LECTURE 5 SUMMARY

1. Van der Pol Equation

See Lecture 14 from lecture slides of Shlomo Sternberg:
http://www.math.harvard.edu/library/sternberg/

2. Poincaré-Bendixson Theorem

Consider $\mathbf{x}' = F(\mathbf{x})$ in $\mathbb{R}^d$. As usual, we assume that the conditions of existence and uniqueness theorem for systems of ODE's are satisfied.

- **ω-limit set of an orbit:** Let $C = \{ \mathbf{x}(t) \}$ be an orbit (trajectory) of the system. The ω-limit set of $C$ is the set of accumulation points, as $t \to +\infty$.

  $$\omega(C) = \{ y \in \mathbb{R}^d : \exists t_n \to \infty, \mathbf{x}(t_n) \to y \}.$$ 

  If instead we let $t \to -\infty$, we obtain the α-limit $\alpha(C)$.

- **Homoclinic orbit:** Let $\mathbf{x}^*$ be a fixed point. An orbit $C = \{ \mathbf{x}(t) \}$ is homoclinic, associated with $\mathbf{x}^*$, if $C$ is not constant and $\lim_{t \to \pm \infty} \mathbf{x}(t) = \mathbf{x}^*$.

- **Heteroclinic orbit:** Let $\mathbf{x}^1 \neq \mathbf{x}^2$ be two fixed points. An orbit $C = \{ \mathbf{x}(t) \}$ is heteroclinic, associated with $\mathbf{x}^1, \mathbf{x}^2$, if $\lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{x}^1$ and $\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^2$.

- **Invariant sets:** A set $Y \subset \mathbb{R}^d$ is called invariant if for any $\mathbf{y}_0 \in Y$ the solution $\mathbf{y}(t)$ of the initial value problem $\mathbf{y}' = F(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$, satisfies $\mathbf{y}(t) \in Y$ for all $t \in \mathbb{R}$. In other words, all trajectories starting in $Y$ stay in $Y$, both forward and backward. If they stay in $Y$ for $t > 0$, the set is called forward-invariant, and if they stay in $Y$ for $t < 0$, then the set is called backward-invariant.

**Proposition 2.1** (General properties of ω-limit sets). Let $C = \{ \mathbf{x}(t) \}$ be an orbit of the system.

(i) $\omega(C)$ is a closed subset of $\mathbb{R}^d$;
(ii) $\omega(C)$ is invariant;
(iii) if $\omega(C)$ is bounded, then it is connected.

**Theorem 2.2** (Poincaré-Bendixson). Consider $\mathbf{x}' = F(\mathbf{x})$ in $\mathbb{R}^2$ and an orbit $C$.

(i) If $C$ is forward-bounded and $\omega(C)$ contains no fixed points, then either (a) $C = \omega(C)$ is itself a periodic orbit, or (b) $\omega(C)$ is a periodic orbit (limit cycle), which $C$ approaches spirally, from inside or from outside.

(ii) Suppose that the system has finitely many fixed points and $\omega(C)$ contains a fixed point $p$. Then (a) if $\omega(C)$ consists only of fixed points, then in fact $\omega(C) = \{ p \}$; (b) otherwise, $\omega(C)$ consists of a union of fixed points and finitely many homoclinic and heteroclinic orbits connecting them.

For the proof of the Proposition and the Theorem see Lecture 14 from lecture slides of Shlomo Sternberg: http://www.math.harvard.edu/library/sternberg/
3. Bifurcations for systems in $\mathbb{R}^2$

3.1. Saddle-node, transcritical, and pitchfork. These are similar to the ones in $\mathbb{R}$.

Representative examples:

saddle-node: \[
\begin{aligned}
x' &= \mu - x^2 \\
y' &= -y
\end{aligned}
\]

transcritical: \[
\begin{aligned}
x' &= \mu x - x^2 \\
y' &= -y
\end{aligned}
\]

supercritical pitchfork: \[
\begin{aligned}
x' &= \mu x - x^3 \\
y' &= -y
\end{aligned}
\]

subcritical pitchfork: \[
\begin{aligned}
x' &= \mu x + x^3 \\
y' &= -y
\end{aligned}
\]

3.2. Hopf bifurcation. Occurs when the eigenvalues of the Jacobian at the fixed point are complex conjugate and cross the imaginary axis at the bifurcation parameter value. The fixed point changes from a stable focus to an unstable focus, and in addition (if some other conditions are satisfied), there appears a limit cycle. If the limit cycle is stable and the fixed point is unstable, we call it the supercritical Hopf bifurcation. If, on the other hand, the limit cycle is unstable and the fixed point is stable, we call it the subcritical Hopf bifurcation.

Example 3.1. Let

(i) \[
\begin{aligned}
x' &= -y + (a - x^2 - y^2) \\
y' &= x + (a - x^2 - y^2)
\end{aligned}
\]

(ii) \[
\begin{aligned}
x' &= -y + (a + x^2 + y^2) \\
y' &= x + (a + x^2 + y^2)
\end{aligned}
\]

The system undergoes a Hopf bifurcation at $a = 0$: (i) supercritical, (ii) supercritical.

Example 3.2 (From materials of J. Schiff). Consider the Van der Pol equation

\[x'' + \mu(x^2 - 1)x' + x = 0.\]

Supercritical Hopf bifurcation occurs at $\mu = 0$. Figure 1 on the next page shows phase portraits for $\mu = -1, -0.5, 0, 0.2, 0.5, 1$.

3.3. Homoclinic bifurcation.

Example 3.3 (From materials of J. Schiff). Consider the system

\[
\begin{aligned}
x' &= y \\
y' &= ay + x - x^2 + xy
\end{aligned}
\]

There are two fixed points for all $a$: $(0,0)$ and $(1,0)$. Moreover, $(0,0)$ is always a saddle, and there is a homoclinic orbit associated with it. The fixed point $(1,0)$ is a stable node for $a < -3$, becomes stable focus for $-3 < a < -1$, and then unstable focus for $-1 < a < 1$ and unstable node for $a > 1$. The global behavior is hard to analyze without a computer. From numerics we deduce:

- at $a = -1$ a supercritical Hopf bifurcation occurs;
- at $a \approx -0.865$ the limit cycle “collides” with the fixed point at $(0,0)$, momentarily becoming a homoclinic orbit, and then disappears for good — this is called homoclinic bifurcation.

Figure 2 on the next page shows phase portraits for $a = -1.1, -1, -0.9, -0.865, -0.86, -0.82$. 
Figure 1. *Hopf bifurcation in the Van der Pol equation* (figures by Jeremy Schiff)

Figure 2. *Hopf and Homoclinic bifurcations* (figures by Jeremy Schiff)