Lecture 5: Hyperbolic plane, geodesic and horocyclic flows

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- Hyperbolic plane: $\mathbb{H} = \{x + iy \in \mathbb{C} : \ y > 0\}$.
- Tangent bundle:

$$\mathrm{T}\mathbb{H} = \mathbb{H} \times \mathbb{C} = \bigsqcup_{z \in \mathbb{C}} \mathrm{T}_z\mathbb{H},$$

  where

$$\mathrm{T}_z\mathbb{H} = \{z\} \times \mathbb{C} \quad \text{(the tangent plane at } z\text{)}.$$

- If $\phi : [0, 1] \to \mathbb{H}$ is differentiable, then

$$D\phi(t) = (\phi(t), \phi'(t)) \in \mathrm{T}_z\mathbb{H}, \quad \text{where } z = \phi(t).$$

Hyperbolic Riemannian metric

- *Inner product in the tangent planes:*

$$\langle \mathbf{v}, \mathbf{w} \rangle_z = \frac{1}{y^2}(\mathbf{v} \cdot \mathbf{w}), \quad \text{where} \quad z = x + iy \in \mathbb{H}, \quad \mathbf{v}, \mathbf{w} \in \mathrm{T}_z\mathbb{H}.$$

Here $(\mathbf{v} \cdot \mathbf{w}) = \mathrm{Re}(\mathbf{v}\,\overline{\mathbf{w}})$ is the usual inner product under the identification $\mathbb{C} \cong \mathbb{R}^2$.

- *Length of a path (piecewise differentiable curve)*:
for a differentiable $\phi : [0, 1] \to \mathbb{H}$,

$$L(\phi) = \int_0^1 \|D\phi(t)\|_{\phi(t)}\, dt = \int_0^1 \frac{|\phi'(t)|}{\mathrm{Im}\,\phi(t)}\, dt.$$

- For $\phi(t) = x(t) + iy(t)$ we have $\|D\phi(t)\|_{\phi(t)} = \frac{\sqrt{x'(t)^2+y'(t)^2}}{y(t)}$.

Hyperbolic metric

- *Hyperbolic metric on $\mathbb{H}$:*

$$\mathrm{d}(z_0, z_1) := \inf\{L(\phi) : \ \phi \text{ is a path connecting } z_0 \text{ and } z_1\}.$$

Lemma 1

- *The hyperbolic metric on $\mathbb{H}$ is indeed a metric.*
- *This metric induces the same topology on $\mathbb{H} \subset \mathbb{C}$ as the Euclidean norm.*

The proof is left as an *exercise*.

Boundary of the hyperbolic plane: $\partial\mathbb{H} := \mathbb{R} \cup \{\infty\}$.

The distance from any point $z \in \mathbb{H}$ to any point $\alpha \in \partial\mathbb{H}$ is *infinite*, where

$$\begin{aligned} \mathrm{dist}(z, \alpha) \ &:= \ \inf\{L(\phi) : \ \phi : [0,1] \to \mathbb{H} \cup \partial\mathbb{H} \text{ is a path}, \\ &\quad \phi(t) \in \mathbb{H}, \ t \in [0,1), \ \phi(0) = z, \ \phi(1) = \alpha\}. \end{aligned}$$

Isometric action on $\mathbb{H}$

- The group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by the Möbius transformations:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; z \mapsto \frac{az+b}{cz+d}\,.$$

- Note that $cz + d \neq 0$ for $z \in \mathbb{H}$ since $c, d \in \mathbb{R}$.
- $\mathrm{Im}\, g(z) = \mathrm{Im} \frac{az+b}{cz+d} = \mathrm{Im} \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \mathrm{Im} \frac{(ad-bc)y}{|cz+d|^2} = \frac{y}{|cz+d|^2}$, thus $g(\mathbb{H}) \subseteq \mathbb{H}$.
- Why action? Consider projectively $z \sim \begin{pmatrix} z \\ 1 \end{pmatrix}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \sim \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} \sim \frac{az+b}{cz+d}$$

Isometric action on $\mathbb{H}$

- Since $-I_2$ (minus the identity matrix) acts trivially, we actually have an action of $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I_2\}$.
- *Derivative action of $PSL_2(\mathbb{R})$ on $\mathrm{T}\mathbb{H}$:* for $g \in PSL_2(\mathbb{R})$ we have $Dg : \mathrm{T}\mathbb{H} \to \mathrm{T}\mathbb{H}$:
$$Dg(z, \mathbf{v}) = (g(z), g'(z)\mathbf{v}).$$
- For a fixed $z \in \mathbb{H}$ we get a linear map $(Dg)_z : \mathrm{T}_z\mathbb{H} \to \mathrm{T}_{g(z)}\mathbb{H}$, that is, $\mathbf{v} \mapsto g'(z)\mathbf{v}$.

