Lecture 9: Examples of lattices, Poincaré's Theorem

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Lattices in $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$

- Recall that we have the hyperbolic area $dm = \frac{dx\,dy}{y^2}$ on $\mathbb{H}$ and the hyperbolic volume $dA = \frac{dx\,dy\,d\theta}{y^2}$ on $\mathrm{T}^1\mathbb{H} \cong PSL_2(\mathbb{R})$, invariant under the action of $PSL_2(\mathbb{R}) \cong$ the group of Möbius transformations.
- This gives a Haar measure on $PSL_2(\mathbb{R})$.
- The $d\theta$-component is just the Haar measure of $\mathbb{T}$ of finite measure $\implies$ $\Gamma < PSL_2(\mathbb{R})$ is a lattice $\iff$ a fundamental domain for the action $\Gamma \curvearrowright \mathbb{H}$ (a Dirichlet domain) has finite hyperbolic measure.
- The canonical projection $SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ is 2-to-1. Under this projection, Haar measure on $SL_2(\mathbb{R})$ is mapped into the Haar measure on $PSL_2(\mathbb{R})$.
- It follows that any lattice in $PSL_2(\mathbb{R})$ gives a lattice in $SL_2(\mathbb{R})$ by taking a pre-image.

Lemma 1

1. *If* $\Gamma' < \Gamma < SL_2(\mathbb{R})$, *where* Γ *is a discrete subgroup and* Γ' *is a lattice, then* Γ *is a lattice.*
2. *If* $\Gamma' < \Gamma < SL_2(\mathbb{R})$, *where* Γ *is a lattice and* $[\Gamma : \Gamma'] < \infty$ *(finite index subgroup), then* Γ' *is a lattice.*

Proof sketch.

1. Dirichlet regions satisfy:

$$D(\Gamma, p) \subset D(\Gamma', p).$$

2. A union of finitely many copies of a fundamental domain for Γ will form a fundamental domain for Γ'.

□

Congruence lattices

Definition 2

A principal congruence lattice of $SL_2(\mathbb{R})$ is a discrete subgroup of $SL_2(\mathbb{R})$ of the form

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \; a \equiv d \equiv 1; \; b \equiv c \equiv 0 \pmod{N} \right\}$$

for some $N \geq 1$.

For example,

$$\Gamma(2) = \left\{ \begin{pmatrix} 2k+1 & 2m \\ 2n & 2\ell+1 \end{pmatrix} \in SL_2(\mathbb{Z}) : \; k, l, m, n \in \mathbb{Z} \right\}.$$

Proposition 3

1. *The image of the lattice* $\Gamma(2)$ *in* $PSL_2(\mathbb{R})$ *is freely generated by*
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$
2. *Its Dirichlet region for the point* $p = i$*:*
$$D = \{z \in \mathbb{H} : \operatorname{Re} z \in (-1, 1),\ |z - 1/2| > 1/2,\ |z + 1/2| > 1/2\}.$$
3. *This lattice is non-uniform.*

Example of a uniform lattice

Recall that $\Gamma < PSL_2(\mathbb{R})$ is a uniform lattice $\iff$ $\Gamma \backslash PSL_2(\mathbb{R})$ is compact $\iff$ any Dirichlet domain has compact closure in $\mathbb{H}$.

Proposition 4

There is a uniform lattice in $PSL_2(\mathbb{R})$.

Proof sketch.

- Use the disc model of $\mathbb{H}$.
- Construct a regular 4-gon D with all internal angles $= \pi/3$.
- A convex hyperbolic n-gon is *regular* if all of its internal angles are equal and all its sides have the same length.
- 6 copies of D isometric under the action of $PSL_2(\mathbb{R})$ can be put together edge-to-edge to cover a neighborhood a vertex.
- By iterating this, we obtain a *tiling*, or *tesselation* of $\mathbb{H}$ by tiles isometric to D.

□

Example of a uniform lattice (cont.)

Proof sketch (cont.)

- Now consider the group Γ of all matrices in $PSL_2(\mathbb{R})$ that map the tiling into itself.
- Γ is discrete, since the set of vertices of the copies of D in the tiling is discrete and must be mapped into itself by Γ.
- Notice that any copy of D in the tiling can be mapped to any other by some element of Γ.
- It follows that D contains a fundamental region for $D \implies \Gamma$ is a uniform lattice.

