TILINGS AND DYNAMICS

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ABSTRACT. We discuss tilings of $\mathbb{R}^d$ of translational finite local complexity and associated dynamical systems—translation $\mathbb{R}^d$ actions. The main focus is on self-similar and self-affine tilings and their spectral properties. This is linked to "aperiodic order."

1. INTRODUCTION

A tiling (or tesselation) of $\mathbb{R}^d$ is a collection of sets, called tiles, which have nonempty disjoint interiors and whose union is the entire $\mathbb{R}^d$. Tilings of the hyperbolic space or other spaces are also considered, but we restrict ourselves to the Euclidean space.

Tiles are often assumed to be polygons (polyhedra), or at least topological balls, but for us they are just compact sets that are closures of their interiors. To get a meaningful theory, it is usually assumed that there are finitely many "prototiles" up to a group of transformations acting on the space. The two natural choices in $\mathbb{R}^d$ are the group of translations and the group of all Euclidean isometries. We will be concerned with the former class of tilings, which are called *translationally-finite*. Usually there are additional constraints, such as "face-to-face" for polyhedral tilings, or "matching rules" which specify how the tiles can fit together.

We start with a few historical remarks, mostly taken from the book chapter by E. A. Robinson, Jr. [81].

Question (Tiling Problem). Is there an algorithm that, upon being given a set of prototiles, with matching rules, decides whether a tiling of the entire space exists?

Hao Wang (1961) considered this problem for squares with colored edges, which became known as "Wang tilings." If we ignore the coloring, this is just the periodic tiling of $\mathbb{R}^2$ by the square tiles in a grid. The edges are colored in finitely many colors, and if two tiles touch each other, the colors of their common edge should match.

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When $d = 1$, the "Wang tiles" are just intervals with colored endpoints, and there is an easy algorithm to answer the Tiling Problem. Draw a graph whose vertices are prototiles and directed edges indicate which pairs are allowed. A tiling of $\mathbb{R}$ exists if and only if there is an infinite path in this graph, which is equivalent to existence of a cycle.

Definition 1.1. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is called a *periodic tiling* if its *translation group* $\Gamma_{\mathcal{T}} = \{\mathbf{t} \in \mathbb{R}^d : \mathcal{T} - \mathbf{t} = \mathcal{T}\}$ is a *lattice*, that is, a subgroup of $\mathbb{R}^d$ with d linearly independent generators. A tiling is called *aperiodic* if $\Gamma_{\mathcal{T}} = \{0\}$.

From the discussion above it follows that if a tiling of $\mathbb{R}$ with a given prototile set exists, then there is a periodic tiling. Wang conjectured that the same holds for $d > 1$. More precisely, he conjectured that (1) there is an algorithm that decides the Tiling Problem; (2) if a tiling exists, then there exists a periodic tiling. Wang proved that (2) implies (1). However, the conjecture turned out to be false! Wang's student, Robert Berger (1966) proved that the Tiling Problem is undecidable and constructed an "aperiodic tiling system," that is, a prototile set which tile the plane but only aperiodically. Berger's prototile set was very large; it had more than 20,000 prototiles. Later, Raphael Robinson (1971) found a simpler example with 32 prototiles.

The problem of finding small aperiodic sets of prototiles has attracted a lot of attention (see [39] for the history up to 1987). The exact formulation depends on whether the prototiles are counted up to translation or up to isometries.

One of the most interesting aperiodic sets is the set of *Penrose tiles*, discovered by Roger Penrose [69]. Penrose tilings play a central role in the theory because they can be generated by any of the three main methods: local matching rules, tiling substitutions, and the projection method. The Penrose tiling has two prototiles up to isometries; below they are defined precisely. It is an open problem whether there exists an aperiodic prototile set consisting of a single prototile in the plane.

The discovery of quasicrystals in 1984 had a profound influence on this subject. A quasicrystal is a solid (usually, metallic alloy) which, like a crystal, has a sharp X-ray diffraction pattern, but unlike a crystal, has an aperiodic atomic structure. Aperiodicity was inferred from a "forbidden" 5-fold symmetry of the diffraction picture. Since the Penrose tilings have this symmetry (not literally, but in an appropriate sense—statistically or for the tiling space), they became a focus of many investigations, both by physicists and by mathematicians. See [89] for an introduction to the mathematics of quasicrystals addressed to a general audience.

Penrose tilings comes in several different versions. The simplest to describe has two rhombs—a thick and a thin one—as prototiles, shown in Figure 1. Their smaller angles are $2\pi/5$ and $2\pi/10$ respectively. The "markings" of the boundary define the matching rules. A part of a Penrose tiling is shown in Figure 2. The tiles appear in 10 orientations, so there are 20 prototiles up to translation.

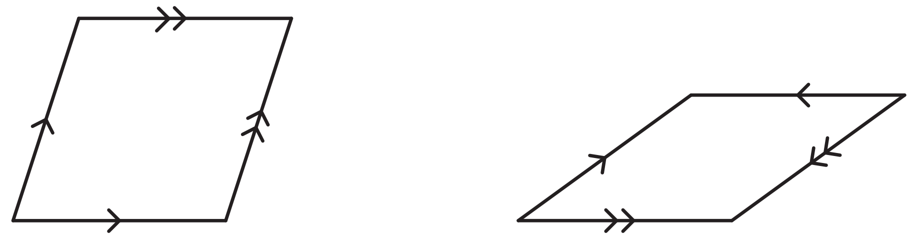

Figure 1. The Penrose tiles (rhombs)

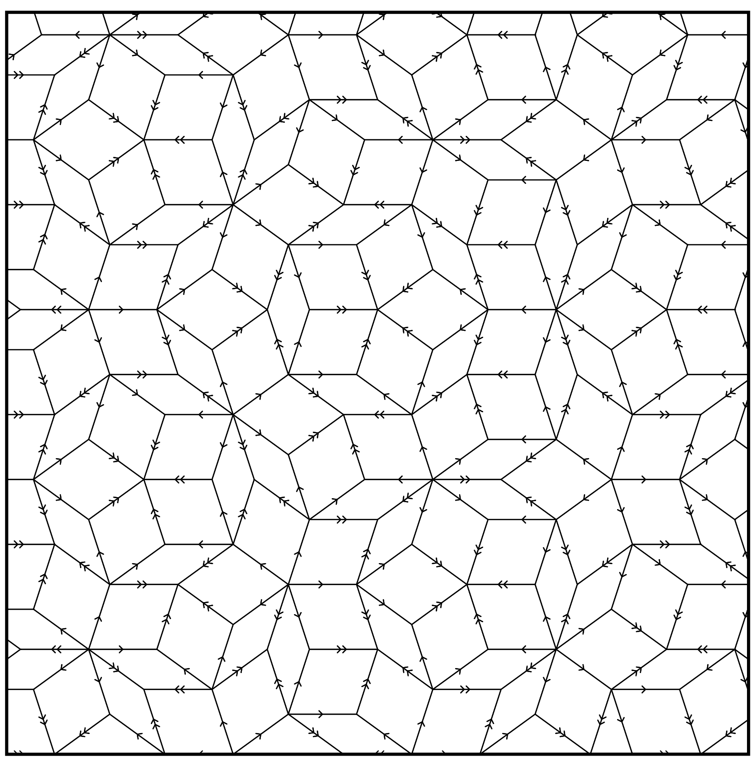

Figure 2. A part of a Penrose tiling

How do we know that a Penrose tiling exists? In other words, why is it possible to tile the whole plane? This is not obvious; in fact, when one starts playing with the Penrose tiles as with a "jigsaw puzzle," it becomes clear that there are many non-extendable configurations. Penrose proved the existence of a tiling using *inflation*. It is easiest to explain this using *triangular Penrose tiles* introduced by R. Robinson, see Figure 3.

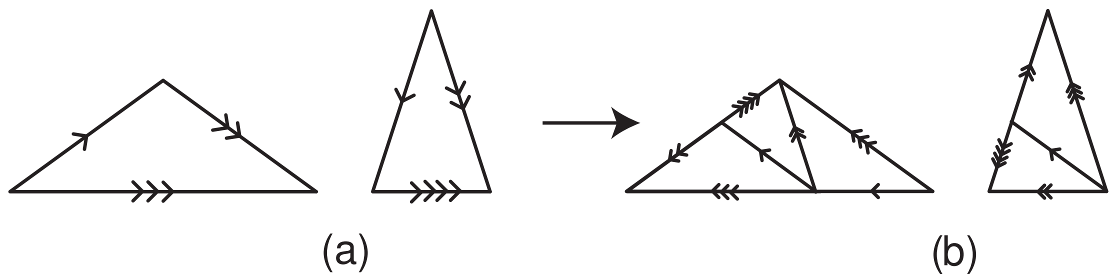

FIGURE 3. The *triangular Penrose tiles*.

They are obtained by cutting each rhomb into two triangles, and the markings are chosen in such a way that any tiling by triangles can be converted into a Penrose tiling by combining adjacent triangles. Now we can do the following: inflate one of the triangles by a factor of $(1+\sqrt{5})/2$ (the golden ratio) and subdivide it according to the rule indicated in the figure, then repeat this with the entire patch. Note that some of the triangles in the subdivision are obtained using a reflection; the subdivision rule respects this.

Exercise. Verify that when we inflate and subdivide repeatedly, adjacent triangles in the patch satisfy the matching rules on the boundary.

Thus we can iterate this procedure producing larger and larger patches. In the limit (appropriately defined) we get a tiling of the entire plane. This inflation-subdivision procedure is a powerful mechanism to create hierarchical structures, and it will be one of the main topics of the lectures.

The use of **dynamical systems** has been a major ingredient in the study of aperiodic tilings. Given a tiling in $\mathbb{R}^d$, one associates with it a space which is the closure of its $\mathbb{R}^d$-translation orbit, the closure being in the "local" topology, which compares tilings for more or less exact match in regions around the origin. We call it the *tiling space*; in the literature on mathematical quasicrystals it is often called the *dynamical hull*. Tiling spaces provide a new point of view; many properties of tilings are really properties of the tiling space. Moreover, many of these properties can be interpreted in dynamical terms. Explaining this will be a major topic of these lectures. We do not list all the relevant literature here, but mention four important early papers where this approach was used: [83, 77, 25, 79], as well as the book [74].

Overview of these Notes. The core of these lectures is Sections 2-5. In Section 2 we introduce the basics of tilings and tiling dynamical systems. We include two "digressions" to briefly review the relevant background in dynamics. In Section 3 we introduce tile-substitutions and self-affine tilings, of which the triangular Penrose tiling is an example. Section 4 is devoted to the problem of characterizing expansion maps for self-similar tilings studied by W. Thurston and R. Kenyon. Geometric and number-theoretic ideas play an important role here. In Section 5 we investigate the eigenvalues of self-affine tiling dynamical systems, emphasizing number-theoretic connections. In Section 6, added for completeness, we discuss some conditions for pure discrete spectrum. It contains only a brief glimpse into the topic, without many details. Section 7 is devoted to "aperiodic order" and "mathematical quasicrystals." This is also a huge topic, of which we can only scratch the surface; I tried to emphasize the relevance of dynamical systems. We cannot possibly prove everything; some statements are left as an exercise, some proofs are outlined, and others are omitted altogether. However, I included some longer proofs, that are, perhaps, difficult to extract from the literature, in the Appendix (Section 8).

Disclaimer. There is a huge literature on tilings, tiling dynamical systems, and related questions. Our reference list has almost 100 items, and it is far from being complete. Many important topics are missing, because of time and space limitations. (In particular, we do not return to the "Tiling Problem" or "matching rules" from the Introduction.) Additional information may be found in excellent surveys [81, 14], which also treat many of the same topics as these notes. I apologize for any inadvertent omissions and wrong attributions. Please let me know if you find mistakes.

Acknowledgment. I would like to thank Michel Rigo and Valérie Berthé for inviting me to present these lectures. I am grateful to Rick Kenyon, Jeong-Yup Lee, Robert V. Moody, and Robbie Robinson for many helpful discussions. Many thanks to Robbie for allowing me to use his figures (Fig. 1-4,7). Some material from [61, 67, 4, 81] was used in these notes.

2. Tilings and tiling dynamical systems

A *tile* is a compact set $T \subset \mathbb{R}^d$ which is a closure of its interior. A *tiling* is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \cup\{T : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors.

We will assume that the tiling has finitely many tiles up to translation. Each tile will be labeled with an element of $\mathcal{A} = \{1, \dots, m\}$, which may be regarded as a "tile type" or a "color." Any two tiles of the same type must be translates of each other (by definition).

Formally, we should say that a tile is a pair (A, i) where A is the support of the tile and i is its label. However, it is usually not a problem to think about tiles as sets, keeping in mind that they have the label.

A $\mathcal{T}$-*patch* is a finite union of $\mathcal{T}$-tiles. Two tiles, or patches, are said to be equal if they have the same collection of tiles (including the labels). Two tiles, or patches, are *equivalent* if they are translates of each other. We write $\approx$ to indicate equivalence. When translating a tile, we move the support and keep the label. The *support* of a patch is the union of tiles in the patch (so a patch is a set of tiles, a subset of $\mathcal{T}$, whereas its support is a subset of $\mathbb{R}^d$).

Two common assumptions are:

- the tiling $\mathcal{T}$ has **finite local complexity** (FLC), that is, for any $R > 0$ there are finitely many $\mathcal{T}$-patches of diameter less than R up to equivalence.
- the tiling $\mathcal{T}$ is **repetitive**, that is, for any $\mathcal{T}$-patch P there exists $R > 0$ such that every ball of radius R contains a translated copy of P.

Tiling space: $X_{\mathcal{T}} = \overline{\{-g + \mathcal{T} : g \in \mathbb{R}^d\}}$, where the closure is in the "local" topology: two tilings are close if after a small translation they *agree* on a large ball around the origin. More precisely, two tilings $\mathcal{T}_1, \mathcal{T}_2$ agree on a set $K \subset \mathbb{R}^d$ if $\mathrm{supp}(\mathcal{T}_1 \cap \mathcal{T}_2) \supset K$. Let B_ε be the open ball of radius ε centered at the origin. For $\mathcal{T}_1, \mathcal{T}_2 \in X_{\mathcal{T}}$ define

$$\widetilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2) := \inf\{r \in (0, 2^{-1/2}) : \ \exists\, g,\ \|g\| \le r \text{ such that } \mathcal{T}_1 - g, \mathcal{T}_2 \ \text{ agree on } B_{1/r}\}.$$

Finally, let

$$\varrho(\mathcal{T}_1, \mathcal{T}_2) := \min\{2^{-1/2}, \varrho(\mathcal{T}_1, \mathcal{T}_2)\}.$$

Exercise. Verify that ϱ is a metric. (*See* [81] *for a solution.*)

Theorem 2.1. [83] (see also [81]). $(X_{\mathcal{T}}, \varrho)$ *is a complete metric space. It is compact, whenever $\mathcal{T}$ has finite local complexity.*

2.1. **Topological Dynamics.** Consider the translation action on $X_{\mathcal{T}}$ defined by

$$T^{\mathbf{t}}(\mathcal{S}) := \mathcal{S} - \mathbf{t} \quad \text{for } \mathcal{S} \in X_{\mathcal{T}} \text{ and } \mathbf{t} \in \mathbb{R}^d.$$

It is easy to see that $T^{\mathbf{t}}$ is a homeomorphism, and we get a continuous action of $\mathbb{R}^d$ on $X_{\mathcal{T}}$. We call it the (topological) tiling dynamical system associated with the tiling $\mathcal{T}$. It will be denoted by $(X_{\mathcal{T}}, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d}$, or just $(X_{\mathcal{T}}, \mathbb{R}^d)$ abusing the notation a little. Next we review briefly the relevant background.

Digression. A topological dynamical system is a pair $(X, T^{\mathbf{t}})_{\mathbf{t}\in G}$ where X is a compact metric space and $T^{\mathbf{t}}$, $\mathbf{t} \in G$, is a continuous action of a topological (semi)group.