Isometric action on $\mathbb{H}$ (cont.)

Lemma 2

The actions defined above have the following properties:

1) $(Dg)_z : \mathrm{T}_z\mathbb{H} \to \mathrm{T}_{g(z)}\mathbb{H}$ *preserves the Riemannian metric, and the action of* $PSL_2(\mathbb{R})$ *on* $\mathbb{H}$ *preserves the hyperbolic metric, that is,*

$$\mathrm{d}(g(z_0), g(z_1)) = \mathrm{d}(z_0, z_1);$$

Proof of the 1st claim; the 2nd claim is an exercise.

$Dg(z, \mathbf{v}) = (g(z), g'(z)\mathbf{v})$. We have

$$g'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

$$\|\mathbf{v}\|_z = \frac{|\mathbf{v}|}{y}, \quad \|g'(z)\mathbf{v}\|_{g(z)} = \frac{|g'(z)||\mathbf{v}|}{\mathrm{Img}(z)} = \frac{|cz+d|^{-2}|\mathbf{v}|}{y/|cz+d|^2} = \|\mathbf{v}\|_z.$$

□

Summary of useful formulas

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g(z) = \frac{az+b}{cz+d},$$

$$\boxed{g'(z) = \frac{1}{(cz+d)^2}}$$

Let $z = x + iy$. Then

$$\operatorname{Re} g(z) = \frac{ac|z|^2 + bd + (ad+bc)x}{|cz+d|^2},$$

$$\boxed{\operatorname{Im} g(z) = \frac{y}{|cz+d|^2}}$$

Definition 3

An action of a group G on a set X is *transitive* if for every $x_1, x_2 \in X$ there exists $g \in G$ such that $g \cdot x_1 = x_2$. If such g is unique for all x_1, x_2, we say that the action is *simply transitive*.

Lemma 4

2) *The action of $PSL_2(\mathbb{R})$ on $\mathbb{H}$ is transitive;*

Proof.

Enough to show that for any $z = x + iy, y > 0$, there exists $g \in PSL_2(\mathbb{R})$ such that $g(i) = z$.
We can take

$$g = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \quad \frac{\sqrt{y}i + x/\sqrt{y}}{1/\sqrt{y}} = iy + x = z.$$

□

Lemma 5

3) *The stabilizer of i under this action is*

$$\mathrm{Stab}(i) := \{g \in PSL_2(R) : \ g(i) = i\} = PSO(2) = SO(2)/\{\pm I_2\},$$

where

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Proof.

- $g(i) = i \implies |ci + d| = 1$ (recall that $\mathrm{Img}(z) = \frac{y}{|cz+d|^2}$). Thus we can find $\theta \in \mathbb{R}$ such that $c = \sin\theta$ and $d = \cos\theta$.
- Then

$$\frac{ai + b}{i \sin\theta + \cos\theta} = i \implies ai + b = -\sin\theta + i\cos\theta.$$

□

Isometric action on $\mathbb{H}$ (cont.)

- Consider the unit tangent bundle:

$$\mathrm{T}^1\mathbb{H} = \{(z, \mathbf{v}) \in \mathrm{T}\mathbb{H} : \ \|\mathbf{v}\|_z = 1\}.$$

- The group $PSL_2(\mathbb{R})$ acts on $\mathrm{T}^1\mathbb{H}$ because it preserves the Riemannian metric.

Lemma 6

- *The action of $PSL_2(\mathbb{R})$ on $\mathrm{T}^1\mathbb{H}$ is simply transitive.*
- *We thus have $PSL_2(\mathbb{R}) \cong \mathrm{T}^1\mathbb{H}$.*
- *The standard identification is $g \mapsto Dg(i, i)$.*

Action on $\mathrm{T}^1\mathbb{H}$.

Proof.

