1. Unitary representations and Ergodicity (see [1, 11.3])

1.1. Three types of actions. We consider a lattice $\Gamma \leq SL_2(\mathbb{R})$ and the quotient space $X = \Gamma \backslash SL_2(\mathbb{R})$, equipped with a metric and finite measure, invariant under the action of $SL_2(\mathbb{R})$. The action is by right multiplication:

$$x = \Gamma h \in X, \ R_g(x) = xg^{-1} = \Gamma hg^{-1}.$$ 

Consider the following subgroups of $SL_2(\mathbb{R})$: the diagonal group $A$, the groups of unipotents $U^\pm$, and the special orthogonal group $SO_2(\mathbb{R})$:

$$A = \left\{ \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} : t \in \mathbb{R} \right\}, \quad U^- = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \quad U^+ = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} : s \in \mathbb{R} \right\},$$

$$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$ 

The geodesic flow on $X$ is given by the action of $A$, and the stable/unstable horocycle flows are given by the action of $U^-$ and $U^+$ respectively.

Lemma 1.1. Every $g \in SL_2(\mathbb{R})$ is conjugate to an element of $\pm A$, or an element of $\pm U^-$, or an element of $SO_2(\mathbb{R})$.

Lemma 1.2. Let $\Gamma \leq G$ be a discrete subgroup of a closed linear group. Suppose that $g_2 = hg_1h^{-1}$ for some $g_1, g_2, h \in G$. Then $R_{g_1}$ and $R_{g_2}$ on $X = \Gamma \backslash G$ are conjugate via $R_h$. In particular, if $\Gamma$ is a lattice, then $(X, \mathcal{B}_X, m_X, R_{g_1})$ and $(X, \mathcal{B}_X, m_X, R_{g_2})$ are measurably conjugate.

From this lemmas it follows that the study of the dynamics of $SL_2(\mathbb{R})$ on $X$ reduces to the three cases:

- diagonalizable elements, called hyperbolic;
- elements conjugate to an element of $\pm U^-$, called parabolic; and
- elements conjugate to an element of $SO_2(\mathbb{R})$, called elliptic.
It turns out that elliptic elements exhibit little interesting dynamics (see Homework 6), so we focus on the rest.

**Theorem 1.3.** Let $\Gamma \leq SL_2(\mathbb{R})$ be a lattice and $X = \Gamma \backslash G$. Let $g \in SL_2(\mathbb{R})$ be an element that is not conjugate to an element of $SO_2(\mathbb{R})$. Then $R_g$ is an ergodic measure-preserving transformation on $(X, \mathcal{B}_X, m_X)$.

We didn’t prove this in full generality; but we did show the case of $g$ hyperbolic, and also the ergodicity of the action of the entire $U^-$ (that is, ergodicity of the horocycle flow rather than individual elements of $U^-$).

**1.2. Unitary representations.**

**Definition 1.4.** Let $\mathcal{H}$ be a Hilbert space, and $U(\mathcal{H})$ denotes the group of unitary linear operators on $\mathcal{H}$. A strongly continuous unitary representation of a metric group $G$ is a homomorphism $\Pi : G \to U(\mathcal{H})$, which is continuous in the strong operator topology, that is,

$$g_n \to g \text{ in } G \implies \Pi(g_n)\xi \to \Pi(g)\xi, \quad \text{for all } \xi \in \mathcal{H}.$$

**Definition 1.5.** Suppose that $G$ is a metric group acting on a finite measure space $(X, \mathcal{B}_X, m_X)$ by measure-preserving transformations. For $g \in G$ consider

$$U_g \phi(x) = \phi(g^{-1} \cdot x) \quad \text{for } \phi \in L^2(X, m_X).$$

Then $U_g$ is a unitary linear operator on $\mathcal{H} := L^2(X, m_X)$ (this follows by a change of variable and the measure-preserving property) and $\Pi : g \mapsto U_g$ is a homomorphism from $G$ to $U(\mathcal{H})$. The unitary representation $\Pi$ is called the Koopman representation associated with the action.

**Lemma 1.6.** Suppose that a locally compact $\sigma$-compact metric group $G$ acts on a $\sigma$-compact metric space $X$, equipped with a finite Borel measure, by measure-preserving transformations. Then the associated Koopman representation is strongly continuous.

_Idea of the proof._ We need to show that if $g_n \to g$ in $G$, then $U_{g_n} \phi \to U_g \phi$ in $L^2(X, m_X)$ for every $\phi \in L^2$. Since all unitary operators have norm one, it is enough to consider test functions $\phi$ from a dense set and then do an approximation; a convenient one is the set of compactly supported continuous functions on $X$. A careful proof requires some work, but it is relatively straightforward, using joint continuity of the action.

**1.3. Mautner lemma and its consequences.**

**Lemma 1.7** (Mautner). Let $G$ be a locally compact group and $\Pi : G \to U(\mathcal{H})$ a strongly continuous unitary representation on a Hilbert space $\mathcal{H}$. Suppose that $g, h \in G$ are such that

$$\lim_{n \to \infty} g^n h g^{-n} = e.$$
Then every $\xi \in \mathcal{H}$ that is fixed by $g$ is also fixed by $h$, that is,

$$\Pi(g)\xi = \xi \implies \Pi(h)\xi = \xi.$$  

**Lemma 1.8.** Let $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in A \leq SL_2(\mathbb{R})$ and $h \in U^- \cup U^+$. Then

$$\lim_{n \to \infty} g^ng^{-n} = e \text{ if } a < 1 \text{ and } h \in U^-;$$

$$\lim_{n \to \infty} g^ng^{-n} = e \text{ if } a > 1 \text{ and } h \in U^+.$$

This is a straightforward verification.