The most basic set-up is an action of a single continuous map $T : X \to X$, which leads to $\mathbb{N}$- or $\mathbb{Z}$-action (if T is invertible). Continuous-time systems are also studied; they correspond to $\mathbb{R}$-actions (if they are invertible). They are even more classical, since such systems arise in physics, whenever there is an autonomous system of ODEs. The theory has been generalized to $\mathbb{Z}^d$- and $\mathbb{R}^d$-actions (which will be our set-up), and to actions of other (even nonabelian) topological groups. A $\mathbb{Z}^d$-action is essentially the same as having d commuting invertible transformations.

Suppose we have two G-actions, on X and on Y. A continuous map $\phi : X \to Y$ commuting with the action is called a *factor map*. If a factor map is a bijection, it is called a *topological conjugacy.*

Given a dynamical system $(X, T^{\mathbf{t}})_{\mathbf{t}\in G}$, the *orbit* of $x \in X$ is the set $O(x) := \{T^{\mathbf{t}}x : \mathbf{t} \in G\}$. The system is called *minimal* if every orbit is dense, or equivalently, if there are no proper closed G-invariant subsets of X.

Theorem 2.2. *The tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is minimal if and only if the tiling $\mathcal{T}$ is repetitive.*

This is proved by a well-known argument in topological dynamics, which goes back to Gottschalk [37]; see [81, Sec. 5] for details.

Next we discuss the topology of tiling spaces. Recall the definition of periodic and aperiodic tilings, Def. 1.1.

Exercise. Prove that a tiling $\mathcal{T}$ is periodic if and only if its orbit $O(\mathcal{T})$ is closed.

For a periodic tiling $\mathcal{T}$, the tiling space is homeomorphic to $\mathbb{R}^d/\Gamma_{\mathcal{T}} = \mathbb{T}^d$, the d-dimensional torus. In order to understand the topology of the tiling space for aperiodic tilings we consider *cylinder sets.* Suppose a tiling $\mathcal{T}$ is given. For a patch $P \subset \mathcal{T}$ let

$$X_P := \{\mathcal{S} \in X_{\mathcal{T}} : \ P \subset \mathcal{S}\},$$

i.e., X_P consists of all tilings in the tiling space containing a given patch. For $W \subset \mathbb{R}^d$ let

$$X_{P,W} := \bigcup_{\mathbf{t}\in W} T^{\mathbf{t}} X_P = \{\mathcal{S} \in X_{\mathcal{T}} : \ \exists \mathbf{t} \in W, \ -\mathbf{t} + P \subset \mathcal{S}\}.$$

Cylinder sets $\{X_{P,B_{\varepsilon_n}} : P \subset \mathcal{T}, \varepsilon_n \to 0\}$ form a basis for the tiling topology on X. For small $\varepsilon > 0$, X_{P,B_ε} is homeomorphic to $X_P \times B_\varepsilon$.

For more information on the topology of tiling spaces see [86, 85] and references therein. A tiling dynamical system can be viewed as a $\mathbb{G}$-*solenoid*, with a structure of *lamination* [13].

Exercise. Let $\mathcal{T}$ be an aperiodic repetitive FLC tiling. Show that X_P is homeomorphic to the Cantor set, i.e., it is compact, totally disconnected, and has no isolated points. In particular, X_P is uncountable.

Instead of tilings, it is often convenient to consider discrete point sets in $\mathbb{R}^d$.

Definition 2.3. A set $\Lambda \subset \mathbb{R}^d$ is called a *Delone set* (sometimes spelled "Delonay") if Λ is uniformly discrete and relatively dense. This means that there exist $c_1, C_2 > 0$ such that every ball of radius c_1 contains at most one point of Λ, and every ball of radius C_2 contains at least one point of Λ. In geometric analysis, Delone sets are known as "uniformly separated nets."

The theory of Delone sets parallels tiling theory in many respects. For instance, we define FLC and repetitivity in a similar way. A finite subset of a Delone set is called a *cluster*. A Delone set Λ is FLC if $\Lambda - \Lambda$ is closed and discrete, which is equivalent to having finitely many clusters of any given size, up to translations (see [51] where the term "finite type" is used instead of FLC). The *Delone dynamical system* associated to Λ is defined, similarly to tiling dynamical systems. First we introduce a metric analogous to ϱ defined above; the space $X_\Lambda := \overline{\{-g + \Lambda : g \in \mathbb{R}^d\}}$ is compact whenever Λ has FLC. The translation action $(X_\Lambda, \mathbb{R}^d)$ is a topological $\mathbb{R}^d$-action; it is minimal if and only if Λ is repetitive. Similarly to tilings, where tiles have a "color" or a "label," it is often convenient to consider "colored" point sets, or more formally, Delone multisets $(\Lambda_1, \ldots, \Lambda_m)$.

Some of the papers in this area have been couched in terms of tilings, and others in terms of Delone sets. It is usually an easy exercise to transfer the proofs from one context to another. This can also be done in a formal way, using the notion of *local derivability*.

We have already seen an example in Section 1—the Penrose tiling with rhombic tiles and the corresponding tiling with triangular tiles are *mutually locally derivable*, because there is a specific local rule allowing one to pass from one kind of patch to another kind, and vice versa. More generally, it can be any transformation rule which only depends on the neighborhood of uniform fixed radius in a translation invariant way. Here is the precise definition.

Definition 2.4. (See [9, 8]) Let Λ_1 and Λ_2 be two Delone sets. We say that Λ_2 is locally derivable (LD) from Λ_1 with a radius $R > 0$ if for all $x, y \in \mathbb{R}^2$ and some $\delta > 0$,

$$B_{R+\delta}(x) \cap \Lambda_1 = (B_{R+\delta}(y) \cap \Lambda_1) + (x - y) \;\Rightarrow\; B_\delta(x) \cap \Lambda_2 = (B_\delta(y) \cap \Lambda_2) + (x - y).$$

If Λ_1 is LD from Λ_2 and Λ_2 is LD from Λ_1, then we say that Λ_1 and Λ_2 are mutually locally derivable (MLD).

The definition of LD and MLD is extended to tilings, as well as to pairs (tiling–Delone set) in an obvious way.

Remark. LD is the tiling analog of a *sliding block code in symbolic dynamics*, see [62]. LD from $\mathcal{T}_1$ to $\mathcal{T}_2$ induces a factor map from $(X_{\mathcal{T}_1}, \mathbb{R}^d)$ to $(X_{\mathcal{T}_2}, \mathbb{R}^d)$ (an easy exercise). If two tilings or two Delone sets are MLD, then the associated translation dynamical systems are topologically conjugate, hence they have the same dynamical and ergodic-theoretic properties. However, in contrast with symbolic dynamics, where any factor map is implemented by a sliding block code (this is Curtis-Lyndon-Hedlund Theorem, see e.g., [62]), a topological conjugacy between tiling systems does not, in general, imply that the tilings are MLD ([70, 76]).

Any tiling $\mathcal{T}$ can be converted into a Delone multiset by choosing a "reference point" in each prototile $x_i \in T_i$ and then taking the points with the same relative position in all $\mathcal{T}$-tiles equivalent to T_i. This results in a Delone multiset $(\Lambda_1, \ldots, \Lambda_m)$, which is easily seen to be MLD with $\mathcal{T}$. Thus, all the concepts and results transfer from the language of tilings to the language of Delone (multi)sets and vice versa.

2.2. Ergodic Theory studies dynamics on measure spaces; in the classical theory the dynamics should preserve the measure. We review briefly the relevant background.

Digression. Let X be a compact metric space. We consider Borel probability measures on X. Recall that Borel sets are the elements of the smallest σ-algebra containing all open (and closed) sets. The measure μ assigns a number $0 \le \mu(E) \le 1$ to a Borel set $E \subset X$, with $\mu(\emptyset) = 0$ and $\mu(X) = 1$. One way to interpret μ is as a probability law in which $\mu(E)$ measures the probability that a randomly chosen point $x \in X$ belongs to E. The integral of a function with respect to a measure μ is denoted $\int_X f\, d\mu$. If we think of f as a random variable on X, then the integral is its expectation. In order to consider the integral, we need to assume that f is measurable and either $f \ge 0$, or $\int_X |f|\, d\mu < \infty$. We will need Lebesgue spaces $L^p(X, \mu)$ for $p \ge 1$, whose elements are equivalence classes of measurable functions f such that $\|f\|_p := (\int_X |f|^p\, d\mu)^{1/p} < \infty$. Two functions are equivalent if they agree on a set of full measure. It is common to consider elements of L^p as functions, keeping in mind this identification.

Now suppose that we have a topological $\mathbb{R}^d$ action $(X, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d}$. Of interest to us will be invariant measures. These are the measures that satisfy $\mu(T^{\mathbf{t}}E) = \mu(E)$ for all Borel sets E and all $\mathbf{t} \in \mathbb{R}^d$.

An invariant measure μ is called *ergodic* if $T^{\mathbf{t}}E = E$ for all $\mathbf{t}$ implies $\mu(E) = 0$ or $\mu(E) = 1$. An ergodic invariant measure always exists (see [100]). Denote

$$Q_r = [-r/2, r/2]^d.$$

If μ is ergodic, then Birkhoff's Ergodic Theorem asserts that for all $f \in L^1(X, \mu)$,

$$\lim_{r\to\infty} r^{-d} \int_{Q_r} f(T^{\mathbf{t}}x)\, d\mathbf{t} = \int_X f\, d\mu \quad \text{for } \mu\text{-a.e. } x \in X. \tag{2.1}$$

In the left-hand side the integration is over the Lebesgue measure in $\mathbb{R}^d$. Instead of the cubes Q_r one can take balls of radius r centered at the origin; then r^{-d} should be replaced by $1/\mathrm{Vol}(B_r)$ where Vol denotes the volume (Lebesgue measure in $\mathbb{R}^d$). The formula (2.1) is often interpreted as follows: the time average is equal to the space average "typically." Note that here the "time" is d-dimensional.

We will be especially interested in the case when there is a unique invariant measure μ, which is then necessarily ergodic. Then the system $(X, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d}$ is called *uniquely ergodic*. Such systems satisfy a stronger version of Ergodic Theorem, namely the convergence in (2.1) is not just almost everywhere, but everywhere, and it is uniform in x.

Now we return to the tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. For a patch $P \subset \mathcal{T}$ let

$$L_P(\mathcal{T}, A) := \#\{\mathbf{t} \in \mathbb{R}^d :\ -\mathbf{t} + P \subset \mathcal{T},\ -\mathbf{t} + \mathrm{supp}(P) \subset A\} \tag{2.2}$$

denote the number of $\mathcal{T}$-patches equivalent to P that are contained in A. The "statistics" of patches turns out to be relevant.

Definition 2.5. A tiling $\mathcal{T}$ has *uniform patch frequencies* (UPF) if for any non-empty patch P, the limit

$$\mathrm{freq}(P, \mathcal{T}) := \lim_{r\to\infty} \frac{L_P(\mathcal{T}, \mathbf{x} + Q_r)}{r^d} \geq 0$$

exists uniformly in $\mathbf{x} \in \mathbb{R}^d$.

In a similar way, one defines an analogous notion for Delone sets, called the *uniform cluster frequencies* (UCF). The following is a standard result; see [58, Th. 2.7] for a proof of the Delone set version, which is based on [73, Cor. IV.14(a)].

Theorem 2.6. **(i)** *Let $\mathcal{T}$ be a tiling with FLC. Then the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic if and only if $\mathcal{T}$ has UPF.*

(ii) *Suppose that $\mathcal{T}$ is a tiling with FLC and UPF. Then there exists $\eta > 0$ such that for any Borel set W with $\mathrm{diam}(W) < \eta$, we have*

$$\mu(X_{P,W}) = \mathrm{Vol}(W) \cdot \mathrm{freq}(P, \mathcal{T}),$$

where μ is the unique invariant measure for $(X_{\mathcal{T}}, \mathbb{R}^d)$.

In order to see how part (ii) follows from part (i), consider $f = \mathbf{1}_{X_{P,W}}$, the indicator function of a cylinder set. Then the integral in the left-hand side of (2.1) reduces to (with $x = \mathcal{T}$):

$$\mathrm{Vol}(\{\mathbf{t} \in Q_r : \ P \subset -\mathbf{t} - \mathbf{g} + \mathcal{T} \text{ for some } \mathbf{g} \in W\}).$$

The "boundary effects" become negligible in the limit, so the left-hand side of (2.1) is just $L_P(\mathcal{T}, Q_r) \cdot \mathrm{Vol}(W)$, whereas the right-hand side of (2.1) is $\mu(X_{P,W})$.

Definition 2.7. A tiling is called *linearly repetitive* (LR) if there exists $C > 0$ such that for any $P \subset \mathcal{T}$ every ball of radius $C\mathrm{diam}(P)$ contains a $\mathcal{T}$-patch equivalent to P. LR Delone sets are defined similarly.

LR tilings were considered in [94] under the name "strongly repetitive tilings." F. Durand [23] studied a similar object in symbolic dynamics, namely, linearly recurrent subshifts. J. Lagarias and P. Pleasants [53] investigated LR Delone sets.

Theorem 2.8. ([53], see also [23, Th. 15]). *If $\mathcal{T}$ is LR, then $\mathcal{T}$ has UPF.*

A proof of this is sketched in Section 8.1. Fabien Durand will say more about LR subshifts and and their higher-dimensional versions in his lectures.

3. Substitution Tilings.

3.1. Word substitutions.

We begin with the more familiar set-up of word substitutions in symbolic dynamics (or morphisms in theoretical computer science), see e.g. [73, 72] for details.

Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^* = \cup_{n \ge 1} \mathcal{A}^n$, the set of finite words, $\mathcal{A}^{\mathbb{N}}$=the set of infinite words with letters in $\mathcal{A}$. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^*$. We will assume that ζ is injective. The substitution is extended to $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$ by concatenation. The length of a word w is denoted by $|w|$.

Assume that $|\zeta^n(\alpha)| \to \infty$, as $n \to \infty$, for all $\alpha \in \mathcal{A}$. Then one can find $k \ge 1$ and $u = u_0 u_1 u_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ such that $\zeta^k(u) = u$ (see [73, Prop. V.1]). The space $\mathcal{A}^{\mathbb{N}}$ is compact in the product topology. The *substitution space* X_ζ is the orbit closure of u under the left shift transformation $\sigma : X_\zeta = \overline{\{\sigma^n u :\ n \ge 0\}}$. The $\mathbb{N}$-action (X_ζ, σ) is called the *substitution dynamical system*.

Alternatively, one can consider a $\mathbb{Z}$-action, the two-sided substitution dynamical system (X'_ζ, σ) where X'_ζ is defined as the set of all sequences $x \in \mathcal{A}^{\mathbb{Z}}$ such that every block of x occurs in u.

It is often assumed that there is a letter $\alpha \in \mathcal{A}$ such that $\zeta(\alpha) = \alpha w$, with $|w| \ge 1$. Then $u = \alpha w \zeta(w) \zeta^2(w) \ldots$ is a fixed point of ζ.

The *substitution matrix* is a matrix $m \times m$ with the entries $M_\zeta(i,j)$ equal to the number of letters i in $\zeta(j)$. The substitution dynamical system is minimal whenever the matrix M_ζ is primitive. (Recall that a non-negative integer matrix M is primitive, or irreducible, if there exists $k \in \mathbb{N}$ such that M^k is strictly positive.) Primitive substitution dynamical systems are uniquely ergodic. The ergodic theory and topological dynamics of substitution systems have been extensively studied, and we do not attempt to review this theory.

3.2. Tile substitutions and self-affine tilings. We study *perfect* (geometric) substitutions, in which a tile is "blown up" by an expanding linear map and then subdivided. Other possibilities, where the substitution is combinatorial, and/or there is no perfect geometric subdivision, have also been considered, see e.g. [28, 29, 33, 18, 96].

Let ϕ be an expansive linear mapping $\mathbb{R}^d \to \mathbb{R}^d$, that is, all its eigenvalues are greater than one in modulus.

Definition 3.1. Let $\mathcal{A} = \{T_1, \ldots, T_m\}$ be a finite set of prototiles in $\mathbb{R}^d$. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches made of tiles each of which is a translate of one of T_i's. A map $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is called a *tile-substitution* with expansion ϕ if

$$\mathrm{supp}(\omega(T_j)) = \phi T_j \quad \text{for } j \le m. \tag{3.1}$$

In plain language, every expanded prototile ϕT_j can be decomposed into a union of tiles (which are all translates of the prototiles) with disjoint interiors.