- Since the action on $\mathbb{H}$ is transitive, it is enough to consider $g \in PSO(2)$ and $\mathbf{v} \in T_i\mathbb{H}$ with base point i.
- We compute

$$(Dg)_i(\mathbf{v}) = (i \sin\theta + \cos\theta)^{-2}\mathbf{v} = (\cos(2\theta) - i\sin(2\theta))\mathbf{v}.$$

  So by varying θ, $(Dg)_i(\mathbf{v})$ could be any vector of modulus 1.
- To see that transitivity is simple, notice that $(Dg)_i(\mathbf{v}) = \mathbf{v}$ implies $2\theta \equiv 0 \pmod{2\pi}$, so $\theta \in \mathbb{Z}\pi$ and $g = \pm I_2$.

□

Geodesics

Lemma 7

Let $z_0 = y_0 i, z_1 = y_1 i;\ 0 < y_0 < y_1$. *Then*

(i) $d(z_0, z_1) = \ln y_1 - \ln y_0$;

(ii) $\phi(t) = y_0(\frac{y_1}{y_0})^t i,\ t \in [0,1]$, *defines a path from* z_0 *to* z_1 *with constant speed* $\ln y_1 - \ln y_0$, *so* $L(\phi) = d(z_0, z_1)$;

(iii) *Moreover,* ϕ *is unique up to parametrization: if* $\psi : [0,1] \to \mathbb{H}$ *has* $L(\psi) = d(z_0, z_1)$, *then* $\psi = \phi \circ f$ *for some increasing* $f : [0,1] \to [0,1]$.

Proof.

$$d(z_0, z_1) \le L(\phi) = \int_0^1 \frac{|\phi'(t)|}{\mathrm{Im}\phi(t)}\, dt = \int_0^1 \frac{y_0(\frac{y_1}{y_0})^t \ln \frac{y_1}{y_0}}{y_0(\frac{y_1}{y_0})^t}\, dt = \ln y_1 - \ln y_0.$$

□

Geodesics (cont.)

Continuation of the proof.

- Suppose $\eta : [0,1] \to \mathbb{H}$ is another path from z_0 to z_1. Let $\eta(t) = \eta_x(t) + i\eta_y(t)$. Then

$$\begin{aligned} L(\eta) &= \int_0^1 \frac{\|\eta'(t)\|_2}{\eta_y(t)}\,dt \ge \int_0^1 \frac{|\eta_y'(t)|}{\eta_y(t)}\,dt \\ &\ge \int_0^1 \frac{\eta_y'(t)}{\eta_y(t)}\,dt = \ln(\eta_y(1)/\eta_y(0)) = \ln y_1 - \ln y_0. \end{aligned}$$

  Thus, $d(z_0, z_1) = \ln(y_1/y_0)$.

- Equality in the formula above implies $\eta_x'(t) = 0$ for all $t \in [0,1]$ and $\eta_y'(t) \ge 0$.

$\square$

Geodesics (cont.)

Definition 8

- **Geodesic path** is a path of minimal length joining two points.
- **Geodesic** (or geodesic curve) is a curve whose every segment is a geodesic path.

We showed that the vertical half-line $\{yi : \ y > 0\}$ is a geodesic in $\mathbb{H}$.

- From now on, we will always parametrize geodesic paths to have constant speed 1, so the domain will be $[0, d(z_0, z_1)]$.
- By the lemma, the unique path of unit speed from $z_0 = iy_0, z_1 = iy_1$ is

$$\phi(t) = y_0 e^t i.$$

- It is clear that an isometry $g \in PSL_2(\mathbb{R})$ sends a geodesic path (or curve) to another geodesic path (or curve).

Lemma 9

1. *For any two distinct points $z_0, z_1 \in \mathbb{H}$ there is a unique path $\phi : [0, d(z_0, z_1)] \to \mathbb{H}$ of unit speed connecting z_0 to z_1.*
2. *Moreover, there is a unique isometry $g \in PSL_2(\mathbb{R})$ such that $\phi(t) = g(e^t i)$.*

Claim. *For any $z_0 \neq z_1$ in $\mathbb{H}$ there exists $g \in PSL_2(\mathbb{R})$ with $g^{-1}(z_0) = i$ and $g^{-1}(z_1) = iy_1$ for some $y_1 > 1$.*

Geodesics (cont.)

Proof of the claim.

