite measure is pure discrete if and only if its Fourier transform is almost periodic (in the sense of H. Bohr), see Corollary 4.15 in Burckel's book [1]. For some substitution dynamical systems a similar proof was sketched by Bernard Host [personal communication], see also Lemma VI.25 in Queffelec's book [17] for the case of $\mathbf{Z}$-actions. We should note that the statement and proof of Theorem 3.2 easily extend to the setting of locally compact abelian group actions.

It is unlikely that the condition in Theorem 3.2 is necessary for pure discrete spectrum; a partial converse is given below.

Proposition 3.3 *Suppose that $(\mathcal{X}, \mu, F_y)$ is a minimal $\mathbf{R}^d$-action with pure discrete spectrum, and all measurable eigenfunctions can be chosen to be continuous. Then the set of measure-theoretic ϵ-almost periods for the $\mathbf{R}^d$-action is relatively dense for all $\epsilon > 0$.*

4 Continuous eigenfunctions

In this section we prove Theorem 3.1 and then use it to deduce Theorem 2.1.

Proof of Theorem 3.1 Necessity. Suppose that there exists a non-zero continuous eigenfunction f_α satisfying (9). Since $(\mathcal{X}, F_y)$ is minimal, $|f_\alpha|$ is constant; assume that it is equal to one. The space $\mathcal{X}$ is compact, so f_α is uniformly continuous. Given $\epsilon > 0$, one can find $\delta > 0$ such that if $\rho(\xi_1, \xi_2) < \delta$ then $|f_\alpha(\xi_1) - f_\alpha(\xi_2)| < \epsilon$. Let ξ be any element of $\mathcal{X}$. For $y \in \Phi_\delta(\xi)$ we have $\rho(F_y\xi, \xi) < \delta$, so

$$|e^{2\pi i\langle y,\alpha\rangle} - 1| = |f_\alpha(\xi)||e^{2\pi i\langle y,\alpha\rangle} - 1| = |f_\alpha(F_y\xi) - f_\alpha(\xi)| < \epsilon,$$

as desired.

Sufficiency. Suppose that (10) holds for some $\xi \in \mathcal{X}$. We let $\Phi_\delta := \Phi_\delta(\xi)$ to simplify the notation. Let

$$f_\alpha(F_y\xi) = e^{2\pi i\langle y,\alpha\rangle}.$$

This defines a function on the orbit of ξ; this orbit is dense by minimality. If we prove that f_α is continuous on the orbit, it can be extended to a continuous function on $\mathcal{X}$ satisfying the eigenfunction equation (9). Fix $\epsilon > 0$. By (10), we can find $\delta_1 > 0$ so that

$$|e^{2\pi i\langle y,\alpha\rangle} - 1| < \epsilon/2 \quad \text{for all} \quad y \in \Phi_{\delta_1}. \tag{12}$$

The set $\Phi_{\delta_1/2}$ is relatively dense by (8), so for some $R > 0$ every ball of radius R contains a point in $\Phi_{\delta_1/2}$. Since our action is continuous, there exists $\delta_2 > 0$ such that

$$\rho(\xi_1, \xi_2) < \delta_2 \;\Rightarrow\; \rho(F_y\xi_1, F_y\xi_2) < \delta_1/2 \quad \text{for all} \quad y \in B_R(0). \tag{13}$$

Now we can show that f_α is continuous on $\{F_y\xi : \ y \in \mathbf{R}^d\}$. Suppose that $\rho(F_{y_1}\xi, F_{y_2}\xi) < \delta_2$. One can find $z \in B_R(0)$ such that $y_1 + z \in \Phi_{\delta_1/2}$, that is, $\rho(F_{y_1+z}\xi, \xi) < \delta_1/2$. Observe that $\rho(F_{y_1+z}\xi, F_{y_2+z}\xi) \le \delta_1/2$ by (13), so $\rho(F_{y_2+z}\xi, \xi) < \delta_1$. We obtain that $y_1 + z \in \Phi_{\delta_1/2} \subseteq \Phi_{\delta_1}$ and $y_2 + z \in \Phi_{\delta_1}$. Thus, (12) implies