The substitution ω is extended to all translates of prototiles by $\omega(x + T_j) = \phi x + \omega(T_j)$, and to patches by $\omega(P) = \cup\{\omega(T) :\ \ T \in P\}$. This is well-defined due to (3.1). The substitution ω also acts on the space of tilings whose tiles are translates of those in $\mathcal{A}$.

To the tile-substitution ω we associate its $m \times m$ substitution matrix S, with S_{ij} being the number of tiles of type i in the patch $\omega(T_j)$. The substitution ω is called *primitive* if the substitution matrix is primitive. We say that $\mathcal{T}$ is a fixed point of the substitution if $\omega(\mathcal{T}) = \mathcal{T}$.

Definition 3.2. A repetitive FLC fixed point of a primitive tile-substitution is called a *self-affine tiling*. It is called *self-similar* if the expansion ϕ is a similarity (or similitude). A self-affine tiling of $\mathbb{R}$ is always self-similar, with expansion map $x \mapsto \lambda x$, and we will call λ the expansion constant. Let $\mathcal{T}$ be a self-similar tiling of the plane $\mathbb{R}^2$ for which the expansion map ϕ is orientation-preserving. An orientation-preserving similarity map of the plane can be expressed as multiplication by $\lambda \in \mathbb{C}$ if we identify $\mathbb{R}^2 \cong \mathbb{C}$. Then $\lambda \in \mathbb{C}$ is called the *complex expansion constant* of the tiling $\mathcal{T}$.

Lemma 3.3. *For any primitive tile-substitution ω there exists $n \in \mathbb{N}$ such that ω^n has a fixed point.*

Proof. Take any prototile, say, T_1, and consider the patches $\omega^n(T_1)$ for $n \in \mathbb{N}$. Since ϕ is expansive, these patches will eventually have tiles contained in the interior of the support of the patch (and as "deep inside" as we wish). By primitivity, for n sufficiently large, these tiles will include all tile types, T_1 in particular. Thus, for some $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $r > 0$,

$$T_1 + x \in \omega^n(T_1), \quad T_1 + x \subset (\phi^n T_1)^\circ = (\mathrm{supp}(\omega^n(T_1)))^\circ,$$

where K° denotes the interior of K. Then

$$T_1 - y \in \omega^n(T_1 - y), \quad \text{where } y = (\phi^n - I)^{-1}x, \tag{3.2}$$

which is well-defined, since ϕ is expansive. Moreover, $T_1 - y \subset (\phi^n(T_1 - y))^\circ$, hence $\phi^{-n}(T_1 - y) \subset (T_1 - y)^\circ$, which implies that the origin, the unique fixed point of ϕ^{-n}, is in the interior of $T_1 - y$. It follows from (3.2) by induction that

$$\omega^{in}(T_1 - y) \subset \omega^{(i+1)n}(T_1 - y), \quad \text{for } i \geq 1.$$

Now,

$$\mathcal{T} := \bigcup_{i=0}^{\infty} \omega^{in}(T_1 - y)$$

is well-defined, it is a tiling of the entire space, and $\omega^n(\mathcal{T}) = \mathcal{T}$. $\square$

Remark. Do we really need to assume FLC in Definition 3.2? The situation is a little subtle. It turns out that a fixed point of a primitive tile-substitution is not necessarily of finite local complexity, see [47, 20, 32].

The next lemma is helpful for checking repetivity.

Lemma 3.4. (see [71]). *Let $\mathcal{T}$ be an FLC fixed point of a tile-substitution with substitution matrix* S.

(i) *$\mathcal{T}$ is repetitive (and therefore self-affine) if and only if every $\mathcal{T}$-patch occurs in $\omega^n(T)$ for every $T \in \mathcal{T}$ for all $n \in \mathbb{N}$ sufficiently large.*

(ii) *$\mathcal{T}$ is repetitive if and only if* S *is primitive and a $\mathcal{T}$-patch, containing the origin in the interior of its support, occurs in $\omega^n(T)$ for some $T \in \mathcal{T}$ and some $n \in \mathbb{N}$ (in particular, if the origin lies in the interior of a tile).*

Examples and remarks. (a) The simplest examples of self-similar tilings of the plane are periodic (lattice) tilings of the plane (i) by parallelograms; (ii) by triangles. They are both self-similar with any expansion constant $k > 1$, $k \in \mathbb{N}$. However, the lattice tiling by hexagons ("honeycombs") is not self-affine (there is no perfect decomposition).

(b) There are many aperiodic self-similar tilings with polygonal tiles congruent to each other. Such tilings were considered by Grünbaum and Shephard [39, 10.1] (who called them similarity tilings). Examples include the "chair" tiling, the "sphinx" tiling, see [39, p. 529], the domino tiling, and many others. Three examples of such tilings are shown in Figures 4-6. Actually, we only show the subdivision rule which generates the tile-substitution. There is no "canonical" chair tiling, domino tiling, etc. One should either speak about an element of the tiling space, or about a fixed point—self-similar tiling, which is not unique either.

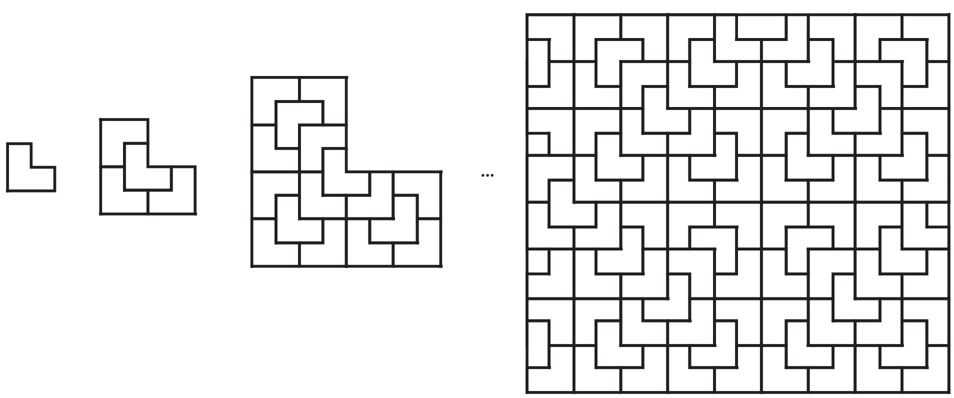

FIGURE 4. The chair tiling

Let us explain the construction in more detail for a chair tiling. In Figure 4 you see the first two iterations of the tile-substitution: T_1, $\omega(T_1)$, $\omega^2(T_1)$, and a larger patch of a tiling. The expansion constant is 2 (that is, the expansion map is a pure dilation by a factor of 2). There are four tile types up to translation, but only one up to isometry, and the subdivision rule respects this. In order to get a self-similar chair tiling, let T_1 be located so that the lower left corner is the origin. Then $\omega(T_1)$ contains T_1, $\omega^n(T_1)$ contains $\omega^{n-1}(T_1)$ as a subpatch, so there is a natural limit $\lim_{n\to\infty}\omega^n(T_1)$, which is a tiling of the quarter-plane (the 1st quadrant). Taking the reflections of this tiling around the axes yields a tiling of $\mathbb{R}^2$ which is a fixed point of ω. In order to show that it is self-similar, we need to verify that it is repetitive. The substitution matrix S is primitive, so we can use Lemma 3.4(ii) to do this.

Exercise. (i) Find another self-similar chair tiling; enumerate ALL self-similar tilings for the chair tile-substitution; (ii) write down the substitution matrix S; (iii) verify that the fixed point of the chair tile-substitution described above is repetitive; for which n does the property in Lemma 3.4(ii) hold?

In Fig. 5-6 we show two other examples of tile substitutions; the first one is sometimes called "domino" (or "table"). In both cases the expansion is $\phi(x) = 2x$. Thick lines indicate "second order" tiles.

FIGURE 5. The "domino" tiling

FIGURE 6. A polyomino tiling

(c) Charles Radin [75] and his collaborators investigated the "pinwheel" tile-substitution which is indicated in Figure 7. There is only one prototile up to isometry: the right triangle with sides of lengths $1, 2, \sqrt{5}$. Note, however, that the angle θ is irrational modulo π which implies that in tilings of the whole plane the triangles appear in infinitely many orientations. Thus, such a tiling is not FLC up to translations, and we do not consider it here. (There are many interesting results about the pinwheel and analogous tilings; see e.g. [84]; we do not review the literature about them.)

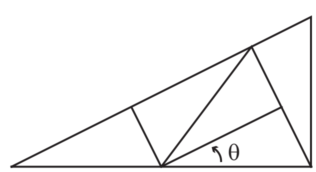

FIGURE 7. The pinwheel decomposition

(d) There are connections between the theory of self-affine tilings and fractal geometry. The prototiles of a self-affine tiling can be viewed as the attractor of a graph-directed iterated function system (see [10]). In the special case when there is only one prototile up to isometry, they have been studied under the name of "reptiles" or self-affine tiles (see e.g. [54]).

It appears that the boundary of self-affine tiles in $\mathbb{R}^d$ is either flat, or fractal; moreover, in the latter case it has dimension strictly between $d-1$ and d. This is a kind of "rigidity" phenomenon. This statement is an **open problem** in full generality [personal communication from M. Urbański], however, it is known in special cases, see e.g. [24, 55, 2]. Note the following easy fact:

Exercise. Let $\mathcal{T}$ be a self-affine tiling with expansion map ϕ whose rotational part is of infinite order (in $\mathbb{R}^2$ this means that the rotation is irrational modulo π). Show that the tiles cannot be polyhedral. *Hint: use the FLC property.*

(e) Many more examples and beautiful pictures may be found in the on-line "Tiling Encyclopedia" developed by E. Harriss and D. Frettlöh [40].

(f) Self-affine tilings are related to many other topics, which we cannot discuss here. Among them: numeration systems (see e.g. [36, 78, 1, 2, 14, 15]), wavelets (see e.g. [27]), Markov partitions (see e.g. [12, 71, 45, 50, 91]), and adic transformations (see [99]).

Lemma 3.5. (see [71, Prop. 1.1]) *For any tile T of a self-affine tiling,* $\mathrm{Vol}(\partial T) = 0$.

By the Perron-Frobenius theory (see [90]), a primitive matrix M has a dominant eigenvalue $\theta > 0$, which is greater in modulus than all other eigenvalues. We call θ the PF eigenvalue of M. The matrix M has strictly positive right and left eigenvectors corresponding to θ, called PF eigenvectors. Moreover, M has no other positive eigenvectors.

Corollary 3.6. *The PF eigenvalue for the substitution matrix of a self-affine tiling with expansion map ϕ is $|\det\phi|$. The vector $\mathbf{e}_l = (\mathrm{Vol}(T_j))_1^m$ is a left PF eigenvector.*

Proof. We only need to check that $\mathbf{e}_l\mathsf{S} = |\det\phi|\mathbf{e}_l$, since $\mathbf{e}_l$ is strictly positive. By Lemma 3.5 and the definition of the substitution matrix,

$$|\det\phi|\mathrm{Vol}(T_j) = \mathrm{Vol}(\phi T_j) = \sum_{i=1}^m \mathsf{S}_{ij}\mathrm{Vol}(T_i),$$

which implies the desired statement. $\square$

3.3. Self-similar tilings of $\mathbb{R}$. Suppose that we have a tiling of $\mathbb{R}$ by connected tiles. Then the tiles are closed intervals; they are distinguished by their labels — "tile types"— and possibly, but not necessarily, their lengths. We label the tile types by elements of $\mathcal{A} = \{1, 2, \ldots, m\}$ and let s_i be their lengths. Such a tiling of $\mathbb{R}$ can be identified with a pair (x, t) where $x \in \mathcal{A}^{\mathbb{Z}}$ and $t \in [0, s_{x(0)})$ (t is the distance from 0 to the left endpoint of the tile which covers 0). Consider the set of tilings arising from the substitution space X'_ζ (the two-sided version). On this set there is a natural $\mathbb{R}$-action by translations. It is easy to see that the resulting tiling dynamical system is a flow under the function $f(x) = s_{x_0}$, built over the substitution dynamical system. Much of the theory of substitutions can be carried over to this setting. This tiling space arises from a self-similar tiling if and only if $(s_i)_{i=1}^m$ is a left PF eigenvector for the substitution matrix M_ζ. The map ϕ is the multiplication by a $\pm\theta$, where θ is the PF eigenvalue of M_ζ.

3.4. Dynamical systems from self-affine tilings of $\mathbb{R}^d$. Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ. We consider the associated tiling space $X_{\mathcal{T}}$ as defined in Section 2, and the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d) = (X_{\mathcal{T}}, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d}$ where the action is by translations. Recall that FLC is always assumed; repetitivity implies that the dynamical system is minimal, see Theorem 2.2.

On the self-affine tiling space $X_{\mathcal{T}}$ we also have the action of the tile-substitution map $\omega: X_{\mathcal{T}} \to X_{\mathcal{T}}$. Indeed, tilings in $X_{\mathcal{T}}$ are characterized by the property that their every patch occurs in $\mathcal{T}$. Since $\omega(\mathcal{T}) = \mathcal{T}$, the property of being an element of $X_{\mathcal{T}}$ is preserved after applying ω. It is easy to see that ω is continuous in the tiling metric.

Digression: in the system $(X_{\mathcal{T}}, \omega)$ we observe some phenomena of hyperbolic dynamics. Recall that there are neighborhoods of $\mathcal{T}$ which look like $X_P \times B_\varepsilon$, where B_ε represents a piece of the orbit—those tilings which are obtained from $\mathcal{T}$ by a small translation, and X_P is the set of tilings with a given patch P around the origin. Then B_ε is a local expanding manifold, and X_P is a local contracting set (which is not a manifold) for ω. To see this, think how the distance between tilings changes if you apply ω in these two cases. For more on this see [3].

Lemma 3.7. *The substitution map $\omega: X_{\mathcal{T}} \to X_{\mathcal{T}}$ is surjective.*

The proof is left as an (easy) exercise.

A natural question arises: is ω injective? If it is, we say that the substitution ω is *invertible*. Other terms used for the same concept are *recognizability* and *unique composition property*. Mossé [68] proved that aperiodic primitive word substitutions are *bilaterally recognizable*, which is equivalent to ω being invertible in the one-dimensional case.

Theorem 3.8. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$. Then the following are equivalent:*
(i) *ω is invertible on $X_{\mathcal{T}}$;*
(ii) *$\mathcal{T}$ is aperiodic, that is, $\Gamma_{\mathcal{T}} = \{0\}$.*

The more difficult direction $(ii) \Rightarrow (i)$ was proved in [94] by a generalization of Mossé's argument; see also [42] for a recent proof of an extension to substitution tilings which are not necessarily translationally finite. The direction $(i) \Rightarrow (ii)$ is easy and well-known (see e.g. [39, 10.1.1]).

Proof of $(i) \Rightarrow (ii)$ in Theorem 3.8. Suppose that $\mathcal{T} - g = \mathcal{T}$ for some $g \in \mathbb{R}^d$. Then we have $\omega^{-1}(\mathcal{T}) - \phi^{-1}g = \omega^{-1}(\mathcal{T} - g) = \omega^{-1}(\mathcal{T})$. Since $\omega^{-1}(\mathcal{T}) = \mathcal{T}$, we get that $\mathcal{T} - \phi^{-1}g = \mathcal{T}$. It follows that $\phi^{-1}(\Gamma_{\mathcal{T}}) \subset \Gamma_{\mathcal{T}}$, which is only possible if $\Gamma_{\mathcal{T}} = \{0\}$, since ϕ^{-1} is contracting, and a small nonzero translation of a $\mathcal{T}$-tile cannot be a $\mathcal{T}$-tile. □

Theorem 3.9. *Let $\mathcal{T}$ be a self-affine tiling. Then the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic.*

This is proved using Theorem 2.6(i) by establishing the existence of UPF (uniform patch frequencies). We refer to [59, 81] for an argument which uses Perron-Frobenius theory (it is a generalization of a similar proof for word substitutions, see [73]). For self-similar tilings we can take advantage of the following.