- By transitivity, we can find $\widetilde{g} \in PSL_2(\mathbb{R})$ with $\widetilde{g}^{-1}(z_0) = i$.
- Then also $h\widetilde{g}^{-1}(z_0) = i$ for any $h \in PSO(2)$.
- Let $h = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\widetilde{g}^{-1}(z_1) = \widetilde{x}_1 + i\widetilde{y}_1$.
- Then $g^{-1} = h\widetilde{g}^{-1}$ satisfies

$$\begin{aligned} \mathrm{Re}\, g^{-1}(z_1) &= \mathrm{Re}\, h(\widetilde{z}_1) = \frac{\frac{1}{2}\sin(2\theta)(|\widetilde{z}_1|^2 - 1) + \cos(2\theta)\widetilde{x}_1}{|cz+d|^2} \\ &= 0 \iff \tan(2\theta) = \frac{2\widetilde{x}}{1 - |\widetilde{z}_1|^2}, \end{aligned}$$

so we get two solutions $\theta_1 \in (-\frac{\pi}{4}, \frac{\pi}{4}]$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$. One of the corresponds to $y_1 > 1$. □

Geodesics (cont.)

Proof of the lemma.

- $\phi_0(t) = e^t i$ is a unique geodesic path of unit speed from i to iy_1, hence $\phi(t) = g(e^t i)$ is the unique geodesic path of unit speed from z_0 to z_1.
- Finally, we claim that $g \in PSL_2(\mathbb{R})$ is unique with this property. This follows from Lemma 7(iii).

□

Proposition 10

The geodesics in $\mathbb{H}$ are precisely vertical half-lines and upper half-circles orthogonal to $\mathbb{R}$ (equivalently, with centers in $\mathbb{R}$).

Proof sketch.

It is well-known that Möbius transformations map lines and circles into lines and circles. This can also be seen as follows:

- Check that this holds for the maps $z \mapsto z + b,\ b \in \mathbb{R}$, $z \mapsto az,\ a > 0$, and $z \mapsto -\frac{1}{z}$, which generate the group of Möbius transformations.

□

$\mathbb{H}$ as a model of non-Euclidean geometry

- The points and geodesics in $\mathbb{H}$ satisfy all the classical axioms of geometry apart from the parallel axiom, thus showing that the parallel axiom is not a consequence of other axioms.
- For instance: for any two different points in $\mathbb{H}$ there is a unique geodesic through them, and any two different geodesic intersect in at most one point. But
- For any geodesic ℓ and $z \in \mathbb{H} \setminus \ell$ there are infinitely many geodesics through z that do not intersect ℓ and are therefore "parallel" to ℓ.

Geodesic flow

- A geodesic ℓ is uniquely determined by a base point $z \in \mathbb{H}$ and a unit vector $\mathbf{v} \in T_z^1\mathbb{H}$ by the requirement that ℓ passes through z in the direction of $\mathbf{v}$.
- In fact, there is a unique $g \in PSL_2(\mathbb{R})$ with $Dg(i,i) = (z, \mathbf{v})$, and ℓ is the image of $\{iy : \ y > 0\}$ under the Möbuis transformation corresponding to g.
- Moreover, the unit speed parametrization of the geodesic ℓ starting at z is $g(e^t i)$.

Definition 11

- The *geodesic flow* $\Gamma_t : T^1\mathbb{H} \to T^1\mathbb{H}$ is defined by following the uniquely defined geodesic, as above, for time t.
- It is called a "flow" because it is a continuous-time dynamical system, or $\mathbb{R}$-action $\{\Gamma_t\}_{t\in\mathbb{R}}$.

Geodesic flow (cont.)

- We have $\Gamma_t(i,i) = (e^t i, e^t i)$, where we write $\Gamma_t(z,\mathbf{v})$ for the action of the geodesic flow.
- Hence

$$\Gamma_t(i,i) = D\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}(i,i),$$

where D is the derivative action: $Dg(z,\mathbf{v}) = (g(z), g'(z)\mathbf{v})$.

- Therefore, for an arbitrary $(z,\mathbf{v}) = g(i,i)$ we get

$$\Gamma_t(z,\mathbf{v}) = Dg(\Gamma_t(i,i)) = Dg\left(D\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}(i,i)\right) = D(ga_t^{-1})(i,i),$$

where $a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$.

Geodesic flow (cont.)

$$\begin{array}{ccc} \mathrm{T}^1\mathbb{H} & \xrightarrow{\Gamma_t} & \mathrm{T}^1\mathbb{H} \\ {\scriptstyle D|_{(i,i)}}\Big\uparrow & & \Big\uparrow{\scriptstyle D|_{(i,i)}} \\ PSL_2(\mathbb{R}) & \xrightarrow{R_{a_t}} & PSL_2(\mathbb{R}) \end{array}$$

Here

$$R_{a_t}(g) = g a_t^{-1}, \quad a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}.$$

- Note that the geodesic flow on $\mathrm{T}^1\mathbb{H} \cong PSL_2(\mathbb{R})$ acts by *right multiplication*, whereas the derivative action on $\mathrm{T}^1\mathbb{H} \cong PSL_2(\mathbb{R})$ corresponds to *left multiplication*.