$$\begin{aligned} |f_\alpha(F_{y_1}\xi) - f_\alpha(F_{y_2}\xi)| &= |e^{2\pi i\langle y_1,\alpha\rangle} - e^{2\pi i\langle y_2,\alpha\rangle}| \\ &= |e^{2\pi i\langle y_1+z,\alpha\rangle} - e^{2\pi i\langle y_2+z,\alpha\rangle}| \\ &\le |e^{2\pi i\langle y_1+z,\alpha\rangle} - 1| + |e^{2\pi i\langle y_2+z,\alpha\rangle} - 1| < 2(\epsilon/2) = \epsilon, \end{aligned}$$

and the proof is complete. ■

Proof of Theorem 2.1. By the definition of the metric (1), if $\Lambda \cap B_{1/\delta}(0) = (\Lambda - y) \cap B_{1/\delta}(0)$, then $\rho(\Lambda - y, \Lambda) \le \delta$, hence $\Psi_\delta(\Lambda) \subseteq \Phi_\delta(\Lambda)$, see (7) and (3). By Theorem 3.1, (4) is necessary for α to be an eigenvalue.

Let us establish sufficiency. Since Λ is locally finite, for any $t > 0$, there exists $\delta(t) \in (0, 1/t)$, such that if

$$d_H[(\Lambda - y) \cap B_t(0), \Lambda \cap B_t(0)] \le \delta(t),$$

then

$$(\Lambda - y') \cap B_t(0) = \Lambda \cap B_t(0)$$

for some y' with $||y' - y|| \le \delta(t)$. Thus, $\Phi_{\delta(t)}(\Lambda) \subseteq \Psi_{1/t}(\Lambda) + B_{\delta(t)}(0)$. It follows that (4) implies (10), so α is an eigenvalue by Theorem 3.1. ■

5 Pure discrete spectrum

Here we prove Theorem 3.2 and Proposition 3.3, and then deduce Theorem 2.2.

Proof of Theorem 3.2. For any $f \in L^2(\mathcal{X}, \mu)$ the *spectral measure* corresponding to f is a finite measure ν_f on $\mathbf{R}^d$ such that

$$\widehat{\nu}_f(y) = \int_{\mathbf{R}^d} e^{-2\pi i \langle y, t\rangle}\, d\nu_f(t) = \int_{\mathcal{X}} f(F_y \xi)\overline{f(\xi)}\, d\mu(\xi) \quad \text{for } \ y \in \mathbf{R}^d. \tag{14}$$

The $\mathbf{R}^d$-action $(\mathcal{X}, \mu, F_y)$ has pure discrete spectrum if and only if all the spectral measures ν_f are pure discrete. It is sufficient to show this for f from a set whose closed linear span is the whole space $L^2(\mathcal{X}, \mu)$. It is well-known that the collection of indicator functions of open sets has this property.

So let U be an open set in $\mathcal{X}$ and let f be its indicator (characteristic) function. It is a fact from harmonic analysis that a finite measure is pure discrete if its Fourier transform is almost periodic (see Hewitt [5] or Corollary 4.15 in Burckel's book [1] for general groups; Katznelson in [9] gives a proof for the case of $\mathbf{R}$). Recall that a continuous function g on $\mathbf{R}^d$ is almost periodic if the set of its ϵ-almost-periods is relatively dense for all $\epsilon > 0$, and a vector $x \in \mathbf{R}^d$ is an ϵ-almost-period for g if $|g(y + x) - g(y)| < \epsilon$ for all $y \in \mathbf{R}^d$. Thus, it suffices to prove that given $\epsilon > 0$, the set of ϵ-almost-periods for $\widehat{\nu}_f$ is relatively dense.

Since μ is a Borel probability measure on a metric space, it is regular, see e.g. Theorem 6.1 in [25]. This implies that there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \epsilon/4. \tag{15}$$

Since $\mathcal{X}$ is compact, the distance between K and the boundary of U is a positive number which we denote by η. Let $\delta = \min\{\epsilon/4, \eta\}$. By assumption, the set of measure-theoretic δ-almost-periods for the system $(\mathcal{X}, \mu, F_y)$ is relatively dense. Consequently, the proof will be complete once we prove that if x is a measure-theoretic δ-almost-period for $(\mathcal{X}, \mu, F_y)$, then x is an ϵ-almost-period for the function $\widehat{\nu}_f$.

