1. Hyperbolic plane and the isometric action (see [1, 9.1])

1.1. Upper half-plane model: \( \mathbb{H} = \{ x + iy \in \mathbb{C} : y > 0 \} \). Tangent bundle:
\[
\mathbb{T} = \mathbb{H} \times \mathbb{C}, \quad T_z \mathbb{H} = \{ z \} \times \mathbb{C} \quad \text{(the tangent plane at } z).\]
If \( \phi : [0, 1] \to \mathbb{H} \) is differentiable, then
\[
D\phi(t) = (\phi(t), \phi'(t)) \in T_z \mathbb{H}, \quad \text{where } z = \phi(t).
\]

Hyperbolic Riemannian metric:
\[
\langle v, w \rangle_z = \frac{1}{y^2} (v \cdot w), \quad \text{where } z = x + iy \in \mathbb{H}, \quad v, w \in T_z \mathbb{H}.
\]
Here \((v \cdot w) = \Re(v \overline{w})\) is the usual inner product under the identification \( \mathbb{C} \cong \mathbb{R}^2 \).

Length of a path = piecewise differentiable curve: if \( \phi : [0, 1] \to \mathbb{H} \), then
\[
L(\phi) = \int_0^1 \|D\phi(t)\|_{\phi(t)} \, dt = \int_0^1 \frac{|\phi'(t)|}{3\phi(t)} \, dt.
\]

Hyperbolic metric on \( \mathbb{H} \):
\[
d(z_0, z_1) := \inf\{ L(\phi) : \phi \text{ is a path connecting } z_0 \text{ and } z_1 \}.
\]

Boundary of the hyperbolic plane: \( \partial \mathbb{H} := \mathbb{R} \cup \{ \infty \} \).

1.2. Isometric action. The group \( SL_2(\mathbb{R}) \) acts on \( \mathbb{H} \) by the Möbius transformations:
\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad z \mapsto \frac{az + b}{cz + d}.
\]
Since \(-I_2\) (minus the identity matrix) acts trivially, we actually have an action of \( PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{ \pm I_2 \} \).

Derivative action of \( PSL_2(\mathbb{R}) \) on \( \mathbb{T} \): for \( g \in PSL_2(\mathbb{R}) \) we have \( Dg : \mathbb{T} \to \mathbb{T} \):
\[
Dg(z, v) = (g(z), g'(z)v).
\]

Definition 1.1. An action of a group \( G \) on a set \( X \) is called transitive if for every \( x_1, x_2 \in X \) there exists \( g \in G \) such that \( g \cdot x_1 = x_2 \). If such a \( g \) is unique for all \( x_1, x_2 \), we say that the action is simply transitive.

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Lemma 1.2. The actions defined above have the following properties:

(i) \( (Dg)_z : T_z \mathbb{H} \to T_{g(z)} \mathbb{H} \) preserves the Riemannian metric, and the action of \( \text{PSL}_2(\mathbb{R}) \) on \( \mathbb{H} \) preserves the hyperbolic metric, that is,

\[
d(g(z_0), g(z_1)) = d(z_0, z_1);
\]

(ii) The action of \( \text{PSL}_2(\mathbb{R}) \) on \( \mathbb{H} \) is transitive;

(iii) The stabilizer of \( i \) under this action is

\[
\text{Stab}(i) := \{ g \in \text{PSL}_2(\mathbb{R}) : g(i) = i \} = \text{PSO}(2) = \text{SO}(2)/\{\pm I_2\},
\]

where \( \text{SO}(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \).

Consider the unit tangent bundle:

\[
T^1 \mathbb{H} = \{(z, v) \in T \mathbb{H} : \|v\|_z = 1\}.
\]

The group \( \text{PSL}_2(\mathbb{R}) \) acts on \( T^1 \mathbb{H} \) because it preserves the Riemannian metric.

Lemma 1.3. The action of \( \text{PSL}_2(\mathbb{R}) \) on \( T^1 \mathbb{H} \) is simply transitive. We thus have \( \text{PSL}_2(\mathbb{R}) \cong T^1 \mathbb{H} \). The standard identification is \( g \mapsto Dg(i, i) \).