Lemma 3.10. [94, Lem. 2.3] *A self-similar tiling $\mathcal{T}$ is linearly repetitive: there exists $C > 0$ such that for every $\mathcal{T}$-patch P, any ball of radius $C\,\mathrm{diam}(P)$ contains a translate of P.*

Together with Theorem 2.8 and Theorem 2.6(i) this implies unique ergodicity in the self-similar case.

Proof of the lemma. For a $\mathcal{T}$-tile T, consider the patch consisting of tiles which have a common boundary point with T (including T itself). This is usually called the *first corona* of T. By FLC, there are finitely many first coronas up to translation. By repetivity (which is assumed for all self-affine tilings), there exists $C_1 > 0$ such that every ball of radius C_1 contains translates of all first coronas. Again using FLC, we observe that there exists $c_2 > 0$ such that two $\mathcal{T}$-tiles must have a common boundary point whenever the distance between them is less than c_2. This implies that a set $F \subset \mathbb{R}^d$ of diameter less than c_2 is covered by the first corona of any tile it meets. Since the expansion map ϕ is a similarity, for any $k \in \mathbb{N}$, the same properties hold for the tiling $\phi^k\mathcal{T} := \{\phi^k T : \ T \in \mathcal{T}\}$ with the constants C_1 and c_2 replaced by $\lambda^k C_1$ and $\lambda^k c_2$, where $\lambda > 1$ is the norm (expansion rate) of ϕ.

Now let P be a $\mathcal{T}$-patch. Find $k \in \mathbb{N}$ so that $\lambda^{k-1}c_2 \le \mathrm{diam}(P) < \lambda^k c_2$. Then P is covered by some first $\phi^k\mathcal{T}$-corona. Every ball of radius $\lambda^k C_1$ contains a translate of this corona, and its k-times decomposition contains a translate of P. Since

$$\frac{\lambda^k C_1}{\mathrm{diam}(P)} \le \frac{\lambda^k C_1}{\lambda^{k-1}c_2} = \frac{\lambda C_1}{c_2} =: C$$

does not depend on k, we are done. $\square$

4. Characterization of expansions

Here we discuss the following question: which ϕ may be expansion maps of self-affine tilings? This was investigated by Thurston [98]. So far, we know from Corollary 3.6 that $|\det\phi|$ is a *Perron number*, that is, an algebraic integer $\lambda > 1$ whose Galois conjugates (i.e., other zeros of the minimal polynomial) are strictly less than λ in absolute value.

Theorem 4.1. *There is a self-similar tiling of the line $\mathbb{R}$ with expansion λ if and only if $|\lambda|$ is a Perron number.*

Proof. We already know the necessity. Sufficiency follows from Lind's Theorem [61] which asserts that every Perron number is the PF eigenvalue of a primitive matrix. In fact, we need a little bit more.

Theorem 4.2 (D. Lind [61])**.** *If $\lambda > 1$ is a Perron number, then there is a primitive non-negative integral matrix M with the PF eigenvalue equal to λ. Moreover, M can be chosen so that it has a column with a positive diagonal entry and the sum of all entries ≥ 3.*

This follows by a minor modification of the argument in [61], sketched in Section 8.2. Now we deduce Theorem 4.1.

First suppose that the expansion constant λ is positive. Then λ is a Perron number and we fix a matrix M from Theorem 4.2. Let m be the size of M; consider the alphabet $\mathcal{A} = \{1, \ldots, m\}$. Define a word substitution with the substitution matrix M. The words $\zeta(j)$ for $j \ge 1$ are chosen so that they contain exactly M_{ij} letters i. Without loss of generality, we can assume that the first column of M has the property stated in the theorem. Then we can take $\zeta(1) = V1W$ where V and W are nonempty words. The order of letters in the words $\zeta(j)$ for $j \geq 2$ is irrelevant. Now we consider a bi-infinite word defined by

$$\ldots\zeta^n(V)\ldots\zeta^2(V)\zeta(V)V1W\zeta(W)\zeta^2(W)\ldots\zeta^n(W)\ldots$$

Let $(s_j)_{j=1}^m$ be the left PF eigenvector for M. The prototiles will be closed intervals (line segments) of length $s_1, \ldots, s_m$ (if some of the lengths are equal, we distinguish the prototiles by a color). Each bi-infinite word in $\mathcal{A}$ corresponds to a tiling if we string together the intervals in the same order as the symbols. This is not exactly right though, since we also need to specify the location of the origin. We can put the origin in the interior of the "central tile" T_1 of type 1 in such a way that the patch corresponding to $\zeta(1) = V1W$ is obtained by linearly expanding T_1 by λ. Then it is straightforward to verify that the resulting tiling is self-similar with expansion λ.

Now let us show that we can find a self-similar tiling with expansion $-\lambda$ for Perron λ. We start with the same substitution ζ as above, but consider a different bi-infinite word. First define

$$\zeta^*(w_1 \ldots w_\ell) = \zeta(w_\ell)\ldots\zeta(w_1).$$

Let $U^{(1)} = 1$ and $U^{(2)} = \zeta(1) = V1W$ where V and W are nonempty words. Next let $V^{(1)} = V,\ W^{(1)} = W$, and define inductively for $n \geq 3$,

$$U^{(n)} = V^{(n-1)}U^{(n-1)}W^{(n-1)}, \quad \text{where } \ V^{(n-1)} = \zeta^*(W^{(n-2)}),\ \ W^{(n-1)} = \zeta^*(V^{(n-2)}).$$

For such a word we consider the corresponding patch, with the same prototiles as in the first case. We can put the origin in the interior of the central tile T_1 marked by 1 in such a way that the patch corresponding to $\zeta(1) = V1W$ is obtained by linearly expanding T_1 by $-\lambda$. Then the patch corresponding to $U^{(n)}$ contains the patch corresponding to $U^{(n-1)}$ and can be naturally identified with $\omega(U^{(n-1)})$ for a tile-substitution ω. The limit of these patches is the desired self-similar tiling of the line. □

Now we consider the case of self-similar tilings of the plane with orientation preserving expansions, so we can speak about complex expansion constants. A complex number λ, $|\lambda| > 1$, is said to be a *complex Perron number* if all its Galois conjugates, except $\overline{\lambda}$, are strictly less that $|\lambda|$ in modulus.

Theorem 4.3. (Thurston, Kenyon) *There is a self-similar tiling of the plane with expansion constant λ if and only if λ is a complex Perron number.*

This was announced by Thurston in his lecture notes [98] with a proof of necessity. The proof of sufficiency was published by Kenyon [49].

I would like to sketch a proof of necessity, which, as far as I know, was never "officially" published. The ideas are very beautiful, and moreover, the method will be useful for us in the study of eigenvalues. I will only say a few words about sufficiency. There are some analogs in higher dimensions, but they are harder, and I will not discuss them in a systematic way.

The following general lemma is a starting point of the proof. The rest of the proof of necessity (in fact, covering the case $d > 2$ to some extent) is given in Section 8.3.

Lemma 4.4. [46, 98] *Let ϕ be an expanding linear map on $\mathbb{R}^d$, and suppose that there exists a self-affine tiling of $\mathbb{R}^d$ with expansion ϕ. Then all the eigenvalues of ϕ are algebraic integers.*

First we introduce an important notion of *control points* (due to Thurston).

Definition 4.5. (see [71]) Let $\mathcal{T}$ be a fixed point of a primitive substitution with expansive map ϕ. For each $\mathcal{T}$-tile T, fix a tile γT in the patch $\omega(T)$; choose γT with the same relative position for all tiles of the same type. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the *tile map.* Then define the *control point* for a tile $T \in \mathcal{T}$ by

$$\{c(T)\} = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma^n T).$$

The control points are not uniquely defined; they depend on the choice of γ.

Let $\mathcal{C} = \mathcal{C}(\mathcal{T}) = \{c(T) : \ T \in \mathcal{T}\}$ be the set of control points for all tiles. The control points have the following properties:

(a) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type;

(b) $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$.

Therefore,

(c) $\phi(\mathcal{C}) \subset \mathcal{C}$.

Proof of Lemma 4.4. Consider $J = \langle \mathcal{C} \rangle$, the subgroup of $\mathbb{R}^d$ generated by $\mathcal{C} = \mathcal{C}(\mathcal{T})$. We claim that J is finitely generated. Indeed, let

$$\Psi := \{c(T') - c(T) : \ T, T' \in \mathcal{T}, \ T \neq T', \ T \cap T' \neq \emptyset\}. \tag{4.1}$$

The set Ψ is finite by FLC, and J is generated by Ψ and an arbitrary control point (we can get from it to any control point by moving "from neighbor to neighbor").

It is a basic fact in algebra that J, as a finitely generated free Abelian group, has a finite set of free generators. That is, there exist vectors $v_1, \ldots, v_N \in \mathbb{R}^d$ such that every element of J can be uniquely written as an integral linear combination of $v_j's$. These vectors span $\mathbb{R}^d$ since the control points are relatively dense in $\mathbb{R}^d$, hence $N \geq d$. Define the matrix $V = [v_1 \ldots v_N]$. This is a $d \times N$ matrix of rank d. By the definition of free generators, for every $\xi \in J$ there exists a unique $a(\xi) \in \mathbb{Z}^N$ such that $\xi = Va(\xi)$. Now, $\phi(\mathcal{C}) \subset \mathcal{C}$ implies $\phi J \subset J$, hence there exists an integer $N \times N$ matrix M such that $\phi V = VM$. For every eigenvalue λ of ϕ we can find a left eigenvector $\mathbf{e}_\lambda$ of ϕ corresponding to λ. Then $\mathbf{e}_\lambda V$ is a left eigenvector for M corresponding to λ (note that $\mathbf{e}_\lambda V \neq 0$ since V has maximal possible rank d). This proves the lemma since λ is an eigenvalue of M, hence it is an algebraic integer. □

Now we do hope the reader will get curious and skips to Section 8.3!

About the proof of sufficiency [49]. Here we only make a few comments, to give a flavor of the proof. There are some parallels with the proof of Lind's Theorem, and it is helpful to take a look at Section 8.2 first. Let B be the companion matrix of the complex Perron number λ, suppose that B is n-dimensional. Consider the decomposition of $\mathbb{R}^n$ into a direct sum of real eigenspaces. The "dominant eigenspace" D is now 2-dimensional. The main part of the proof consists in finding a tiling $\mathcal{T}$ with non-exact subdivisions. Then the boundaries are redrawn using a recursive process, so that in the limit the subdivisions are exact and we obtain a self-similar tiling $\mathcal{T}'$ which is MLD with $\mathcal{T}$. The method of this last step goes back to Dekking [22]; it was later used in [33].

Let $\pi_{_D}$ be the projection to D along the complementary direct sum. Let $\pi_{_C} = I - \pi_{_D}$ be the projection to the invariant complement C of D. Consider all triangles F with vertices in $\mathbb{Z}^n$ with some uniform bounds on the diameter of F and on the diameter of the projection $\pi_{_C}(F)$. Then consider the projection $\pi_{_D}(F)$. There are finitely many such triangles up to translation; these are going to be the tile types of $\mathcal{T}$. First one constructs a non-exact subdivision with expansion λ^ℓ for a large ℓ. Consider $B^\ell F$ for $\ell \in \mathbb{N}$. If ℓ is sufficiently large, then $B^\ell F$ is almost parallel to D by the Perron property. Note that $\pi_{_D}(B^\ell F) = \lambda^\ell \pi_{_D}(F)$. Consider all points in $\mathbb{Z}^n$ within some fixed distance from $B^\ell F$, project them to D, consider a triangulation (the Delone triangulation, see [49], is a natural choice), and choose which triangles "belong" to $\lambda^\ell \pi_{_D}(F)$. It should be done consistently, so that the subdivisions agree on the common boundary. For this, it is necessary to increase the number of tile types, to keep track of a triangle and its immediate neighbors. Additional work is needed (a) to make sure that the subdivision is primitive (for this purpose an additional "special" tile T_0 is introduced which occurs in the subdivision of every tile); (b) to pass from a tiling with expansion λ^ℓ to one with expansion λ. □

Remark. It should be mentioned that in many special cases there exist simpler and more direct constructions of self-similar tilings, especially for Pisot expansion constants, see e.g. [78, 98, 71, 1, 2, 14, 15]. Some of them use β-expansions and other numeration systems. Another method uses free group endomorphisms and covers some non-Pisot expansions as well, see [44, 49] and especially [34].

5. Eigenvalues

Definition 5.1. Let $(X, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d}$ be a continuous $\mathbb{R}^d$-action on a compact metric space. A vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is said to be an eigenvalue for the continuous $\mathbb{R}^d$-action if there exists an eigenfunction $f_\alpha \in C(X)$, that is, $f_\alpha \not\equiv 0$ and for all $\mathbf{t} \in \mathbb{R}^d$ and all $x \in X$,

$$f_\alpha(T^{\mathbf{t}}x) = e^{2\pi i\langle \mathbf{t},\alpha\rangle} f_\alpha(x). \tag{5.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$.

Note that this "eigenvalue" is actually a vector. In physics it might be called a "wave vector." More generally, for an action of a locally compact Abelian group G, the eigenvalues are elements of the dual group $\widehat{G}$.

The set of eigenvalues of a dynamical system is an important conjugacy invariant. They always form a group: if f_α and f_β are the eigenfunctions corresponding to α and β respectively, then the product $f_\alpha \cdot f_\beta$ satisfies the eigenfunction equation for $\alpha + \beta$. Note that $\alpha = 0$ is always a trivial eigenvalue corresponding to the constant eigenfunction. A system without non-trivial eigenvalues is called *topologically weak-mixing.*

Definition 5.2. Let $(X, T^{\mathbf{t}}, \mu)_{\mathbf{t}\in\mathbb{R}^d}$ be a measure-preserving $\mathbb{R}^d$-action. A vector $\alpha \in \mathbb{R}^d$ is called an eigenvalue for this action if there exists an eigenfunction $f_\alpha \in L^2(X, \mu)$, that is, f_α is not the zero function in L^2 and for all $\mathbf{t} \in \mathbb{R}^d$, the equation (5.1) holds for μ-a.e. $x \in X$.

If the measure-preserving system is ergodic, then there are no non-constant invariant functions, which implies that the eigenfunctions have constant modulus a.e., so they belong to L^∞. Therefore, we can multiply the eigenfunctions (as in the topological setting), and the set of eigenvalues is a subgroup of $\mathbb{R}^d$. If there are no non-constant eigenfunctions, the system is called *(measurably) weak-mixing.* The set of eigenvalues (with δ-masses on them) is the discrete part of the spectrum.

To distinguish between the measure-theoretic and topological settings, we can speak about measurable and continuous eigenfunctions. Of course, every continuous eigenfunction is measurable, but the converse is not true for dynamical systems in general.

However, for tiling dynamical systems we have the following:

Theorem 5.3 ([97])**.** *If $\mathcal{T}$ is a self-affine tiling, then every measurable eigenfunction for the system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ coincides with a continuous function μ-a.e.*

This, in a sense, extends the result of Host [43] on $\mathbb{Z}$-actions associated to primitive one-dimensional symbolic substitutions. As a consequence of Theorem 5.3, for our systems measure-theoretic weak-mixing is equivalent to topological weak-mixing.

Remark. Continuous and measurable eigenfunctions for linearly recurrent Cantor systems (which include LR subshifts) were recently investigated in [19, 17]. In the latter paper necessary and sufficient conditions for being an eigenvalue are established, and it is proved that not every measurable eigenfunction is continuous for such systems.

The *proof of Theorem 5.3* proceeds via a characterization of eigenvalues. Recall that $\Gamma_{\mathcal{T}}$ denotes the group of translation symmetries (i.e. periods) for the tiling $\mathcal{T}$. We also need the set of translation vectors between tiles of the same type:

$$\Xi = \Xi(\mathcal{T}) := \{x \in \mathbb{R}^d : \ \exists T, T' \in \mathcal{T}, \ T' = T + x\}.$$

These vectors are sometimes called "return vectors," they are analogous to return words in symbolic dynamics.

