LECTURE 7 SUMMARY

1. Discrete dynamical systems — introduction

- Let \( f : X \to X \) be a map of a space into itself (usually, a metric space). Given \( x_0 \in X \), the orbit of \( x_0 \) under \( f \) is the sequence \( \{x_n\}_{n=0}^{\infty} \), where \( x_{n+1} = f(x_n) \). The iteration of \( f \) is often denoted as follows: \( f^n = f \circ f \circ \cdots \circ f \) (\( n \) times). In Dynamical Systems theory, one is usually interested in the long-term behavior of the orbits. Do they exhibit regularity? Converge to a stable or periodic behavior? Or are they chaotic, unpredictable?

- A point \( x^* \) is a fixed point for \( f \) if \( f(x^*) = x^* \). Fixed points represent equilibria, from the physics point of view. Fixed points may be attracting or unstable (repelling). There are slightly different definitions in the literature, but the following is one standard version.

- A fixed point \( x^* \) for \( f : X \to X \) on a metric space is attracting if there is a neighborhood \( B_{\delta}(x^*) \) such that for all \( x_0 \in B_{\delta}(x^*) \), the orbit converges to \( x^* \): \( \lim_{n \to \infty} f^n(x_0) = x^* \).

- A fixed point \( x^* \) for \( f : X \to X \) on a metric space is repelling if there is a neighborhood \( B_{\delta}(x^*) \) such that for all \( x_0 \in B_{\delta}(x^*) \), \( x_0 \neq x^* \), there exists \( n \) such that \( f^n(x_0) \not\in B_{\delta}(x^*) \).

- There are other possibilities for fixed points as well: for instance, \( x^* \) may have a neighborhood such that the orbits starting there neither converge to \( x^* \), nor “escape” (I don’t make it precise). Sometimes such points are also called stable, but not asymptotically stable. On the plane, or in higher directions, the fixed point can also be a saddle point: attracting in some directions, but repelling in other directions.

- **Theorem.** Suppose that \( f : [a,b] \to [a,b] \) is a continuous function. Then \( f \) has at least one fixed point in \([a,b]\).

  The proof is easy, using the Intermediate Value Theorem.

- **Proposition.** Suppose that \( f : [a,b] \to [a,b] \) is a continuous function and \( x^* \in (a,b) \) is a fixed point. Suppose also that \( f \) is increasing on an interval \((x^*-\varepsilon,x^*+\varepsilon)\) and

  \[
  f(x) > x \quad \text{for all } x \in (x^*-\varepsilon,x^*)
  \]

  \[
  f(x) < x \quad \text{for all } x \in (x^*,x^*+\varepsilon).
  \]

  Then \( f^n(x_0) \to x^* \) for all \( x_0 \in (x^*-\varepsilon,x^*+\varepsilon) \), hence \( x^* \) is attracting.

  There are similar claims in the cases when (a) \( x^* \) is repelling; (b) \( x^* \) is attracting on one side and repelling on the other side.
• **Theorem.** Suppose that \( f : U \to U \) is a differentiable function, where \( U \) is an interval in \( \mathbb{R} \). Suppose that \( z^* \) is an interior point of \( U \) and \( f(z^*) = z^* \). Then
  
  (i) if \( |f'(z^*)| < 1 \), then \( z^* \) is an attracting fixed point;
  (ii) if \( |f'(z^*)| > 1 \), then \( z^* \) is a repelling fixed point.

  If \( |f'(z^*)| = 1 \), there is not enough information; however, if we know something about the behavior of \( f \) near \( z^* \), we may be able to determine the nature of the fixed point, for example, using the proposition above or similar considerations. If \( f'(z^*) = 1 \) and \( f \) has higher derivatives, we can look at the Taylor expansion near \( z^* \).

• A point \( x_0 \) is called **periodic, of period** \( k \) for \( f \), if \( f^k(x_0) = x_0 \). If \( k \) is minimal such number, then we say that \( x_0 \) has **minimal period** \( k \), and the orbit \( \{x_0, \ldots, x_{k-1}\} \) is called a \( k \)-cycle (note that \( x_k = x_0 \) in this orbit). If a point \( x_0 \) has period \( k \) for \( f \), then it is a fixed point for \( f^k \). The definition of attracting/repelling is extended to periodic points — simply pass from \( f \) to \( f^k \). For instance, a \( k \)-cycle is attracting if all nearby orbits approach this cycle in the limit.

• Stability of periodic points for differentiable functions can be checked by the theorem above, also passing from \( f \) to \( f^k \). For this purpose, the following lemma is useful, which is immediate from the Chain Rule:

• **Lemma.** Suppose that \( f : U \to U \) is a differentiable function, where \( U \) is an interval in \( \mathbb{R} \). Suppose that \( \{z_0, \ldots, z_{k-1}\} \) is a \( k \)-cycle in the interior of \( U \). Then
  
  \[
  |(f^k)'(z_0)| = |f'(z_0)| \cdots |f'(z_{k-1})|.
  \]

2. **Chaotic systems (see [1, 2])**

What is “chaos”? There is no one, universally agreed upon, definition, but there are several features which are usually considered to be associated with chaotic systems. One of the commonly used definitions, due to R. Devaney, is as follows.

**Definition 2.1.** Let \( f : X \to X \) be a continuous “onto” map on a compact metric space. We say that \( f \) has **has sensitive dependence on initial conditions** if there is a \( \beta > 0 \) such that for any \( x \in X \) and any \( \varepsilon > 0 \) there exists \( k \) such that the distance between \( f^k(x) \) and \( f^k(y) \) is at least \( \beta \).

The dynamical system \((X, f)\) is called **chaotic** if the following three properties are satisfied:

(i) periodic points are dense in \( X \);
(ii) there is a dense orbit;
(iii) \( f \) has sensitive dependence on initial conditions.

**Example 2.2 (Doubling function).** The doubling function is defined on \([0, 1)\) as follows:

\[
D(x) = \begin{cases} 
2x, & 0 \leq x < 1/2 \\
2x - 1, & 1/2 \leq x < 1.
\end{cases}
\]
This function has a discontinuity at 1/2, but we actually want to consider it on the circle, identifying 0 and 1, and then it becomes continuous. It is possible to view it on the circle $S^1$ as “angle-doubling map” $\theta \mapsto 2\theta$, or on the complex plane: $z \mapsto z^2$, $|z| = 1$.

We will show that this doubling map is chaotic in the sense of Devaney with the help of the binary expansion: for $x \in [0, 1)$ let

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n \in \{0, 1\}, \quad x = 0.b_1b_2b_3\ldots$$

Then

$$D(x) = 0.b_2b_3b_4\ldots$$

acts simply as a “shift”.

References