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ERGODIC THEORY — LECTURE 4 SUMMARY

1. Weak mixing

Definition 1.1. Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. It is called *weakly mixing* if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(T^{-j}A \cap B) - \mu(A)\mu(B)| = 0.$$

Remark 1.2. Note that mixing (which requires that $\mu(T^{-n}A \cap B) - \mu(A)\mu(B) \to 0$ for all $A, B \in \mathcal{B}$) implies mixing. We have seen (follows from Homework 2, #3) that weak mixing implies ergodicity. As we will see, the converse is not true; e.g., irrational rotation is ergodic but not weakly mixing. As for examples of weakly mixing, but not mixing systems, there are many — both "generic" and "specific", but this is a bit harder to prove.

The *density* of a subset of $\mathbb{N}$ is defined by $d(J) := \lim_{n\to\infty} \frac{1}{n}\#\{j \in J : \ j \le n\}$, assuming the limit exists.

Theorem 1.3. *Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. It is weakly mixing if and only if for all $A, B \in \mathcal{B}$ there exists $J = J_{A,B} \subset \mathbb{N}$, with $d(J) = 0$, such that*

$$\lim_{J \not\ni n \to\infty} \mu(T^{-n}A \cap B) = \mu(B)\mu(B).$$

This follows immediately from

Lemma 1.4 (Koopman-von Neumann). *Let $0 \le a_n \le K$ be a sequence of positive real numbers. Then the following are equivalent:*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j = 0$*;*

(ii) *there exists $J = J(\{a_n\}) \subset \mathbb{N}$ of density zero, such that $a_n \to 0$ for $n \notin J$, $n \to \infty$.*

(iii) $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j^2 = 0$

Theorem 1.5. *Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. The following are equivalent:*

(i) *T is weakly mixing.*

(ii) *T has continuous spectrum (which means, by definition, that T has no non-constant measurable eigenfunctions).*

(iii) *$T \times T$ is ergodic.*

(iv) *$T \times T$ is weakly mixing.*

Scheme of the proof. The implications (iv) ⇒ (i) and (iv) ⇒ (iii) are obvious. For the impication (i) ⇒ (iv) we use the fact that it is sufficient to check the definition for sets from a generating semi-algebra, and then work with measurable rectangles and Theorem 1.3. For the implication (i) ⇒ (ii) we argue by contradiction and make use of the following Lemma.

Lemma 1.6. *Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. and $U = U_T$ the corresponding Koopman operator on L^2_0. The following are equivalent:*
(i) *T is weakly mixing.*
(ii) *for any $f, g \in L^2_0(X, \mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\langle U^j f, g\rangle| = 0.$$

(ii) *for any $f \in L^2_0(X, \mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\langle U^j f, f\rangle| = 0.$$

For the remaining implication (ii) ⇒ (i) we use Spectral Theory (see Hand-out II): let $f \in L^2_0(X, \mu)$; we want to show that the spectral measure $\nu := \sigma_f$ is continuous (that is, every point has measure zero). We have $\widehat{\nu}(n) = \widehat{\sigma}_f(n) = \langle U^n f, f\rangle$. By Lemma 1.6, weak mixing implies $\frac{1}{n}\sum_{j=0}^{n-1} |\widehat{\nu}(j)| \to 0$, which in view of Lemma 1.4 is equivalent to $\frac{1}{n}\sum_{j=0}^{n-1} |\widehat{\nu}(j)|^2 \to 0$. It remains to apply the following classical

Lemma 1.7 (Wiener)**.** *Let ν be a positive finite measure on the circle $\mathbb{T}$, and $\nu(n) = \int_{\mathbb{T}} z^n \, d\nu(z)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\widehat{\nu}(j)| = \sum_{x\in\mathbb{T}} \nu(\{x\})^2.$$

□