We have by (14), using that f is the indicator of U:

$$\begin{aligned} |\widehat{\nu}_f(y+x) - \widehat{\nu}_f(y)| &= \left| \int_{\mathcal{X}} (f(F_{y+x}\xi) - f(F_y\xi)) f(\xi)\, d\mu(\xi) \right| \\ &\le \mu(F_{-x-y}U \triangle F_{-y}U) \\ &= \mu(U \triangle F_{-x}U) \\ &= 2[\mu(U) - \mu(U \cap F_{-x}U)]\,. \end{aligned} \tag{16}$$

In the last two equalities we used the F_y-invariance of the measure μ. Now observe that if $\xi \in K$ and $\rho(F_x\xi, \xi) \le \delta < \rho(K, \partial U)$, then $F_x\xi \in U$ and so $\xi \in U \cap F_{-x}U$. It follows that

$$\mu(U) - \mu(U \cap F_{-x}U) \le \mu(U) - \mu(K) + \delta < \epsilon/2.$$

This, together with (16), shows that x is an ϵ-almost-period for $\widehat{\nu}_f$, as desired. ∎

Proof of Proposition 3.3. Fix $\epsilon > 0$. Let $B_1, \ldots, B_m$ be a finite covering of the compact space $\mathcal{X}$ by balls of radius $< \epsilon/2$. Let h_j denote the indicator function of B_j. By assumption, h_j can be approximated by a linear combination of continuous eigenfunctions. Thus, for all $j \le m$, there exist $c_{j,k} \in \mathbf{C}$, $f_{j,k} \in C(\mathcal{X})$, and $\alpha_{j,k} \in \mathbf{R}^d$, with $k = 1, \ldots, n_j$, such that

$$f_{j,k}(F_y\xi) = e^{2\pi i \langle y, \alpha_{j,k}\rangle} f_{j,k}(\xi) \quad \text{for} \;\; y \in \mathbf{R}^d,\; \xi \in \mathcal{X}, \tag{17}$$

and

$$||h_j - \sum_{k=1}^{n_j} c_{j,k} f_{j,k}||_2 < (1/3)(\epsilon/m)^{1/2}\,. \tag{18}$$

The eigenfunctions are normalized so that $||f_{j,k}||_2 = 1$. Choose any $\xi_0 \in \mathcal{X}$ and let $\Phi_\delta = \Phi_\delta(\xi_0)$, see (7) for the definition. By Theorem 3.1, there exists $\delta > 0$ such that

$$x \in \Phi_\delta \implies |e^{2\pi i \langle x, \alpha_{j,k}\rangle} - 1| < (\epsilon/m)^{1/2} (3 \sum_{k=1}^{n_j} |c_{j,k}|)^{-1} \quad \text{for} \;\; k = 1, \ldots, n_j;\; j = 1, \ldots, m. \tag{19}$$

Since our system is minimal, the set Φ_δ is relatively dense by (8), so it suffices to show that any $x \in \Phi_\delta$ is a measure-theoretic ϵ-almost-period for $(\mathcal{X}, \mu, F_y)$.

Let $x \in \Phi_\delta$. It follows from (17) and (19) that

$$||f_{j,k}(F_x\xi) - f_{j,k}(\xi)||_2 < (\epsilon/m)^{1/2}(3\sum_{k=1}^{n_j}|c_{j,k}|)^{-1} \quad \text{for} \;\; k = 1, \ldots, n_j;\; j = 1, \ldots, m.$$

Combined with (18), this implies

$$||h_j(F_x\xi) - h_j(\xi)||_2 < (\epsilon/m)^{1/2} \quad \text{for} \;\; j = 1, \ldots, m, \tag{20}$$

Since h_j is the indicator of B_j,

$$||h_j(F_x\xi) - h_j(\xi)||_2^2 = \mu(B_j \triangle F_{-x}B_j) \quad \text{for} \;\; j = 1, \ldots, m. \tag{21}$$

Observe that

$$\{\xi \in \mathcal{X} :\; \rho(\xi, F_x\xi) > \epsilon\} \subseteq \bigcup_{j=1}^{m}(B_j \triangle F_{-x}B_j).$$

