LECTURE 11 SUMMARY

1. Hyperbolic toral automorphisms (cont.)

Let A be $d \times d$ integer matrix, with $\det A = \pm 1$. Consider $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (the d-dimensional torus).

Definition 1.1. The map $T_A(\mathbf{x}) = A\mathbf{x}$ (mod $\mathbb{Z}^d$) is called the toral automorphism associated with A.

Proposition 1.2. *T_A is a group automorphism; in fact, every automorphism of $\mathbb{T}^d$ has such a form.*

Definition 1.3. The automorphism is called hyperbolic if A is hyperbolic, that is, A has no eigenvalues of absolute value one.

Example 1.4 (Arnold's "cat map")**.** This is T_A for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Theorem 1.5 (see [1, §2.4])**.** *Every hyperbolic toral automorphism is chaotic.*

We will see the proof for $d = 2$, for simplicity, following [1, 2]. The proof will be broken into lemmas. The first one was already covered on December 27.

Lemma 1.6. *The points with rational coordinates $\{(\frac{m}{k}, \frac{n}{k}),\ m, n, k \in \mathbb{N}\}$ (mod $\mathbb{Z}^2$) are dense in $\mathbb{T}^2$, and they are all periodic for T_A.*

Lemma 1.7. *Let A be a hyperbolic 2×2 matrix. Thus it has two real eigenvalues: $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Then the eigenvalues are irrational and the eigenvectors have irrational slopes.*

Let E^s, E^u be the stable and unstable subspaces for A, respectively; they are one-dimensional and are spanned by the eigenvectors.

Consider the point $\mathbf{0} = (0, 0)$ on the torus; it is fixed by T_A. Let $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$ be the stable and unstable manifolds of T_A corresponding to $\mathbf{0}$. They are obtained by considering E^s (mod $\mathbb{Z}^2$) and E^u (mod $\mathbb{Z}^2$).

Lemma 1.8. *Consider $\mathbb{T}^2$ as the unit square $[0, 1]^2$ (with opposite sides identified). Let $t_0 = 0, t_1, t_2, \cdots,$ be the consecutive intersections of $W^u(\mathbf{0})$ with the "base" of the square $[0, 1]$. Then these points form the orbit of 0 under an irrational rotation by α, where α^{-1} is the slope of E^u.*

Lemma 1.9. *Every orbit of an irrational rotation $R_\alpha : t \mapsto t + \alpha$ (mod 1) is dense on the circle $[0, 1] \cong \mathbb{R}/\mathbb{Z}$, hence $W^u(\mathbf{0})$ is dense in $\mathbb{T}^2$. Similarly, $W^s(\mathbf{0})$ is dense. As a consequence, the set of homoclinic points for $\mathbf{0}$ is dense in $\mathbb{T}^2$.*

The statements about density may be made more quantitative.

Definition 1.10. A set F in a metric space X is called *ε-dense* if $\bigcup_{x\in F} B_\varepsilon(x) = X$.

Lemma 1.11. *Let $\alpha \notin \mathbb{Q}$. Then for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$ every orbit of R_α is ε-dense in $\mathbb{T}$.*

Corollary 1.12. *For any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that every segment of $W^u(\mathbf{0})$ of length $\geq M_\varepsilon$ is ε-dense in $\mathbb{T}^2$.*

Definition 1.13. A topological dynamical system (X, T) is called *topologically mixing* if for any $U, V \neq \emptyset$ open, there exists $n_0 \in \mathbb{N}$ such that

$$T^n U \cap V \neq \emptyset, \quad \text{for all } n \geq n_0.$$

Proposition 1.14. *A hyperbolic toral automorphism T_A is topologically mixing.*

Remark. Observe that T_A is NOT typologically exact.

Lemma 1.15. *If (X, T) is topologically mixing, then (X, T) has sensitive dependence on initial conditions.*

Proposition 1.16. *The following are equivalent for a continuous dynamical system (X, T) on a separable complete metric space:*
(i) *there is a dense orbit;*
(ii) *for any $U, V \neq \emptyset$ open, there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$.*
In this case we say that (X, T) is transitive.

Obviously, "topologically mixing" implies "transitive"; thus, combining all of the above we obtain that T_A is chaotic.

2. Strange attractors

Definition 2.1. Suppose $f : X \to X$ is a map. A compact set $K \subset X$ is called an *attractor* for f if there exists a neighborhood U of K such that $f(U) \subset U$ and $\bigcap_{n \in \mathbb{N}} f^n(U) = K$. We usually require K to have no proper subsets with the same property.

The simplest attractors are attracting fixed points. (Attracting limit cycles are attractors for *continuous* dynamical systems, but here we are talking about *discrete* dynamical systems.) "Strange" attractors are attractors that have a complicated *"fractal"* structure (we do not give a formal definition).

2.1. **Solenoid, or Smale attractor, see** [1, §2.5]**.**

Consider the solid torus $M := S^1 \times B^2$, where B^2 is the unit disk in $\mathbb{R}^2$ and S^1 is the unit circle. On it we define the coordinates (θ, p) such that $\theta \in S^1$ and $p \in B^2$, that is, $p = (x, y)$ with $x^2 + y^2 \leq 1$. Using these coordinates we define the map by doubling up and shrinking the thickness by 5.

Proposition 2.2. *The map*

$$f : M \to M, \quad f(\theta, p) = \Big(2\theta, \frac{1}{5}\, p + \frac{1}{2}\, e^{2\pi i\theta}\Big)$$

is well-defined and injective.

Theorem 2.3. *The map f has an attractor $K = \bigcap_{n \geq 0} f^n(M)$. Moreover, $f|_K$ is chaotic.*

Remark. A similar stretching and refolding procedure is used in practice: (a) candy machines making "taffy", which consists of molasses and sugar; the candy has "stringy" structure; (b) production of Japanese swords [2, p.333].

Denote $B(\theta^*) = \{e^{2\pi i\theta}\} \times B$. Observe that

$$f\big(B(\theta^*)\big) \subset B(2\theta^*) \quad \text{and} \quad f\big(B(\theta^* + 1/2)\big) \subset B(2\theta^*).$$

Proposition 2.4 (without proof)**.** (i) *The attractor K is connected, but not locally connected and not path-connected.*

(ii) *Locally, K is homeomorphic to the Cartesian product of a line segment and a Cantor set.*

2.2. **Hénon map.** Let $H = H_{a,b}$ be the map of the plane given by

$$H(x, y) = (a - by - x^2, x).$$

Observe that $\det(DH) = b$.

- For $b \neq 0$ the map H is invertible.
- For $0 < |b| < 1$ the map H is area-contracting.
- For $|b| = 1$ the map H is area-preserving.
- For $|b| < 1$ and $a > c(b)$ a non-linear horseshoe appears.
- For some parameters H has a "strange attractor"; the classical parameters are $b = 0.3$, $a = 1.4$, for which it was observed numerically.

References

[1] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Taylor and Francis, 2003 (2nd edition).

[2] B. Hassellblatt and A. Katok, *A First Course in Dynamics*, Cambridge, 2003.