Theorem 5.4 ([97])**.** *Let $\mathcal{T}$ be a self-affine tiling with expansion map ϕ. Then the following are equivalent for $\alpha \in \mathbb{R}^d$:*

(i) *α is an eigenvalue for the topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$;*

(ii) *α is an eigenvalue for the measure-preserving system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$;*

(iii) *α satisfies the following two conditions:*

$$\lim_{n\to\infty} e^{2\pi i \langle \phi^n z, \alpha\rangle} = 1 \quad \text{for all } z \in \Xi(\mathcal{T}), \tag{5.2}$$

and

$$e^{2\pi i \langle g, \alpha\rangle} = 1 \quad \text{for all } g \in \Gamma_{\mathcal{T}}. \tag{5.3}$$

This theorem is also a generalization of the corresponding result from [43]. Theorem 5.3 is immediate from Theorem 5.4, since for an ergodic system, all eigenvalues are simple.

I will only sketch a proof of the equivalence (i) $\Leftrightarrow$ (iii). The direction (i) $\Rightarrow$ (iii) is very easy, but the converse is not quite so.

Proof of (i) $\Rightarrow$ (iii) in Theorem 5.4. Consider the continuous eigenfunction f_α of modulus 1, corresponding to the eigenvalue $\alpha \in \mathbb{R}^d$. Thus,

$$f_\alpha(\mathcal{T} - \mathbf{t}) = e^{2\pi i \langle \mathbf{t}, \alpha \rangle} f_\alpha(\mathcal{T}) \quad \text{for all } \mathbf{t} \in \mathbb{R}^d.$$

Taking $\mathbf{t} \in \Gamma_{\mathcal{T}}$ we obtain (5.3). To verify (5.2), let $z \in \Xi(\mathcal{T})$, so that for some tile $T \in \mathcal{T}$ we have $T + z \in \mathcal{T}$. Let ξ be any point in the interior of T. Then $\mathcal{T} - \xi$ and $\mathcal{T} - z - \xi$ agree on some neighborhood of the origin B_ε. Applying ω^n we obtain that $\omega^n(\mathcal{T} - \xi) = \mathcal{T} - \phi^n \xi$ and $\omega^n(\mathcal{T} - z - \xi) = \mathcal{T} - \phi^n z - \phi^n \xi$ agree on $\phi^n B_\varepsilon$. By the definition of the tiling metric,

$$\varrho(\mathcal{T} - \phi^n \xi, \mathcal{T} - \phi^n z - \phi^n \xi) \to 0, \quad \text{as } n \to \infty.$$

A continuous function on a compact metric space is uniformly continuous, hence

$$|f_\alpha(\mathcal{T} - \phi^n \xi) - f_\alpha(\mathcal{T} - \phi^n z - \phi^n \xi)| \to 0, \quad \text{as } n \to \infty.$$

Using the eigenfunction equation and the fact that f is unimodular, we obtain that $\left|1 - e^{2\pi i \langle \phi^n z, \alpha \rangle}\right| \to 0$, as $n \to \infty$, proving (5.2). $\square$

Sketch of the proof of (i) $\Rightarrow$ (iii) in Theorem 5.4. Let us assume that $\mathcal{T}$ is aperiodic for simplicity. Suppose that (5.2) holds; we need to show that there exists a continuous eigenfunction corresponding to $\alpha \in \mathbb{R}^d$. Define

$$f_\alpha(\mathcal{T} - \mathbf{t}) = e^{2\pi i \langle \mathbf{t}, \alpha \rangle} \quad \text{for } \mathbf{t} \in \mathbb{R}^d.$$

The orbit $\{\mathcal{T} - \mathbf{t} : \ \mathbf{t} \in \mathbb{R}^d\}$ is dense in $X_{\mathcal{T}}$ by the definition of the tiling space. If we show that f_α is uniformly continuous on this orbit, then we can extend f_α to the entire space, and this extension will satisfy the eigenvalue equation (5.1) by continuity.

Suppose that $\mathcal{T} - x$ is close to $\mathcal{T} - y$. Making a small translation we can assume without loss of generality that $\mathcal{T} - x$ and $\mathcal{T} - y$ agree on a large neighborhood around the origin, which contains a tile of order n for a large n. We will use Theorem 3.8 which says that ω is invertible, since $\mathcal{T}$ is aperiodic. Applying ω^{-n} we obtain that $\mathcal{T} - \phi^{-n}x$ and $\mathcal{T} - \phi^{-n}y$ share a tile covering the origin, hence

$$\phi^{-n}x - \phi^{-n}y =: v \in \Xi(\mathcal{T}).$$

Thus, $x - y = \phi^n v$, and

$$|f_\alpha(\mathcal{T} - x) - f_\alpha(\mathcal{T} - y)| = |e^{2\pi i \langle \phi^n v, \alpha \rangle} - 1|,$$

which should be small when n is large by (5.2). This is cheating, of course! In order to make this precise, we need two facts: that the convergence in (5.2) is exponential, and that it is uniform in $z \in \Xi(\mathcal{T})$. The exponential convergence (that is, if $|e^{2\pi i \langle \phi^n z, \alpha\rangle} - 1| \to 0$, then $|e^{2\pi i \langle \phi^n z, \alpha\rangle} - 1| \le C\rho^n$ for some $\rho \in (0,1)$) follows from Lemma 4.4. The proof of uniform convergence uses a kind of "numeration system" for translation vectors between control points $c(T) - c(S)$ in powers of ϕ applied to a finite set of "digit vectors" in $\Xi(\mathcal{T})$. For details we refer to [60]. $\square$

The characterization of eigenvalues given in Theorem 5.4 does not address the question whether non-trivial eigenvalues actually exist. This turns out to be closely related to number theory, more precisely, to *Pisot numbers* (also called PV-numbers) and their generalizations. Recall that an algebraic integer $\theta > 1$ is a Pisot number if all its Galois conjugates θ' satisfy $|\theta'| < 1$.

A complex algebraic integer λ, with $|\lambda| > 1$, is a *complex Pisot number* if all its Galois conjugates λ', except the complex conjugate $\overline{\lambda}$, satisfy $|\lambda'| < 1$. Extending this further, we say that a family of algebraic integers $\{\lambda_1, \dots, \lambda_r\}$ of modulus greater than 1 is a *Pisot family* if for every Galois conjugate λ' of every λ_i, $i \le r$, we have either $|\lambda'| < 1$ or $\lambda' = \lambda_j$ for some $j \le r$. Thus, λ is a complex Pisot number whenever $\{\lambda, \overline{\lambda}\}$ is a Pisot family.

We get a "clean" answer for self-similar tilings of $\mathbb{R}^d$ whose expansion map is a pure dilation (without rotation), and for self-similar tilings of $\mathbb{R}^2$ with complex expansion constants.

Theorem 5.5. (i) *Let $\mathcal{T}$ be a self-similar tilings of $\mathbb{R}^d$ with expansion $\phi(x) = \theta x$ for $\theta > 1$. Then the associated tiling dynamical system has a non-trivial eigenvalue (is not weak-mixing) if and only if θ is a Pisot number.*

(ii) *Let $\mathcal{T}$ be a self-similar tilings of $\mathbb{R}^2 = \mathbb{C}$ with a complex expansion constant λ. Then the associated tiling dynamical system has a non-trivial eigenvalue (is not weak-mixing) if and only if λ is a complex Pisot number.*

Part (i) was proved in [97] and we sketch the proof below. A similar result was obtained by Gähler and Klitzing [35], where the diffraction spectrum was considered. In connection with eigenvalues of symbolic substitution systems, Pisot numbers and Pisot families appeared in [43, 26], and the link with quasicrystals was observed in [16].

Proposition 5.6. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ, such that the associated tiling dynamical system has a set of eigenvalues which spans $\mathbb{R}^d$. Then the set of eigenvalues of ϕ is a Pisot family.*

This follows from [60], see also [93] and [81]. We do not know if the converse is true.

Sketch of the proof of Theorem 5.5(i). Necessity. Let $\alpha \neq 0$ be an eigenvalue. The set of translation vectors Ξ between tiles of the same type spans $\mathbb{R}^d$, hence we can find $z \in \Xi$ such that $\langle z, \alpha \rangle \neq 0$. By Theorem 5.4, the distance from $\theta^n \langle z, \alpha \rangle$ to the nearest integer tends to zero, as $n \to \infty$. We know that θ is algebraic (see Lemma 4.4), hence θ is a Pisot number by the classical result of Pisot (see e.g. [87]). $\square$

Sufficiency. We need to show that if θ is Pisot, then there are non-zero eigenvalues for the dynamical system. The proof relies on the following result (incidentally, [35] uses it as well).

Theorem 5.7 (Kenyon)**.** *Under the assumptions of Theorem 5.5(i), there exists a basis* $\{b_1, \ldots, b_d\}$ *of* $\mathbb{R}^d$ *such that*

$$\Xi \subset b_1\mathbb{Z}[\theta] + \cdots + b_d\mathbb{Z}[\theta]. \tag{5.4}$$

We sketch the proof of Theorem 5.7 in Section 8.4 (it is not so easy to extract from the literature). Now we finish the sketch of the proof of sufficiency in Theorem 5.5(i) after [97], assuming $\mathcal{T}$ is aperiodic for simplicity.

Let $\{b_1^*, \ldots, b_d^*\}$ be the dual basis for $\{b_1, \ldots, b_d\}$, that is, $\langle b_i, b_j^* \rangle = \delta_{ij}$. We claim that the set

$$b_1^*\mathbb{Z}[\theta^{-1}] + \cdots + b_d^*\mathbb{Z}[\theta^{-1}]$$

is contained in the group of eigenvalues. Indeed, suppose $\alpha = \sum_{j=1}^d b_j^* p_j(\theta^{-1})$ for some polynomials $p_j \in \mathbb{Z}[x]$. Let $z \in \Xi$. By (5.4), we can write $z = \sum_{j=1}^d b_j q_j(\theta)$ for some polynomials $q_j \in \mathbb{Z}[x]$. Then

$$\langle \phi^n z, \alpha \rangle = \theta^n \sum_{j=1}^d q_j(\theta) p_j(\theta^{-1}) = \theta^{n-k} P(\theta)$$

for some $k \in \mathbb{N}$ and $P \in \mathbb{Z}[x]$. We have $\mathrm{dist}(\theta^{n-k}P(\theta), \mathbb{Z}) \to 0$, as $n \to \infty$ (see [87]), so (5.2) is satisfied and α is an eigenvalue by Theorem 5.4. $\square$

6. Pure discrete spectrum.

A measure-preserving system (X, T, μ) is said to have *pure discrete* (or *pure point*) spectrum if the eigenfunctions span a dense subspace of $L^2(X, \mu)$. By the Halmos-von Neumann Theorem (see [100]) any dynamical system with pure discrete spectrum is metrically isomorphic to a *Kronecker system*, that is, a translation action on a compact Abelian group. There has been a lot of interest in substitution and tiling systems with pure discrete spectrum, in particular, because of the connections with quasicrystals. We only discuss this topic briefly.

Let $\mathcal{T}$ be a self-affine tiling. For $x \in \Xi(\mathcal{T})$ consider the infinite set of tiles

$$D_x := \mathcal{T} \cap (x + \mathcal{T}).$$

It is non-empty and relatively dense by repetitivity. Observe that D_x has a well-defined positive density given by

$$\mathrm{dens}(D_x) = \lim_{r\to\infty} \frac{\mathrm{Vol}(D_x \cap B_r)}{\mathrm{Vol}(B_r)} = \sum_{i=1}^{m} \mathrm{freq}(T_i \cup (x + T_i), \mathcal{T}) \cdot \mathrm{Vol}(T_i)\,,$$

where T_i's are the prototiles.

We also need the notion of *Meyer set.* A point set $\Lambda \subset \mathbb{R}^d$ is a Meyer set if Λ is relatively dense and $\Lambda - \Lambda$ is uniformly discrete (equivalently, Λ and $\Lambda - \Lambda$ are Delone sets). Meyer sets play an important role in the theory of aperiodic order; we will say more about them in the next section.

Theorem 6.1 ([93, 59, 60])**.** *Let $\mathcal{T}$ be a self-affine tiling with expansion ϕ. Then $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has pure discrete spectrum iff $\Xi(\mathcal{T})$ is a Meyer set and*

$$\lim_{n\to\infty} \mathrm{dens}(D_{\phi^n x}) = 1,\ \forall\, x \in \Xi(\mathcal{T}). \tag{6.1}$$

Heuristically, it is useful to think about the property (6.1) as *almost periodicity* of the tiling. Recall that a function is (Bohr) almost-periodic if the set of its ε-almost periods is relatively dense for every $\varepsilon > 0$. We can say that $x \in \mathbb{R}^d$ is an ε-almost period for $\mathcal{T}$ if $\mathrm{dens}(D_x) > 1 - \varepsilon$ and call a tiling almost periodic if the set of ε-almost periods is relatively dense in $\mathbb{R}^d$ for every $\varepsilon > 0$. In fact, this is an approach to prove sufficiency in Theorem 6.1, see [95] for details ([93] used a different method). The necessity of the Meyer property is recent: it was proved in [60] and answers a question of J. Lagarias [52].

There is an algorithm to check pure discrete spectrum, called the *overlap algorithm.* It was first introduced in [93] and extended to this setting in [59]; see also [92] for the case of tilings on the line.

Definition 6.2. Let $\mathcal{T}$ be a tiling. A triple (T, y, S), with $T, S \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is an *overlap* if the intersection $(y + T) \cap S$ has non-empty interior. We say that two overlaps (T, y, S) and (T', y', S') are *equivalent* if for some $g \in \mathbb{R}^d$ we have $y + T = g + y' + T',\ S = g + S'$. Denote by $[(T, y, S)]$ the equivalence class of an overlap. An overlap (T, y, S) is a *coincidence* if $y + T = S$. The *support* of an overlap (T, y, S) is $\mathrm{supp}(T, y, S) = (y + T) \cap S$.

Lemma 6.3. *Let $\mathcal{T}$ be a tiling such that $\Xi(\mathcal{T})$ is a Meyer set. Then the number of equivalence classes of overlaps for $\mathcal{T}$ is finite.*

Next we consider the *subdivision graph for overlaps.* Its vertices are equivalence classes of overlaps. It has directed edges as follows: let $\mathcal{O} = (T, y, S)$. We have

$$\mathrm{supp}(\phi y + \omega(T)) \cap \mathrm{supp}(\omega(S)) = Q(\mathrm{supp}(T, y, S)).$$

For each pair of tiles $T' \in \omega(T)$ and $S' \in \omega(S)$ such that $\mathcal{O}' := (T', \phi y, S')$ is an overlap, we draw a directed edge from $[\mathcal{O}]$ to $[\mathcal{O}']$. Denote this graph by $\mathcal{G}_{\mathcal{O}}(\mathcal{T})$.

Lemma 6.4. *Let $\mathcal{T}$ be a self-affine tiling with expansion map ϕ such that $\Xi(\mathcal{T})$ is a Meyer set. Let $x \in \Xi(\mathcal{T})$. The following are equivalent:*

(i) $\lim_{n\to\infty} \mathrm{dens}(D_{\phi^n x}) = 1$*;*

(ii) $1 - \mathrm{dens}(D_{\phi^n x}) \le Cr^n$, $n \ge 1$, *for some* $C > 0$ *and* $r \in (0, 1)$*;*

(iii) *From each vertex of the graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T})$ there is a path leading to a coincidence.*

See [59] for the proof. In the planar case this algorithm was used in [93] to analyze some examples. For instance, the chair tiling (dynamical system) has a pure discrete spectrum, but the domino tiling has a mixed spectrum.

Remarks. (a) The use of coincidences to characterize systems with pure discrete spectrum goes back to Dekking [21] (for constant length symbolic substitutions) and to Livshits [63] (for general symbolic substitutions). For recent work on the *Pure Discrete Spectrum Conjecture* see [11, 14] and references therein. The conjecture is that if $\mathcal{T}$ is a self-similar tiling of the line $\mathbb{R}$ whose expansion constant is a unimodular Pisot number, then $(X_{\mathcal{T}}, \mathbb{R}, \mu)$ has pure discrete spectrum. It is open for the case of ≥ 3 symbols.