Indeed, every $\xi \in X$ lies in some B_j. If $\rho(\xi, F_x\xi) > \epsilon$, then $F_x\xi \notin B_j$ since B_j is a ball of radius $\epsilon/2$. Thus, by (20) and (21),

$$\mu\{\xi \in \mathcal{X} :\; \rho(\xi, F_x\xi) > \epsilon\} < \epsilon,$$

so x is a measure-theoretic ϵ-almost-period. The proof is complete. ∎

Proof of Theorem 2.2. Recall that we now work with Delone sets and the translation action, so we write $\xi - x$ instead of $\Gamma_x(\xi)$. Let

$$\Omega(x, \epsilon) = \{\xi \in \mathcal{X}_\Lambda :\; \rho(\xi, \xi - x) \le \epsilon\}.$$

To deduce Theorem 2.2 from Theorem 3.2 it suffices to show that there exists an absolute constant $C > 0$ such that if

$$\underline{\text{freq}}[\Lambda \triangle (\Lambda - x)] < C\epsilon^{d+1}, \tag{22}$$

then $\mu(\Omega(x, \epsilon)) > 1 - \epsilon$. We will do this for $C = (4\mathcal{L}_d(B_1(0)))^{-1}$ where $\mathcal{L}_d$ denotes Lebesgue measure.

By the pointwise Ergodic Theorem for $\mathbf{R}^d$-actions (see Krengel's book [10]), for any $f \in L^1(\mathcal{X}_\Lambda, \mu)$,

$$\int_X f(\xi)\, d\mu(\xi) = \lim_{L\to\infty} L^{-d}\int_{C_L} f(\xi - y)\, dy$$

for μ-a.e. $\xi \in \mathcal{X}_\Lambda$. Letting f be the indicator function of a Borel set $E \subseteq \mathcal{X}_\Lambda$, we obtain for μ-a.e. $\xi \in \mathcal{X}_\Lambda$,

$$\mu(E) = \lim_{L\to\infty} L^{-d}\mathcal{L}_d\{y \in C_L :\; \xi - y \in E\}.$$

Fix such a ξ for $E = \Omega(x,\epsilon)$. Then for all L sufficiently large

$$|\mu(\Omega(x,\epsilon)) - L^{-d}\,\mathcal{L}_d\{y \in C_L:\ \xi - y \in \Omega(x,\epsilon)\}| < \epsilon/2. \tag{23}$$

Since ξ is in the orbit closure of Λ and $\underline{\mathrm{freq}}$ is the lower uniform frequency, (22) implies

$$\underline{\mathrm{freq}}[\xi \bigtriangleup (\xi - x)] \le \underline{\mathrm{freq}}[\Lambda \bigtriangleup (\Lambda - x)] < C\epsilon^{d+1}.$$

Thus, we can choose L arbitrarily large, so that

$$\#[(\xi \bigtriangleup (\xi - x)) \cap C_L] < 2C\epsilon^{d+1}L^d. \tag{24}$$

Observe that $\xi - y \in \Omega(x,\epsilon)$ means $\rho(\xi - y, \xi - y - x) \le \epsilon$. This is certainly true if $\xi - y = \xi - y - x$ on $B_{1/\epsilon}(0)$ or equivalently, if $\xi = \xi - x$ on $B_{1/\epsilon}(y)$. But the latter holds for

$$y \notin [\xi \bigtriangleup (\xi - x)] + B_{1/\epsilon}(0).$$

Indeed, such a y has a distance of at least $1/\epsilon$ to every point of $\xi \bigtriangleup (\xi - x)$ and hence

$$B_{1/\epsilon}(y) \cap [\xi \bigtriangleup (\xi - x)] = \emptyset$$

which means that ξ and $\xi - x$ agree on $B_{1/\epsilon}(y)$. Thus,

$$\begin{aligned}
\mathcal{L}_d\{y \in C_L:\ \xi - y \in \Omega(x,\epsilon)\} &\ge \mathcal{L}_d(C_L) - \mathcal{L}_d\Big(\Big([\xi \bigtriangleup (\xi - x)] + B_{1/\epsilon}(0)\Big) \cap C_l\Big) \\
&\ge \mathcal{L}_d(C_L) - \#[(\xi \bigtriangleup (\xi - x)) \cap C_L]\,\mathcal{L}_d(B_{1/\epsilon}(0)) \\
&\ge L^d - 2C\epsilon^{d+1}L^d\,\mathcal{L}_d(B_1(0))\epsilon^{-d} \\
&= L^d[1 - 2C\epsilon\,\mathcal{L}_d(B_1(0))] \\
&= L^d(1 - \epsilon/2),
\end{aligned}$$

using (24) in the third inequality and that $C = (4\mathcal{L}_d(B_1(0)))^{-1}$ in the last equality. Combining this with (23) implies $\mu(\Omega(x,\epsilon)) > 1 - \epsilon$, as desired. The proof is complete. ∎

