# FRACTALS (BAR-ILAN, SPRING 2018) – LECTURE 7 SUMMARY

BORIS SOLOMYAK

## 8. Section 8: Frostman's lemma

**Lemma 8.1** (Frostman)**.** *Let* $K \subset \mathbb{R}^d$ *be a Borel set with* $\mathcal{H}^\alpha(K) > 0$. *Then there exists a positive finite measure* $\mu$ *on* $K$ *such that*

$$\mu(B) \le \mathrm{const} \cdot |B|^\alpha \quad \textit{for all Borel sets } B.$$

The proof was given in the set when $K$ **is compact**. We spent considerable time on this in class – see Lecture 6 for the summary and [1, 3.1] for complete details.

## 9. Marstrand's product theorem

**Theorem 9.1** (Marstrand)**.** *Let* $X \subset \mathbb{R}^n$ *and* $Y \subset \mathbb{R}^m$ *be Borel sets. Then*

$$\dim_H(X) + \dim_H(Y) \le \dim_H(X \times Y) \le \dim_H(X) + \overline{\dim}_M(Y).$$

**Corollary 9.2.** *Suppose that either* $X$ *or* $Y$ *has the Hausdorff dimension equal to the Minkowski dimension, for example, one of the sets is self-similar. Then* $\dim_H(X \times Y) = \dim_H(X) + \dim_H(Y)$.

**Example 9.3.** For $\alpha \in (0, \frac{1}{2})$, let $C_\alpha$ be the self-similar Cantor set, which is the attractor of the IFS $f_1(x) = \alpha x,\ f_2(x) = (1-\alpha) + \alpha x$. Then $\dim_H(C_\alpha) = \dim_M(C_\alpha) = \frac{\log 2}{\log(1/\alpha)}$ and

$$\dim_H(C_\alpha \times C_\beta) = \dim_H(C_\alpha) + \dim_H(C_\beta).$$

**Example 9.4.** Let $S = \bigcup_{n=1}^\infty [(2n)!, (2n+1)!]$, and consider the sets $A_S$, $A_{S^c}$ defined by digit restrictions (in binary expansion). Then

$$0 = \dim_H(A_S) + \dim_H(A_{S^c}) < \dim_H(A_S \times A_{S^c}) = \dim_H(A_S) + \dim_M(A_{S^c}) = 1.$$

In order to show that $\dim_H(A_S \times A_{S^c}) \ge 1$ we observed that

$$A_S + A_{S^c} = \{x + y :\ x \in A_S,\ y \in A_{S^c}\} = [0,1],$$

and then that the map $(x,y) \mapsto x + y$ is Lipschitz, so doesn't increase Hausdorff dimension.

---

*Date*: April 29, 2018.

*Sketch of the proof of Theorem 9.1.* (i) For the lower bound we took arbitrary $\alpha < \dim_H(X)$ and $\beta < \dim_H(Y)$. Then $\mathcal{H}^\alpha(X) = \mathcal{H}^\beta(Y) = +\infty > 0$, hence by Frostman's Lemma, there exist finite positive Borel measures $\mu$ on $X$ and $\nu$ on $Y$ such that

$$\mu(A) \le |A|^\alpha \ \text{ for all } A \subset \mathbb{R}^n \quad \text{and} \quad \mu(B) \le |B|^\beta \ \text{ for all } B \subset \mathbb{R}^m.$$

Let $\eta = \mu \times \nu$ be the product measure. It is clearly a positive finite Borel measure on $X \times Y$, and it is easy to see that

$$\eta(D) \le |D|^{\alpha+\beta} \ \text{ for all } D \subset \mathbb{R}^{n+m}.$$

It follows that $\dim_H(X \times Y) \ge \alpha + \beta$, by the Mass Distribution Principle, and the desired estimate follows.

(ii) For the upper bound, we took arbitrary $\alpha > \dim_H(X)$ and $\beta > \overline{\dim}_M(Y)$. Then $\mathcal{H}^\alpha(X) = 0$, and so for any small $\varepsilon > 0$ we can find an $\varepsilon$-cover $\{D_i\}_{i=1}^\infty$ of $X$, with $\sum_i |D_i|^\alpha < \infty$. On the other hand, from $\beta > \overline{\dim}_M(Y)$ and the definition of the upper Minkowski dimension, it follows that for all $\rho > 0$ sufficiently small, there is a cover $\{B_j^\rho\}_1^{N_\rho}$ of $Y$ by sets with $|B_j^\rho| = \rho$, such that

$$N_\rho \le \rho^{-\beta}.$$

Denote $\rho_i = |D_i|$. It follows that, assuming $\varepsilon$ is small enough,

$$\{D_i \times B_j^{\rho_i} : \ 1 \le i < \infty, \ 1 \le j \le N_{\rho_i}\}$$

is a cover of $X \times Y$, by sets of diameters $\le 2|D_i| < 2\varepsilon$, and moreover,

$$\sum_{i,j} |D_i \times B_j^{\rho_i}|^{\alpha+\beta} \le 2^{\alpha+\beta} \sum_i |D_i|^{\alpha+\beta} \cdot \rho_i^{-\beta} = \text{const} \cdot \sum_i |D_i|^\alpha < \infty.$$

Thus $\dim_H(X \times Y) \le \alpha + \beta$, and the desired estimate follows. □

## References

[1] C. Bishop and Y. Peres, *Fractals in Probability and Analysis*, Cambridge University Press, 2017. *Final version is available at* `http://www.math.stonybrook.edu/~bishop/fractalbook_final.pdf`

[2] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, Wiley, first edition 1990; there were later editions.