(b) Tiling systems with mixed spectrum are still poorly understood. For instance, it is an open question whether the domino (="table") dynamical system has pure singular spectrum (see [80] for more on this system). On the other hand, N. P. Frank [30] found a large class of tiling systems with Lebesgue spectrum of even multiplicity, generalizing Rudin-Shapiro substitution systems; see also [31] for an overview of the spectral properties of multidimensional constant length substitution sequences. The only general fact known is that a self-affine tiling system must have a singular spectral component, since it is not mixing [93].

7. Aperiodic order.

The discovery of quasicrystals in the 1980's inspired a lot of research in the area of "aperiodic order" and "mathematical quasicrystals." Roughly speaking, physical quasicrystals are solids (metallic alloys) which exhibit sharp bright spots (called Bragg peaks) in their X-ray diffraction pattern, but have aperiodic structure (usually manifested by the presence of a non-quasicrystallographic symmetry). The presence of Bragg peaks indicates the presence of "long-range order" in the structure, hence the term "aperiodic order."

A mathematical idealization of a large set of atoms is a Delone set Λ in $\mathbb{R}^d$, see Definition 2.3. Recall that we defined the Delone set dynamical system $(X_\Lambda, \mathbb{R}^d)$ in Section 2. It is uniquely ergodic if and only if Λ has UCF. Let μ be an ergodic invariant probability measure. We can then study the *spectrum* of the measure-preserving system $(X_\Lambda, \mathbb{R}^d, \mu)$. More precisely, consider the group of unitary operators $\{U_x\}_{x\in\mathbb{R}^d}$ on $L^2(X_\Lambda, \mu)$:

$$U_x f(\Lambda') = f(-x + \Lambda').$$

Every $f \in L^2(X_\Lambda, \mu)$ defines a function on $\mathbb{R}^d$ by $x \mapsto (U_x f, f)$. This function is positive definite on $\mathbb{R}^d$, so its Fourier transform is a positive measure σ_f on $\mathbb{R}^d$ called the *spectral measure* corresponding to f. If f is an eigenfunction (of norm 1) for an eigenvalue $\alpha \in \mathbb{R}^d$, then $\sigma_f = \delta_\alpha$, the Dirac's measure at α. It is a consequence of Spectral Theory for unitary operators that σ_f is pure discrete for all $f \in L^2(X_\Lambda, \mu)$ if and only if the eigenfunctions for the $\mathbb{R}^d$-action span a dense subspace of $L^2(X_\Lambda, \mu)$. Then we say that the measure-preserving system has pure discrete (or pure point) spectrum. In analogy to ordinary crystallography, quasicrystals are often considered in terms of tilings, but as we discussed in Section 2, it is easy to pass from the tiling setting to the Delone setting and vice versa.

7.1. Diffraction. A sharp diffraction picture has been the hallmark of aperiodic order. How do we describe it mathematically? In our idealized world, when atoms are replaced by points, we consider the so-called *Dirac comb* of Λ,

$$\delta_\Lambda := \sum_{x\in\Lambda} \delta_x.$$

For $r > 0$ we calculate the *autocorrelation* of δ_Λ restricted to the ball $B_r = B_r(0)$:

$$\delta_{\Lambda\cap B_r} * \widetilde{\delta}_{\Lambda\cap B_r} = \sum_{x,y\in\Lambda\cap B_r} \delta_{x-y}.$$

Here the over-tilde indicates changing the sign of the argument. The absolute value of the Fourier transform of the last expression represents the scattering intensity of the diffraction pattern of the *finite* set of scatterers in the ball B_r. In order to consider the diffraction of the entire Λ, we need to let $r \to \infty$, after normalizing by the volume:

$$\gamma = \lim_{r\to\infty} \frac{1}{\mathrm{Vol}(B_r)} \sum_{x,y\in\Lambda\cap B_r} \delta_{x-y}.$$

We will assume that the limit distribution exists in the *vague topology*, i.e., this limit exists when taken against rapidly decreasing test functions. One can show that if Λ has UCF, then

$$\gamma = \sum_{z \in \Lambda - \Lambda} \nu(z)\delta_z,$$

where $\nu(z)$ is the frequency of the cluster $\{x, x+z\}$ in Λ. The distribution γ is called the *auto-correlation measure* of Λ. It is positive-definite, hence its Fourier transform $\widehat{\gamma}$ is a positive measure by Bochner's Theorem. **The measure $\widehat{\gamma}$ gives the diffraction pattern of Λ.** It can be decomposed into the discrete (or pure point) part, called the *Bragg spectrum* and continuous part. See [41] for more details about mathematics of diffraction.

7.2. Connection of the diffraction spectrum to the dynamical spectrum. To relate the autocorrelation of δ_Λ to spectral measures we need to do some "smoothing." Let $\omega \in \mathcal{C}_0(\mathbb{R}^d)$, that is, ω is continuous and has compact support. Denote

$$\rho_{\omega,\Lambda'} := \omega * \nu_{_{\Lambda'}}$$

and let

$$f_\omega(\Lambda') := \rho_{\omega,\Lambda'}(0) \quad \text{for} \quad \Lambda' \in X_\Lambda.$$

Lemma 7.1. $f_\omega \in \mathcal{C}(X_\Lambda)$.

Proof. We have

$$f_\omega(\Lambda') = \int \omega(-x)\, d\nu_{_{\Lambda'}}(x) = \sum_{x \in -\mathrm{supp}(\omega) \cap \Lambda'} \omega(-x).$$

The continuity of f_ω follows from the continuity of ω and the definition of topology on X_Λ. □

Denote by $\gamma_{\omega,\Lambda}$ the autocorrelation of $\rho_{\omega,\Lambda}$. Assuming that there is a unique autocorrelation measure γ, we have

$$\gamma_{\omega,\Lambda} = (\omega * \widetilde{\omega}) * \gamma.$$

Lemma 7.2. ([25], see also [41])

$$\sigma_{f_\omega} = \widehat{\gamma_{\omega,\Lambda}}.$$

Proof. By definition, $f_\omega(-x+\Lambda) = \rho_{\omega,\Lambda}(x)$. Therefore,

$$\begin{aligned}
\gamma_{\omega,\Lambda}(x) &= \lim_{r\to\infty} \frac{1}{\mathrm{Vol}(B_r)} \int_{B_r} \rho_{\omega,\Lambda}(x+y)\overline{\rho_{\omega,\Lambda}(y)}\,dy \\
&= \lim_{r\to\infty} \frac{1}{\mathrm{Vol}(B_r)} \int_{B_r} f_\omega(-x-y+\Lambda)\overline{f_\omega(-y+\Lambda)}\,dy \\
&= \int_{X_\Lambda} f_\omega(-x+\Lambda')\overline{f_\omega(\Lambda')}\,d\mu(\Lambda') \\
&= (U_x f_\omega, f_\omega)\,. \qquad (7.1)
\end{aligned}$$

Here the third equality is the main step; it follows from unique ergodicity and the continuity of f_ω. Thus,

$$\widehat{\gamma_{\omega,\Lambda}} = (\widehat{U_{(\cdot)} f_\omega, f_\omega}) = \sigma_{f_\omega},$$

and the proof is finished. □

The introduction of the function f_ω and the series of equations (7.1) is often called Dworkin's argument.

Lemma 7.2 implies, essentially, that the diffraction spectrum is always a "part" of the dynamical spectrum. In particular, (a) if the dynamical spectrum is pure discrete, then the diffraction spectrum is pure discrete and (b) every Bragg peak must be an eigenvalue. The latter implies that if the dynamical system has no nontrivial eigenvalues, then there are no Bragg peaks except at the origin. The following was proved in [58] and later generalized in [5] and [38].

Theorem 7.3 ([58]). *Suppose that the Delone set* Λ *has FLC and UCF. Then the following are equivalent:*

(i) Λ *has pure discrete dynamical spectrum;*

(ii) δ_Λ *has pure point diffraction spectrum.*

About the proof. (i) ⇒ (ii) This is essentially proved by Dworkin in [25], see also [41] and [7]. By Lemma 7.2, pure point dynamical spectrum implies that $\widehat{\gamma_{\omega,\Lambda}}$ is pure point for any $\omega \in \mathcal{C}_0(\mathbb{R}^d)$. Note that

$$\widehat{\gamma_{\omega,\Lambda}} = |\widehat{\omega}|^2 \widehat{\gamma}.$$

Choosing a sequence $\omega_n \in \mathcal{C}_0(\mathbb{R}^d)$ converging to the delta measure δ_0 in the vague topology, we can conclude that $\widehat{\gamma}$ is pure point as well, as desired. This approximation step requires some care; it is explained in detail in [7].

(ii) $\Rightarrow$ (i) This is proved using the group property of the point spectrum, i.e., the product of eigenfunctions is an eigenfunction. It is largely a generalization of an argument in [73, Prop. IV.21], see [58] for details. $\square$

7.3. Model sets. There is a very general method to construct Delone sets with pure point diffraction. In its original form it was based on projection from lattices in higher dimensional spaces and called the "cut and project" method. Y. Meyer [65] had already considered sets formed by projection under the name "model sets" from the point of view of harmonic analysis long before the discovery of quasicrystals. R. V. Moody [66] (see also [67]) developed the theory much further. Here we only give a brief overview of this subject.

Definition 7.4. A *cut and project scheme* (CPS) consists of a collection of spaces and mappings as follows;

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times G & \xrightarrow{\pi_2} & G \\ & & \bigcup & & \\ & & \mathcal{L} & & \end{array} \tag{7.2}$$

where $\mathbb{R}^d$ is a real Euclidean space, G is some locally compact Abelian group, $\mathcal{L} \subset \mathbb{R}^d \times G$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^d \times G)/\mathcal{L}$ is compact, $\pi_1|_{\mathcal{L}}$ is injective, and $\pi_2(\mathcal{L})$ is dense in G.

Definition 7.5. A *model set* in $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ which, up to translation, is of the form $\Gamma(W) = \{\pi_1(x) : x \in \mathcal{L}, \pi_2(x) \in W\}$ for some cut and project scheme as above, where the *window* $W \subset G$ is compact with $W = \overline{W^\circ}$ (the closure of its interior). The model set $\Gamma(W)$ is *regular* if the boundary ∂W is of (Haar) measure 0. The model set $\Gamma(W)$ is *generic* if $\partial W \cap \pi_2(\widetilde{L}) = \emptyset$.

The LCA group G is called the *internal space*, and $\mathbb{R}^d$ is the *physical space* of the CPS. It is actually quite reasonable to replace $\mathbb{R}^d$ with a general LCAG as well, and this has been done in [88]. Here are some basic facts about model sets:

- Every model set Λ is a Meyer set.
- A generic model set is repetitive.
- A regular model set has UCF.

It follows that for each regular and generic model set Λ, the associated dynamical system $(X_\Lambda, \mathbb{R}^d)$ is minimal and uniquely ergodic.

To each CPS one can associate a Kronecker $\mathbb{R}^d$-action as follows: Let $Y := (\mathbb{R}^d \times G)/\mathcal{L}$; this is a compact group by definition. Define $\phi = \pi \circ i_1 : \mathbb{R}^d \to Y$, where $i_1 : \mathbb{R}^d \to \mathbb{R}^d \times G$ is the coordinate injection and π is the canonical projection. Let R be the $\mathbb{R}^d$-action on Y defined by $R^g y = y + \phi(g)$. The dynamical system $(Y, \mathbb{R}^d)$ is a classical object in dynamics; it is uniquely ergodic and has pure discrete spectrum.

Theorem 7.6 ([88])**.** (M. Schlottmann) *Let Λ be a regular and generic model set. Then there is a continuous factor map from $(X_\Lambda, \mathbb{R}^d)$ to $(Y, \mathbb{R}^d)$ which is almost everywhere 1:1. The corresponding measure-preserving systems are metrically isomorphic. Thus, $(X_\Lambda, \mathbb{R}^d, \mu)$ has pure discrete spectrum, and Λ has a pure point diffraction spectrum.*

Note that the Kronecker $\mathbb{R}^d$-action depends on the CPS but not on the window; it is the factor map (called *torus parametrization*) that depends on the window. See [82] for more on diffraction spectra of model sets from the point of view of Ergodic Theory.

7.4. Consequences of pure discrete spectrum and conditions for being a model set. A natural question is whether there is some kind of a converse to the last statement of Theorem 7.6, namely, given a Delone set dynamical system, which is minimal and uniquely ergodic, can it be obtained from a model set? At least, is this true for self-affine tilings? The classical examples, such as the Penrose tiling, were long known to be obtainable from a regular CPS. The examples studied in [93], namely the chair and the sphinx tiling (although for the latter no details were provided in [93]), turned out to be MLD to model (multi)sets as well, as shown by J.-Y. Lee and R. V. Moody [57]. It is interesting that the "internal space" G is non-Euclidean in these examples; it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the group of 2-adic integers. Building on [57] and [93] the following was obtained:

Theorem 7.7 ([59])**.** *Let $\mathcal{T}$ be a self-affine tiling which is mutually locally derivable with a Delone multiset $(\Lambda_1, \dots, \Lambda_m)$, such that $\cup_{i=1}^m \Lambda_i$ is a lattice. Then $(X_\mathcal{T}, \mathbb{R}^d, \mu)$ has pure discrete spectrum if and only if each Λ_i is a regular model set.*

Using very different methods, Barge and Kwapisz [11] proved an analogous result for 1-dimensional self-similar tilings associated to unimodular Pisot substitutions. More precisely, one of the corollaries of their work is that if such a tiling dynamical system ($\mathbb{R}$-action) has pure discrete spectrum, then the endpoints of a generic tiling in the tiling space form a model set, see [11, Remark 18.6].

Recently, Baake and Moody [7] and Baake, Lenz and Moody [6] investigated conditions for being a model set in terms of dynamical systems in the general (non-substitution) setting. Description of their work is beyond the scope of these lectures. We just state the main result of [6] (in a special case).

Theorem 7.8 ([6]). *Let* $(X_\Lambda, \mathbb{R}^d)$ *be a Delone set dynamical system. It is associated to a repetitive regular model set if and only if the following four conditions are satisfied.*

(1) *All elements of* X_Λ *are Meyer sets;*

(2) $(X_\Lambda, \mathbb{R}^d)$ *is minimal and uniquely ergodic;*

(3) $(X_\Lambda, \mathbb{R}^d)$ *has pure discrete dynamical spectrum with continuous eigenfunctions;*

(4) *The eigenfunctions of* $(X_\Lambda, \mathbb{R}^d)$ *separate almost all points of* X_Λ *in the following sense: the set* $\{\Gamma \in X_\Lambda : \ \exists \Gamma \neq \Gamma' \text{ with } f(\Gamma) = f(\Gamma') \text{ for all eigenfunctions } f\}$ *has measure zero.*

Note that for Λ arising from a primitive substitution, the properties (1)-(3) are known, but (4) is open. On the other hand, J.-Y. Lee [56] obtained a characterization of substitution systems with pure discrete spectrum in terms of *inter-model sets* introduced in [6].

Definition 7.9. An *inter-model set* in $\mathbb{R}^d$ is a set Λ which satisfies

$$\mathbf{t} + \Gamma(W^\circ) \subset \Lambda \subset \mathbf{t} + \Gamma(W)$$

for some $\mathbf{t} \in \mathbb{R}^d$ and compact $W \subset G$ with $W = \overline{W^\circ}$.