2. Geodesic and horocycle flows (first intro.)

2.1. Geodesic curves.

Lemma 2.1. Let \( z_0 = y_0 i, \ z_1 = y_1 i \) with \( 0 < y_1 < y_2 \). Then \( d(z_0, z_1) = \log(y_1/y_0) = L(\phi) \), where \( \phi(t) = iy_0(y_1/y_0)^t \). Moreover, this path of minimal length is unique geometrically, \( L(\psi) = L(\phi) \) iff \( \psi = \phi \circ f \) for some monotone piecewise differentiable \( f : [0, 1] \to [0, 1] \).

The curve of minimal distance is called a **geodesic**. We thus see that \( \{yi : y > 0\} \) is a geodesic. Since \( \text{PSL}_2(\mathbb{R}) \) acts on \( \mathbb{H} \) by isometries, we can find other geodesics applying Möbius transformations.

It is convenient to reparametrize geodesic paths to always have constant speed one.

Lemma 2.2. For any \( z_0, z_1 \in \mathbb{H} \) there is a unique path \( \phi : [0, d(z_0, z_1)] \to \mathbb{H} \) of unit speed connecting \( z_0 \) to \( z_1 \). Moreover, there is a unique isometry \( g \in \text{PSL}_2(\mathbb{R}) \) such that \( \phi(t) = g(e^t i) \).

The geodesic curves are the half-lines \( \{x + iy \in \mathbb{C} : y > 0\}, \ x \in \mathbb{R} \), and upper half-circles orthogonal to the real line: \( \{x + iy \in \mathbb{C} : (x - a)^2 + y^2 = r^2\}, \ x \in \mathbb{R} \) and \( r > 0 \).
2.2. Geodesic flow. A geodesic is uniquely determined by a point \( z \in \mathbb{H} \) and a unit “velocity” vector \( v \), that is, by \((z, v) \in T^1\mathbb{H}\). The geodesic flow \( g_t : T^1\mathbb{H} \rightarrow T^1\mathbb{H} \) is defined by following the parametrization of this uniquely defined geodesic for time \( t \). (It is called a “flow” because it is a continuous-time dynamical system, or an \( \mathbb{R} \)-action.) In the case of the reference point \((i, i) \in T^1\mathbb{H}\) we get a simple formula

\[
g_t((i, i)) = (e^t i, e^t i), \quad t \in \mathbb{R}.
\]

For an arbitrary point \((z, v) = g((i, i))\) we get

\[
g_t((z, v)) = Dg_t((i, i)) = D(ga_t^{-1})(i, i), \quad \text{where} \quad a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}.
\]

On \( \text{PSL}_2(\mathbb{R}) \cong T^1\mathbb{H} \) the geodesic flow is the right multiplication by the inverse of \( a_t \):

\[
R_{a_t}(g) = ga_t^{-1}.
\]

2.3. Horocycle flow. The orbits of the horocycle flows, which we now define, turn out to be the stable and unstable manifolds of a given point for the geodesic flow. Consider the subgroups

\[
U^- = \left\{ u^-(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \quad U^+ = \left\{ u^+(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R} \right\}.
\]

Algebraically (on \( \text{PSL}_2(\mathbb{R}) \)), the stable and unstable horocycle flows are defined by

\[
h^-(s) \cdot g = R_{u^-(s)}(g) = gu^-(s), \quad h^+(s) \cdot g = R_{u^+(s)}(g) = gu^+(s).
\]

In \( T^1\mathbb{H} \) a point \((z, v)\) is moving under the horocycle flow along a horocycle (a line parallel to \( \mathbb{R} \) or a circle in \( \mathbb{H} \) tangent to \( \mathbb{R} \)), which is orthogonal to \( v \) at \( z \), and the unit tangent vector stays orthogonal to the horocycle curve. There exactly two horocycles with this property, corresponding the stable/unstable flows. The stable and unstable sets for the geodesic flow at a given point \((z_0, v_0) \in T^1\mathbb{H}\) are defined, respectively, by

\[
W_{g_t}^-(z_0, v_0) = \{(z, v) \in T^1\mathbb{H} : \text{dist}((z, v), (z_0, v_0)) \to 0, \quad \text{as} \quad t \to +\infty\}
\]

and

\[
W_{g_t}^+(z_0, v_0) = \{(z, v) \in T^1\mathbb{H} : \text{dist}((z, v), (z_0, v_0)) \to 0, \quad \text{as} \quad t \to -\infty\}.
\]

This is not very precise yet, because we have not defined an invariant metric on \( T^1\mathbb{H} \) (only on \( \mathbb{H} \)).

References