Acknowledgments. I am grateful to F. Gähler, A. Hof, and J. Lagarias for helpful discussions. I would also like to thank R.V. Moody and M. Senechal for the opportunity to take part in the NATO-ASI on Mathematics of Aperiodic Order held in Waterloo, ON, August 1995.

References

[1] R.B. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.

[2] S. Dworkin, Spectral theory and X-ray diffraction, *J. Math. Phys.* **34** (1993), 2965–2967.

[3] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, 1978.

[4] F. Gähler and R. Klitzing, The diffraction pattern of self-similar tilings, *The Mathematics of Long-Range Aperiodic Order*, (R. V. Moody, ed.), 175–198, Kluwer, 1997.

[5] E. Hewitt, Representation of functions as absolutely convergent Fourier-Stieltjes transforms, *Proc. Amer. Math. Soc.* **4** (1953), 663–670.

[6] A. Hof, On diffraction by aperiodic structures, *Comm. Math. Phys.* **169** (1995), 25–43.

[7] A. Hof, Uniform distribution and the projection method, Preprint, 1996.
******************** *perhaps, this volume? ********************

[8] A. Hof, Diffraction by aperiodic structures, *The Mathematics of Long-Range Aperiodic Order*, (R. V. Moody, ed.), 239–268, Kluwer, 1997.

[9] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover, New York, 1976.

[10] U. Krengel, *Ergodic Theorems*, W. de Gruyter, Berlin–New York, 1985.

[11] J.C. Lagarias, Meyer's concept of Quasicrystal and quasiregular sets, *Comm. Math. Phys.* **179** (1996), 365–376.

[12] J.C. Lagarias, Delone sets of finite type and Meyer sets I, Preprint, 1996.

[13] J.C. Lagarias and P.A.B. Pleasants, Local isomorphism classes of Delone sets, Preprint, 1996.

[14] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, North Holland, New York, 1972.

[15] Y. Meyer, Quasicrystals, diophantine approximation, and algebraic numbers, *Beyond Quasicrystals*, (F. Axel and D. Gratias, ed.), 3–16, Les editions de physique, Springer, New York, 1995.

[16] R. Moody, Meyer sets and their duals, *The Mathematics of Long-Range Aperiodic Order*, (R. V. Moody, ed.), 403–441, Kluwer, 1997.

[17] M. Queffelec, *Substitution dynamical systems — spectral analysis*, Springer Lecture Notes in Math. **1294**, Springer, 1987.

[18] C. Radin, Space tilings and substitutions, *Geometriae Dedicata* **55** (1995), 257–264.

[19] C. Radin, Aperiodic tilings, ergodic theory and rotations, *The Mathematics of Long-Range Aperiodic Order*, (R. V. Moody, ed.), 499–519, Kluwer, 1997.

[20] C. Radin and M. Wolff, Space tilings and local isomorphism, *Geometriae Dedicata* **42** (1992), 355–360.

[21] E.A. Robinson, Jr., On uniform convergence in the Wiener-Wintner Theorem, *J. London Math. Soc.* **49:2** (1994), 493–501.

[22] E.A. Robinson, Jr., The dynamical properties of Penrose tilings, *Trans. Amer. Math. Soc.* **348** (1996), 4447–4469.

[23] E.A. Robinson, Jr., The dynamical theory of tilings and quasicrystallography, *Ergodic Theory of $\mathbf{Z}^d$-Actions*, (M. Pollicott and K. Schmidt, eds.), 451–473, Cambridge University Press, Cambridge, 1996.

[24] B. Solomyak, Dynamics of self-similar tilings, *Ergodic Th. Dynam. Sys.* **17** (1997), 695–738.

[25] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1982.