Theorem 7.10 (J.-Y. Lee [56]). *Let* Λ *be an aperiodic primitive substitution Delone multiset arising from a self-affine tiling. Then the following are equivalent:*

(i) Λ *has pure discrete spectrum (diffraction or dynamical);*

(ii) Λ *is an inter-model multiset.*

8. Appendix. Selected proofs.

8.1. Linear Repetivity. *Sketch of the proof of Theorem 2.8.* Write $L_P(A) := L_P(\mathcal{T}, A)$ to simplify notation, see (2.2). Fix a $\mathcal{T}$-patch P and define for $n \geq 1$:

$$\overline{\phi}_n := \max_{x \in \mathbb{R}^d} \frac{L_P(x + [0, 2^n]^d)}{2^{nd}}, \quad \underline{\phi}_n := \min_{x \in \mathbb{R}^d} \frac{L_P(x + [0, 2^n]^d)}{2^{nd}}.$$

The max and min exist since the numerators are integers. Note also that there exists $C_1 > 0$ such that for any Borel set W,

$$L_P(W) \leq C_1 \mathrm{Vol}(W). \tag{8.1}$$

This follows from the fact that two equivalent $\mathcal{T}$-patches cannot be too close to each other (they are allowed to have overlapping supports; however, if they are distinct, then the translation vector between them should be at least $\eta > 0$ such that every tile contains a ball of diameter η in its interior). Clearly, $\underline{\phi}_n \le \overline{\phi}_n$ for all n. Moreover, there exists $C_2 > 0$ such that $\underline{\phi}_n \ge C_2$ by ordinary repetitivity. Next observe that

$$\underline{\phi}_{n+1} \ge \underline{\phi}_n \quad \text{for all } n \ge 1. \tag{8.2}$$

This follows by subdividing a cube of side 2^{n+1} containing the minimal number of patches equivalent to P, into 2^d cubes of side 2^n. Therefore, there exists a limit $\underline{\phi} = \lim_{n\to\infty} \underline{\phi}_n$. To get a similar estimate for $\overline{\phi}_n$, consider a cube Q of side 2^{n+1} containing the maximal number of patches equivalent to P. We have

$$\overline{\phi}_{n+1} \le \overline{\phi}_n + R_n/2^{nd}, \tag{8.3}$$

where R_n is the number of patches equivalent to P contained in Q, which intersect one of the d hyperspaces which dissect Q into 2^d cubes of side 2^n. By (8.1),

$$R_n \le C_1 \mathrm{diam}(P) \cdot d \cdot 2^{(d-1)n}.$$

Thus $\overline{\phi}_{n+1} \le \overline{\phi}_n + o(1)$, as $n \to \infty$, and we can deduce that there exists a limit $\overline{\phi} = \lim_{n\to\infty} \overline{\phi}_n$. Clearly, $\underline{\phi} \le \overline{\phi}$. We claim that $\underline{\phi} = \overline{\phi}$. This would imply the existence of uniform patch frequencies in dyadic cubes, and then a standard argument shows the existence of UPF over cubes of all sizes.

Now we use the LR property. Consider a cube Q of side 2^{2n} with the maximal number of patches equivalent to P. By linear repetitivity, there exists $\ell \in \mathbb{Z}^+$ independent of n such that the "pattern of $\mathcal{T}$ in Q" occurs in every cube of side $2^{2n+\ell}$. Now consider any cube Q' of side $2^{2n+\ell}$. Subdivide it into $2^{(n+\ell)d}$ dyadic cubes of side 2^n. Find a translate with the pattern of Q in Q'. It is covered by at most $(2^n+1)^d$ cubes of side 2^n from the subdivision. It follows that

$$L_P(Q') \ge \overline{\phi}_{2n} \cdot 2^{2nd} + \underline{\phi}_n \cdot 2^{nd} \cdot (2^{(n+\ell)d} - (2^n+1)^d),$$

hence

$$\underline{\phi}_{2n} \ge \frac{\overline{\phi}_{2n} \cdot 2^{2nd} + \underline{\phi}_n \cdot 2^{nd} \cdot (2^{(n+\ell)d} - (2^n+1)^d)}{2^{(2n+\ell)d}}.$$

This reduces to

$$\underline{\phi}_{2n} - \underline{\phi}_n \ge 2^{-\ell d}(\overline{\phi}_{2n} - \underline{\phi}_n(1+2^{-n})^d).$$

Letting $n \to \infty$ yields $\underline{\phi} = \overline{\phi}$, as desired. $\square$

8.2. Lind's Theorem. Here we prove Theorem 4.2 which we restate for convenience.

Theorem (D. Lind [61]) *If λ is a Perron number, then there is a primitive non-negative integral matrix M with the PF eigenvalue equal to λ. Moreover, M can be chosen so that it has a column with a positive diagonal entry and the column sum ≥ 3.*

Proof. We follow [61], but leave many estimates as an exercise and make a minor modification.

Let λ be a Perron number of degree $d \geq 2$ over $\mathbb{Q}$. Denote the minimal polynomial of λ by $f(t) = t^d - c_1 t^{d-1} - \cdots - c_d$, where $c_j \in \mathbb{Z}$. Then

$$B = \begin{bmatrix} 0 & 0 & \cdots & c_d \\ 1 & 0 & \cdots & c_{d-1} \\ 0 & 1 & \cdots & c_{d-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_1 \end{bmatrix}$$

is the companion matrix of λ. Of course, it can contain negative entries, otherwise, we could take $M = B$.

The idea of the proof is to find integral points $z_1, \ldots, z_n \in \mathbb{R}^d$ such that

$$Bz_j = \sum_{i=1}^{n} a_{ij} z_i, \quad \text{for } j \leq n, \text{ with } a_{ij} \in \mathbb{Z}^+,$$

and the matrix $A = [a_{ij}]$ has positive trace. Selecting an irreducible component of A will yield the desired matrix.

Now we proceed with the construction. Since $f(t)$ is irreducible, it has no repeated roots. It follows that $\mathbb{R}^d$ splits into the direct sum of three classes of B-invariant subspaces. The first consists of the single 1-dimensional eigenspace D for λ, the dominant eigenvalue. The second class $\mathcal{E}$ contains eigenspaces E corresponding to conjugates of λ strictly outside the unit circle. Note that $\dim E = 1$ if the conjugate is real, and $\dim E = 2$ otherwise. The third class $\mathcal{F}$ contains those 1- or 2-dimensional subspaces F corresponding to conjugates of λ with absolute value ≤ 1. There are norms on these subspaces so that

$$\begin{aligned} \|Bx\|_{D} &= \lambda \|x\|_{D}, & & x \in D, \\ \|Bx\|_{E} &= \tau_{E} \|x\|_{E}, & & x \in E,\ 1 < \tau_{E} < \lambda, \\ \|Bx\|_{F} &= \tau_{F} \|x\|_{F}, & & x \in F,\ \tau_{F} \leq 1. \end{aligned}$$

We equip $\mathbb{R}^d$ with the norm which is the maximum of these norms. If G represents one of the subspaces above, let $\pi_{G} : \mathbb{R}^d \to G$ be the projection to G along the complementary direct sum. Let $\pi_C = I - \pi_D$ be the projection to the invariant complement C of D. By choosing a non-zero vector in D we can identify D with $\mathbb{R}$ and think about $\pi_D(x)$ as the coordinate of x.

We next construct a B-invariant convex region. For $E \in \mathcal{E}$ denote $p(E) = \log \lambda / \log \tau_E > 1$ and $x_E = \pi_E x$. Define $\Phi : \bigoplus_{\mathcal{E}} E \to \mathbb{R}$ by

$$\Phi\Big(\sum_E x_E\Big) = \sum_E \|x_E\|^{p(E)}.$$

For fixed $\xi, \eta > 0$, consider the region

$$\Omega = \Omega_{\xi,\eta} = \Big\{x \in \mathbb{R}^d : \max_F \|\pi_F x\| \le \xi,\ \Phi\Big(\sum_E x_E\Big) \le \eta\, \pi_D x\Big\}.$$

Since $p(E) > 1$ for every E, the region Ω is bowl-shaped, tangent to C at the origin, and curved towards D. A direct verification shows that Φ has an invariant graph and $B\Omega \subset \Omega$. We leave the verification as an exercise.

If $S \subset \mathbb{R}^d$, let $\mathrm{sg}(S)$ be the additive subgroup generated by S. For $\theta > 0$, consider the cone about D:

$$K_\theta := \{x :\ \pi_D x \ge \theta \|\pi_C x\|\}.$$

For $r, s > 0$ define

$$K_\theta(r) = \{x \in K_\theta :\ \pi_D x \le r\}, \qquad K_\theta(r,s) = \{x \in K_\theta :\ r \le \pi_D x \le s\}.$$

The following lemma shows that the semigroup generated by the integral points in a truncated cone contains all the integral points in a slimmer cone.

Lemma 8.1. *Fix $\theta > 0$. For all sufficiently large r,*

$$K_{2\theta} \cap \mathbb{Z}^d \subset \mathrm{sg}(K_\theta(r) \cap \mathbb{Z}^d).$$

Proof of the Lemma. We first claim that there exists $\delta > 0$ such that if $x \in K_{2\theta}$ with $\pi_D x > 4$, then $[x - K_\theta(1,3)] \cap K_{2\theta}$ contains a ball of radius δ. This is intuitively obvious: the point y on the ray $[0,x]$ with $\pi_D y = 2$ lies in $K_{2\theta}$, and its neighborhood of some fixed size is in K_θ, since $K_{2\theta}(1,3)$ is contained in the interior of K_θ. If we move in this neighborhood in the direction of decreasing the slope of the line to x, we will still be in $K_{2\theta}$. Precise estimates are left as an exercise; in [61] it is shown that $\delta = (2\theta + 2)^{-1}$ works.

Now the lemma is proved inductively. First choose ρ such that any ball in $\mathbb{R}^d$ of radius ρ intersects $\mathbb{Z}^d$. Choose r so that $r\delta > \rho$. The claim implies that if $x \in K_{2\theta}$ with $\pi_D x > 4r$, then $[x - K_\theta(r, 3r)] \cap K_{2\theta}$ contains a ball of radius $r\delta > \rho$, hence intersects $\mathbb{Z}^d$.

Let $\Gamma = \mathrm{sg}(K_\theta(4r) \cap \mathbb{Z}^d)$. We show $K_{2\theta} \cap \mathbb{Z}^d \subset \Gamma$. Clearly $K_{2\theta}(4r) \cap \mathbb{Z}^d \subset \Gamma$. Suppose $K_{2\theta}(t) \cap \mathbb{Z}^d \subset \Gamma$ for some $t \geq 4r$; we show that this forces $K_{2\theta}(t+r) \cap \mathbb{Z}^d \subset \Gamma$, which suffices by induction.

Let $z \in [K_{2\theta}(t+r) \setminus K_{2\theta}(t)] \cap \mathbb{Z}^d$. By the claim, there is an element $y \in \mathbb{Z}^d$ contained in $[z - K_\theta(r, 3r)] \cap K_{2\theta}$. We have $\pi_D y \leq t$, hence $y \in \Gamma$ by hypothesis, and $y = z - x$ for some $x \in K_\theta(r, 3r) \cap \mathbb{Z}^d \subset \Gamma$. Therefore $z = x + y \in \Gamma + \Gamma \subset \Gamma$, and the lemma is proved. $\square$

Now we can proceed with the proof of the theorem. First fix any $\theta > 0$. By the lemma, choose $r > 0$ such that $K_{2\theta} \cap \mathbb{Z}^d \subset \mathrm{sg}(K_\theta \cap \mathbb{Z}^d)$. Next find $\xi, \eta > 0$ so that

$$K_\theta(r) \subset \Omega_{\xi,\eta} = \Omega.$$

It is geometrically obvious that this is possible, since Ω is tangent to C at the origin. Precise estimates are left to the reader. Here we are using the Perron property, that is, $p(E) > 1$. For $s > 0$ define

$$\Omega(s) = \{x \in \Omega : \pi_D x \leq s\}, \quad \Omega(s, \infty) = \{x \in \Omega : \pi_D x \geq s\}.$$

We claim that if s is sufficiently large, then

$$(B - I)\Omega(s, \infty) \subset K_{2\theta}.$$

The estimates are not difficult and are left to the reader (note that $\pi_D(Bx - x) = (\lambda - 1)\pi_D x$ and $\|\pi_E(Bx - x)\| \leq (\tau_E + 1)\|\pi_E x\|$).

Now fix $s > \frac{\lambda}{\lambda - 1} r$ so that $(B - I)\Omega(s/\lambda, \infty) \subset K_{2\theta}$. (This is the place where we made a tiny modification: in [61] it is assumed that $s > r$.) Let $\Gamma = \Omega(s) \cap \mathbb{Z}^d$, and order the elements of Γ as $z_1, \ldots, z_n$. The following procedure specifies a rule for writing each Bz_j as a non-negative integral combination

$$Bz_j = \sum_{i=1}^{n} a_{ij} z_i. \tag{8.4}$$

If $\pi_D(z_j) \leq s/\lambda$, then $Bz_j = z_k \in \Gamma$ (because Ω is B-invariant), and put $a_{ij} = \delta_{ik}$. If

$$s/\lambda < \pi_D(z_j) \leq s, \tag{8.5}$$

then

$$Bz_j - z_j \in K_{2\theta} \cap \mathbb{Z}^d \subset \mathrm{sg}[K_\theta(r) \cap \mathbb{Z}^d] \subset \mathrm{sg}(\Gamma),$$

so choose a_{ij} in (8.4) with $a_{jj} \geq 1$ and other non-zero coefficients corresponding to $z_i \in K_\theta(r)$. Note that starting with any $z_i \in K_\theta(r)$ and applying powers of B, we will eventually get a z_j satisfying (8.5). For such z_j we have $\pi_D(Bz_j - z_j) = (\lambda - 1)\pi_D(z_j) > s(\lambda - 1)/\lambda > r$, so we will have $\sum_{i \neq j} a_{ij} \geq 2$ in (8.4). This process yields a matrix A with $a_{jj} \geq 1$ and the sum of non-diagonal entries in jth column ≥ 3. Now starting with z_j we can choose an irreducible component of A. (Draw the directed graph with vertices z_i and the incidence matrix A. Consider all the vertices which can be reached from z_j. This is our irreducible component; it corresponds to a primitive matrix since there is a loop from z_j to itself.) Replace A by this component, so now A is primitive and indexed by $\Gamma_0 \subset \Gamma$. We can of course reorder the variables so that $j = 1$. It remains to prove that λ_A, the PF eigenvalue of A, is equal to λ.

Let A be n-dimensional. Denote by e_i the unit vectors in $\mathbb{R}^n$ and define a linear map $P : \mathbb{R}^n \to \mathbb{R}^d$ by $Pe_i = z_i$. Then (8.4) implies that $PA = BP$. By the Perron-Frobenius theory, A has a strictly positive eigenvector v for λ_A. Then Pv is a positive linear combination of the z_i, hence $\pi_{_D}(Pv) > 0$, and so $Pv \neq 0$. Also, $B(Pv) = P(Av) = \lambda_A(Pv)$, so λ_A is an eigenvalue of B. Thus, $\lambda_A \leq \lambda$.

To prove $\lambda \geq \lambda_A$, we note that $P(\mathbb{R}^d) \supset D$. Indeed, $PA = BP$ implies that $P(\mathbb{R}^d)$ is a B-invariant subspace. Since Pe_i have non-zero D-coordinate and λ is a simple eigenvalue, the claim follows. Thus, there is $u \in \mathbb{R}^n$ such that $Pu = w$, where w is an eigenvector for B corresponding to λ. Then

$$\lambda^n \|w\| = \|B^n w\| = \|B^n Pu\| = \|PA^n u\| \leq \|P\|\|A^n\|\|u\|.$$

The spectral radius formula for A shows that $\lambda_A \geq \lambda$, completing the proof.

8.3. Necessity in Theorem 4.3. We will actually prove something stronger:

Theorem 8.2. *Let ϕ be an expanding linear similarity on $\mathbb{R}^d$, and suppose that there exists a self-similar tiling of $\mathbb{R}^d$ with expansion ϕ. Let λ be an eigenvalue of ϕ, and let γ be a Galois conjugate of λ. Then either $|\gamma| < |\lambda|$, or γ is also an eigenvalue of ϕ of multiplicity greater or equal to that of λ.*

This, of course, includes as a special case $d = 2$, with ϕ being the multiplication by $\lambda \in \mathbb{C}$, which has the eigenvalues λ and $\overline{\lambda}$. Then the conclusion is that λ has all other conjugates of modulus strictly smaller than $|\lambda|$, i.e., it is complex Perron.

Proof. We continue the argument started in the proof of Lemma 4.4. Recall that $J = \langle \mathcal{C} \rangle$ is the free Abelian group generated by the control points of the tiling $\mathcal{T}$, and we fixed the matrix $V = [v_1 \dots v_N]$ whose columns are free generators of J. By the definition of free generators, for every $\xi \in J$ there exists a unique $a(\xi) \in \mathbb{Z}^N$ such that

$$\xi = Va(\xi).$$

We call $\xi \mapsto a(\xi)$ the "address map." Observe that

$$\mathrm{Span}_{\mathbb{R}}\{a(\xi) : \ \xi \in \mathcal{C}\} = \mathbb{R}^N. \tag{8.6}$$

Indeed, J is generated by $\mathcal{C}$, hence every v_j is an integral linear combination of control points, and $a(v_j)$ is the jth unit vector in $\mathbb{R}^N$.