- **Intuition:** the horocycle flows on $\mathrm{T}^1\mathbb{H}$ move along the stable $W^-(z_0,\mathbf{v}_0)$ and unstable $W^+(z_0,\mathbf{v}_0)$ manifolds for the geodesic flow:

$$W^\pm(z_0,\mathbf{v}_0) := \left\{(z,\mathbf{v}) \in \mathrm{T}^1\mathbb{H} : \ d_G(\Gamma_t(z_0,\mathbf{v}_0),\Gamma_t(z,\mathbf{v})) \xrightarrow{t\to\pm\infty} 0\right\}$$

- The problem: we haven't defined the metric d_G on $\mathrm{T}^1\mathbb{H}$ yet!
- **Observation:** the geodesics

$$\Gamma_t(i,i) = (e^t i, e^t i) \ \text{ and } \ \Gamma_t(x+i,i) = (x+e^t i, e^t i)$$

move parallel to each other, and the distance between their base points tends to 0. Indeed,

$$\mathrm{d}(e^t i, x+e^t i) \leq L([e^t i, x+e^t i]) = \int_0^{|x|} \frac{1}{e^t}\,dt = \frac{|x|}{e^t}.$$

Horocycle flows (cont.)

- The set $\{(x+i, i) : \ x \in \mathbb{R}\} \subseteq \mathrm{T}^1\mathbb{H}$ is precisely the orbit of (i, i) under the subgroup

$$U^- := \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} : \ s \in \mathbb{R} \right\}.$$

- More generally, for any $(z, \mathbf{v}) = Dg(i, i)$ we define the *stable horocycle flow* on $\mathrm{T}^1\mathbb{H}$ by

$$u^-(s)(z, \mathbf{v}) = D\left(g \begin{pmatrix} 1 & -s \\ & 1 \end{pmatrix}\right)(i, i),$$

with the corresponding flow on $PSL_2(\mathbb{R})$, given by

$$R_{u^-(s)}(h) = hu^-(-s), \quad h \in PSL_2(\mathbb{R}), \quad u^-(s) = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \in U^-.$$

- Now we apply an arbitrary Möbius transformation to obtain: *horocycles* are lines parallel to $\mathbb{R}$ and circles tangent to $\mathbb{R}$.

Horocycle flows (cont.)

- The *unstable horocycle flow* on $\mathrm{T}^1\mathbb{H}$ can be obtained from the stable one by reversing the direction of the tangent vector $(z, \mathbf{v}) \mapsto (z, -\mathbf{v})$.
- This corresponds to the action of $w(z) = -\frac{1}{z}$ at (i, i), hence to $g \mapsto g \circ w$ for Möbius transformations, or

$$g \mapsto gW, \quad g \in PSL_2(\mathbb{R}), \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim w(z).$$

- Thus the unstable horocycle flow is given by

$$g \mapsto gWu^-(-s)W^{-1} = gu^+(s), \quad \text{where} \quad u^+(s) = \begin{pmatrix} 1 & \\ s & 1 \end{pmatrix},$$

where we used $\begin{pmatrix} 1 & \\ s & 1 \end{pmatrix} = W \begin{pmatrix} 1 & -s \\ & 1 \end{pmatrix} W^{-1}$.

Horocycle flows (cont.)

- Denote the horocycle flows on $\mathrm{T}^1\mathbb{H}$ by $H^{\pm}(z, \mathbf{v})$.
- We obtain for the stable horocycle flow H^-:

$$\begin{array}{ccc} \mathrm{T}^1\mathbb{H} & \xrightarrow{H^- s} & \mathrm{T}^1\mathbb{H} \\ D|_{(i,i)} \uparrow & & \uparrow D|_{(i,i)} \\ PSL_2(\mathbb{R}) & \xrightarrow{R_{u^-(s)}} & PSL_2(\mathbb{R}) \end{array}$$

where $u^-(s) = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$.

- Similarly for the unstable horocycle flow H^+.

Bibliography

M. Einsiedler and T. Ward, *Ergodic Theory, with a view towards Number Theory*, Springer Graduate Texts in Mathematics, 2010.