**Lemma 1.9 (Exercise).** The set $U^- \cup U^+$ algebraically generates all of $SL_2(\mathbb{R})$.

**Corollary 1.10.** Suppose that $SL_2(\mathbb{R})$ acts on $(X, \mathcal{B}_X, m_X)$ (finite measure space) by measure-preserving transformations. Let $U_g : L^2(X, m_X) \to L^2(X, m_X)$ be the unitary operator obtained from the Koopman representation. If $U_g \phi = \phi$ for some $e \neq g \in A$ (the diagonal group) and $\phi \in L^2$, then $U_h \phi = \phi$ for all $h \in SL_2(\mathbb{R})$.

This is a combination of Lemmas 1.6, 1.7, 1.8, and 1.9.

Now we are ready to prove the part of Theorem 1.3 concerning $g$ that are conjugate to an element of $A$.

**Proposition 1.11.** Let $\Gamma \leq SL_2(\mathbb{R})$ be a lattice and $X = \Gamma \backslash G$. Suppose that $e \neq g \in A$. Then $R_g$ on $(X, m_X)$ is ergodic, and hence the geodesic flow on $X$ is ergodic.

**Proof.** In order to show that $R_g$ is ergodic, it is enough (in fact, equivalent) to check that if $\phi \in L^2(X, m_X)$ is such that $U_g \phi = \phi$ then $\phi$ is constant a.e. By the Corollary, we have that $U_h \phi = \phi$ for all $h \in SL_2(\mathbb{R})$, but $SL_2(\mathbb{R})$ acts transitively on $X$, and we are “done”.

Here is a careful proof: we got a function $\phi$ on $X$ with the following property: for every $h \in G := SL_2(\mathbb{R})$,

$$\phi(h^{-1} \cdot x) := \phi(xh) = \phi(x) \quad \text{for } m_X\text{-a.e. } x \in X.$$  

However, the implied set of full measure depends on $h$. Nevertheless, by Fubini Theorem, we obtain that for almost all $x \in X$, for almost all $h \in G$, we have

$$\phi(xh) = \phi(x).$$

Choose one such $x$, denote it by $x_0$. Then $\phi(x_0h) = \phi(x_0)$ for $h \in G_0$, with $m_G(G \setminus G_0) = 0$. Let $X_0 = x_0G_0$. We obtain that $\phi$ is constant on $X_0$.

We claim that $X_0$ has full $m_X$-measure. Recall that $M_X(B) = m_G(\pi^{-1}B \cap F)$, where $\pi : G \to X$ is the natural projection and $F$ is a fundamental domain. Let $x_0 = \Gamma g_0$. Then

$$m_X(X_0) = m_G(\pi^{-1} \Gamma g_0 G_0 \cap F) = m_G(g_0 G_0 \cap F) = m_G(F) = m_X(X),$$

where we used that $m_G(G \setminus g_0 G_0) = m_G(g_0^{-1} G \setminus G_0) = m_G(G \setminus G_0) = 0$. \qed
1.4. Ergodicity of the horocycle flow. For this, we need a more technical version of Mautner’s Lemma:

**Proposition 1.12** (Margulis). Let $G$ be a locally compact group and $\Pi : G \to U(\mathcal{H})$ a strongly continuous unitary representation on a Hilbert space $\mathcal{H}$. Suppose that $\xi \in \mathcal{H}$ is fixed by some subgroup $L \leq G$. Then $\xi$ is also fixed by any other element $h \in G$ with the property that

$$\forall \delta > 0, \quad B^G_\delta(h) \cap LB^G_\delta(e) L \neq \emptyset.$$  

Here $B^G_\delta(h)$ denotes the open ball of radius $\delta$ centered at $h \in G$, using the left-invariant metric $d_G$. Observe that Mautner Lemma 1.7 is a special case of the proposition, with $L$ being the cyclic group generated by $g$, because under the condition (1) we have $h \in LB^G_\delta(e)L$ for every $\delta > 0$.

**Corollary 1.13.** Let $\Gamma \leq SL_2(\mathbb{R})$ be a lattice and $X = \Gamma \backslash G$. Then the action of $U^-$, as well as the action of $U^+$ (that is, the stable and unstable horocycle flows on $(X, \mathcal{B}_X, m_X)$) are ergodic.

**Proof.** Let’s check it for $U^-$, the proof for $U^+$ is the same. We will use the Proposition of Margulis. We need to show that if $\phi$ is a function in $L^2(X, m_X)$ invariant under the action of $U^-$, then it is constant a.e. We will prove that it is invariant under the action of a hyperbolic element $h = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, and then the claim follows from Proposition 1.11.

Naturally, we are going to use $L := U^-$. It remains to check the property (2). For any $\varepsilon > 0$ and $s, t \in \mathbb{R}$ we have

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + s\varepsilon & (1 + s\varepsilon)t + s \\ \varepsilon & 1 + t\varepsilon \end{pmatrix} =: M$$

Taking $s = 1/\varepsilon$ and $t = -1/(2\varepsilon)$ yields $M = \begin{pmatrix} 2 & 0 \\ \varepsilon & 1/2 \end{pmatrix}$. This implies (2), because the left-invariant metric is equivalent (locally bi-Lipschitz) to the norm metric. \qed

**References**