Lemma 8.3. *The address map is uniformly Lipschitz on $\mathcal{C}$: there exists $L_1 > 0$ such that*

$$\|a(\xi) - a(\xi')\| \le L_1 \|\xi - \xi'\|, \quad \forall\ \xi, \xi' \in \mathcal{C}. \tag{8.7}$$

Note that the address map is usually not even continuous on J, since J is usually dense in $\mathbb{R}^d$, whereas the range of the address map is a subset of the integer lattice.

Proof sketch. It is not hard to see that one can move "quasi-efficiently" between control points by moving "from neighbor to neighbor." More precisely, there is a constant $C_1 = C_1(\mathcal{T})$ such that $\forall\ \xi, \xi' \in \mathcal{C}$, there exist $p \in \mathbb{N}$ and $\xi_1 := \xi,\ \xi_2, \ldots, \xi_{p-1} \in \mathcal{C},\ \xi_p := \xi'$ such that $\xi_{i+1} - \xi_i \in \Psi$ for $i = 1, \ldots, p-1$ (see the definition of Ψ in (4.1)), and

$$\sum_{i=1}^{p-1} \|\xi_{i+1} - \xi_i\| \le C_1 \cdot \|\xi - \xi'\|.$$

(See [51, Lem. 2.2] for a proof of this claim.) Let

$$C_2 := \max\{\|a(\zeta) - a(\zeta')\| / \|\zeta - \zeta'\| : \ \zeta' - \zeta \in \Psi\},$$

which is well-defined and finite by FLC. Now we can estimate:

$$\begin{aligned}
\|a(\xi) - a(\xi')\| = \|a(\xi - \xi')\| &= \left\| \sum_{i=1}^{p-1} a(\xi_{i+1} - \xi_i) \right\| \\
&\le \sum_{i=1}^{p-1} \|a(\xi_{i+1} - \xi_i)\| \\
&\le C_2 \sum_{i=1}^{p-1} \|\xi_{i+1} - \xi_i\| \le C_1 C_2 \|\xi - \xi'\|,
\end{aligned}$$

as desired. □

Recall that there exists an integer $N \times N$ matrix M such that

$$\phi V = VM. \tag{8.8}$$

We already saw that every eigenvalue of ϕ must be an eigenvalue of M. Note also that (8.8) implies

$$a(\phi\xi) = Ma(\xi), \quad \forall\ \xi \in J. \tag{8.9}$$

Lemma 8.4. *The matrix M is diagonalizable in $\mathbb{C}^N$.*

Proof. This is a standard fact in algebra; we provide an elementary proof for the reader's convenience.

Recall that ϕ is a similarity, hence it is diagonalizable over $\mathbb{C}$ (it is a unitary linear map times θ for some $\theta > 1$). Decompose $\mathbb{R}^d$ into a direct sum of eigenspaces $E_i, i \le p$, corresponding to eigenvalues λ_i (one-dimensional if λ_i is real and 2-dimensional otherwise). Decomposing the vectors v_j (the generators of J) in terms of E_i yields

$$J \subset J' := \bigoplus_{i=1}^{p} J_i e_i,$$

where $e_i \in E_i$ and J_i is a finitely-generated $\mathbb{Z}[\lambda_i]$-module. Thus,

$$J_i = \bigoplus_{k=1}^{r_i} \mathbb{Z}[\lambda_i] y_k^{(i)}$$

for some $y_k^{(i)} \in E_i$. The transformation ϕ induces an endomorphism of J'. We choose the canonical set of generators for $\mathbb{Z}[\lambda_i]$, namely, $1, \lambda_i, \ldots, \lambda_i^{n_i-1}$, where n_i is the degree of the algebraic integer λ_i, and the corresponding basis for J', namely, $\{\lambda_i^s y_k^{(i)} : 0 \le s \le n_i - 1,\ 1 \le k \le r_i,\ i \le p\}$. In this basis, the endomorphism has a block matrix, whose every block is a companion matrix of the minimal polynomial of one of the λ_i's. This matrix is diagonalizable over $\mathbb{C}$, since the minimal polynomial has no repeated roots. Finally, we note that the endomorphism induced by ϕ on J is a restriction of the one which is induced on J', hence its matrix, M, is diagonalizable as well. $\square$

Now suppose that γ is a conjugate of λ and $|\gamma| \ge |\lambda| > 1$ (otherwise, there is nothing to prove). Then γ is an eigenvalue of M. Let U_γ be the (real) eigenspace for M corresponding to γ. By Lemma 8.4, there is a projection π_γ from $\mathbb{R}^N$ to U_γ commuting with M. By definition, the only eigenvalues of $M|_{U_\gamma}$ are γ and $\overline{\gamma}$ (if γ is nonreal). Thus, we can fix a norm on U_γ satisfying

$$\|My\| = |\gamma|\,\|y\|, \quad y \in U_\gamma. \tag{8.10}$$

Consider the mapping $f_\gamma : \mathcal{C} \to U_\gamma$ given by

$$f_\gamma(\xi) = \pi_\gamma a(\xi), \quad \xi \in \mathcal{C}. \tag{8.11}$$

Our goal is to extend f_γ to the entire space $\mathbb{R}^d$. We let

$$f_\gamma(\phi^{-k}\xi) = M^{-k} f_\gamma(\xi), \quad \xi \in \mathcal{C}. \tag{8.12}$$

This is well-defined since M is invertible on U_γ, and unambiguous by (8.9), since $\pi_\gamma M = M\pi_\gamma$. This way we have f_γ defined on a dense set

$$\mathcal{C}_\infty := \bigcup_{k=0}^{\infty} \phi^{-k}\mathcal{C}.$$

We want to show that f_γ is uniformly continuous on $\mathcal{C}_\infty$, hence can be extended to all of $\mathbb{R}^d$. In fact, it is uniformly Lipschitz. Recall that ϕ is a similarity and λ is its eigenvalue, hence $\|\phi x\| = |\lambda|\|x\|$ for all $x \in \mathbb{R}^d$.

Lemma 8.5. *The map f_γ is uniformly Lipschitz on $\mathcal{C}_\infty$:*

$$\|f_\gamma(\xi_1) - f_\gamma(\xi_2)\| \le L_1\|\xi_1 - \xi_2\| \quad \forall\, \xi, \xi' \in \mathcal{C}_\infty.$$

Moreover, if $|\gamma| > |\lambda|$, then f_γ is a constant function.

Proof. Let $\xi_i = \phi^{-k}c_i$ for $i = 1, 2$, where $c_i \in \mathcal{C}$. We have, using $|\gamma| \ge |\lambda|$, (8.10) and Lemma 8.3:

$$\begin{aligned}
\|f_\gamma(\phi^{-k}c_1) - f_\gamma(\phi^{-k}c_2)\| &= \|M^{-k}(f_\gamma(c_1) - f_\gamma(c_2))\| \\
&= |\gamma|^{-k}\|f_\gamma(c_1) - f_\gamma(c_2)\| \\
&\le L_1|\gamma|^{-k}\|c_1 - c_2\| \\
&= L_1(|\lambda|/|\gamma|)^k\|\phi^{-k}c_1 - \phi^{-k}c_2)\|.
\end{aligned}$$

If $|\gamma| = |\lambda|$, we get the desired property. If $|\gamma| > |\lambda|$, then we obtain that $f_\gamma(\xi_1) = f_\gamma(\xi_2)$ by letting $k \to \infty$ (this is possible since $\phi(\mathcal{C}) \subset \mathcal{C}$). Since ξ_1 and ξ_2 were arbitrary, we obtain that f_γ is a constant function. $\square$

Lemma 8.5 implies that $|\gamma| > |\lambda|$ is impossible. Indeed, f_γ is non-constant by (8.11) and (8.6). Thus, it remains to consider the case $|\gamma| = |\lambda|$. By Lemma 8.5, we can extend f_γ by continuity to obtain a function $f_\gamma : \mathbb{R}^d \to U_\gamma$. Observe that

$$f_\gamma \circ \phi = M \circ f_\gamma, \tag{8.13}$$

since this holds on the dense set $\mathcal{C}_\infty$.

Lemma 8.6. *The function f_γ depends only on the tile type in $\mathcal{T}$ up to an additive constant: if $T, T + x \in \mathcal{T}$ and $\xi \in T$, then*

$$f_\gamma(\xi + x) = f_\gamma(\xi) + \pi_\gamma a(x). \tag{8.14}$$

Proof. It is enough to check (8.14) on a dense set. Suppose $\xi = \phi^{-k} c(S) \in T$ for some $S \in \omega^k(T)$. Then $S + \phi^k x \in \omega^k(T + x)$ and we have

$$\begin{aligned} f_\gamma(\xi + x) &= f_\gamma(\phi^{-k} c(S) + x) \\ &= f_\gamma(\phi^{-k} c(S + \phi^k x)) \\ &= M^{-k} f_\gamma(c(S + \phi^k x)) \\ &= M^{-k} f_\gamma(c(S)) + M^{-k} \pi_\gamma a(\phi^k x) \\ &= f_\gamma(\xi) + \pi_\gamma a(x), \end{aligned}$$

as desired. Here we used the definition of f_γ on $\mathcal{C}$ and (8.9). □

Conclusion of the proof of Theorem 8.2. We mimic the argument of Thurston [98] but provide more details.

The function $f_\gamma : \mathbb{R}^d \to U_\gamma$ is Lipschitz, hence it is differentiable almost everywhere by Rademacher's Theorem. Let x be a point where the total derivative $D = Df_\gamma(x)$ (a linear map from $\mathbb{R}^d$ to U_γ) exists. Then

$$f_\gamma(x + u) = f_\gamma(x) + Du + \Psi(u) \quad \text{for all} \quad u \in \mathbb{R}^d,$$

where $\Psi : \mathbb{R}^d \to U_\gamma$ satisfies $\|\Psi(u)\| = o(\|u\|)$, as $u \to 0$. Multiplying by M^n, using (8.13) and substituting $v = \phi^n u$, we obtain

$$f_\gamma(\phi^n x + v) = f_\gamma(\phi^n x) + M^n D\phi^{-n} v + M^n \Psi(\phi^{-n} v) \quad \text{for all} \quad v \in \mathbb{R}^d.$$

For a set $A \subset \mathbb{R}^d$ denote by $[A]^{\mathcal{T}}$ the $\mathcal{T}$-patch of tiles which intersect A. By repetitivity, there exists $R > 0$ such that $B_R(0)$ contains a translate of the patch $[B_1(\phi^n x)]^{\mathcal{T}}$ for all $n \in \mathbb{N}$. This implies, in view of Lemma 8.6, that there exist $x_n \in B_R(0)$, for $n \geq 1$, such that

$$f_\gamma(x_n + v) = f_\gamma(x_n) + M^n D\phi^{-n} v + M^n \Psi(\phi^{-n} v) \quad \text{for all} \quad v \in B_1(0).$$

Recall that ϕ is a similarity with expansion factor $|\lambda|$, and the expression above is in U_γ where M expands by $|\gamma| = |\lambda|$, see (8.10). Thus, $\|M^n D\phi^{-n}\| = \|D\|$ and $\|M^n \Psi(\phi^{-n} v)\| = |\gamma|^n \cdot o(|\lambda|^{-n} \|v\|) \to 0$, as $n \to \infty$. Passing to a subsequence, we can assume that $x_{n_k} \to x'$ and $M^{n_k} D\phi^{-n_k} \to D'$, where $D' : \mathbb{R}^d \to U_\gamma$ is a linear map. Then by continuity of f,

$$f_\gamma(x' + v) = f_\gamma(x') + D'v \quad \text{for all} \quad v \in B_1(0).$$

Thus, f_γ is flat on some neighborhood. Multiplying by M^n and applying (8.13), we obtain that f_γ is flat on an arbitrarily large neighborhood. By repetitivity, a translate of $[B_1(0)]^{\mathcal{T}}$ occurs in every sufficiently large neighborhood, therefore, by Lemma 8.6, the function f_γ

is flat on $B_1(0)$. Using (8.13) again, we conclude that f_γ is flat everywhere, and since $f_\gamma(0) = 0$ by (8.12), we obtain that f_γ is linear. By (8.6) and (8.11), the range of f_γ is the entire U_γ. Thus, in view of (8.13), there exists a linear subspace E of $\mathbb{R}^d$, invariant under ϕ, such that f_γ restricted to E is an isomorphism onto U_γ. It follows that the restriction of ϕ to E is isomorphic, as a linear map, to $M|_{U_\gamma}$. This implies the desired conclusion: γ is an eigenvalue of ϕ, and if λ has multiplicity k as an eigenvalue of ϕ, then γ (being its Galois conjugate) has multiplicity greater or equal to k as an eigenvalue of M and hence of ϕ as well. □

8.4. Sketch of the proof of Theorem 5.7. Here we only present an outline; see [97] for more details. The proof is based on [98, 48] and a personal communication from Rick Kenyon.

Instead of the set Ξ, it is more convenient to work with control points, see Definition 4.5. Let $\mathcal{C} = \mathcal{C}(\mathcal{T}) = \{c(T) : T \in \mathcal{T}\}$ be the set of control points for all tiles. Clearly, $\Xi(\mathcal{T}) \subset \mathcal{C} - \mathcal{C}$. Observe that it is enough to prove the inclusion

$$\mathcal{C} \subset e_1\mathbb{Q}(\theta) + \cdots + e_d\mathbb{Q}(\theta) \tag{8.15}$$

for some basis $\{e_1, \ldots, e_d\}$. Indeed, the Abelian group $\langle \mathcal{C} \rangle$ is finitely generated. Let $\{w_1, \ldots, w_N\}$ be a set of free generators. By (8.15), $w_j = \sum_{j=1}^d e_j \frac{p_j^{(i)}(\theta)}{q_j^{(i)}(\theta)}$, for $i \le N$, for some polynomials $p_j^{(i)}, q_j^{(i)} \in \mathbb{Z}[x]$. Then we obtain (5.4), with

$$b_j = e_j \left(\prod_{j=1}^{d} \prod_{i=1}^{N} q_j^{(i)}(\theta) \right)^{-1}.$$

Now pick any set $\{e_1, \ldots, e_d\} \subset \mathcal{C}$ which spans $\mathbb{R}^d$. Consider the vector space $\widetilde{\mathcal{C}} := \mathrm{Span}_{\mathbb{Q}(\theta)}\mathcal{C}$ over the field $\mathbb{Q}(\theta)$. We want to show that $\{e_1, \ldots, e_d\}$ is a basis for $\widetilde{\mathcal{C}}$. Let π be any linear projection from $\widetilde{\mathcal{C}}$ onto $\mathrm{Span}_{\mathbb{Q}(\theta)}\{e_1, \ldots, e_d\}$. Since the vector space is over $\mathbb{Q}(\theta)$, we obviously have

$$\pi(\theta^n x) = \theta^n \pi(x) \quad \text{for } x \in \widetilde{\mathcal{C}},\ n \in \mathbb{Z}. \tag{8.16}$$

Thus we have a map

$$f(x) = \pi(x) \quad \text{for } x \in \mathcal{C}_\infty, \quad \text{where } \mathcal{C}_\infty := \bigcup_{n=0}^{\infty} \theta^{-n}\mathcal{C}.$$

This is consistent in view of (8.16). Now f defined on a dense subset of $\mathbb{R}^d$. The rest of the proof is almost identical to that of Theorem 8.2: it is shown that f is uniformly Lipschitz on this set, hence f can be extended to $\mathbb{R}^d$ by continuity. This extension satisfies the equation

$f(\theta x) = \theta f(x)$ for all $x \in \mathbb{R}^d$. Then it is proved that f is linear. But $f(e_j) = \pi(e_j) = e_j$, hence f is the identity map, $\pi(\xi) = \xi$ for all $\xi \in \mathcal{C}$, and we conclude that (8.15) holds, as desired. □

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