□

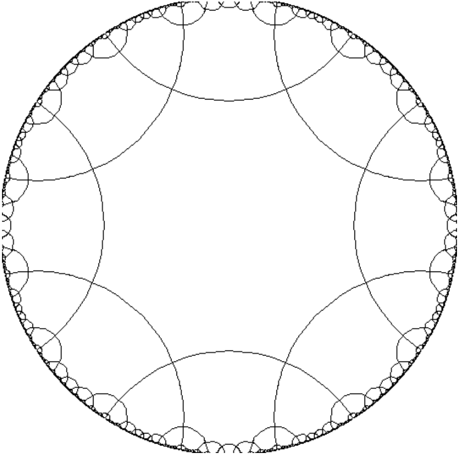
Hyperbolic tesselations

- Regular $\{n, k\}$ tiling: n sides of the polygon, k tiles meet at each vertex $\implies$ the angles $= 2\pi/k$.
- For which n and k does it exist? Need

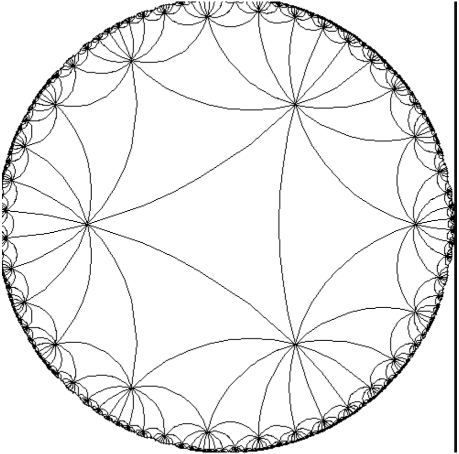
$$\boxed{2\pi n/k < (n-2)\pi \iff \frac{1}{n} + \frac{1}{k} < \frac{1}{2}}.$$

- Regular tilings of the plane are *parabolic*: $\{3, 6\}$ – triangles, $\{4, 4\}$ – squares, and $\{6, 3\}$ – hexagons. Note that $\frac{1}{n} + \frac{1}{k} = \frac{1}{2}$.
- Regular tilings of the sphere are *elliptic*; they correspond to the five regular solids: $\{3, 3\}$ – tetrahedron, $\{3, 4\}$ – octahedron, $\{3, 5\}$ – icosahedron, $\{4, 3\}$ – cube, $\{5, 3\}$ – dodecahedron. Note that $\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$.

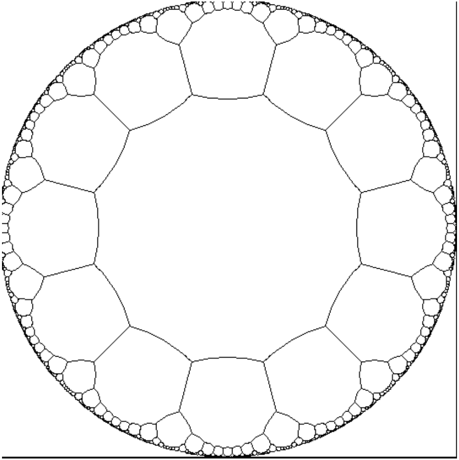
Hyperbolic tesselation $\{8, 4\}$ from https://mathcs.clarku.edu/ djoyce/poincare/tilings.html



Hyperbolic tesselation $\{3, 12\}$ from https://mathcs.clarku.edu/ djoyce/poincare/tilings.html



Hyperbolic tesselation $\{12, 3\}$ from https://mathcs.clarku.edu/ djoyce/poincare/tilings.html



Question: given a hyperbolic polygon, when is it a Dirichlet region for a uniform lattice?

The answer is given by **Poincaré Theorem**.

- We will discuss (a part of) Poincaré Theorem without proof.
- It appeared in the paper by Poincaré (1882) [4]
- Rigorous proof was given much later by Maskit (1971) [3].
- See also [2] and [5].

- M. Einsiedler and T. Ward, *Ergodic Theory, with a view towards Number Theory*, Springer Graduate Texts in Mathematics, 2010.
- S. Katok, *Fuchsian Groups*, Chicago Lecture Notes in Mathematics, 1992
- B. Maskit, *On Poincaré's Theorem for fundamental polygons*, Adv. Math. **7** (1971), 219–230.
- H. Poincaré, *Théorie des groupes Fuchsiens*, Acta Math. **1**, 1–62.
- C. Walkden, *Hyperbolic Geometry*, Lecture Notes for the course 32052, Univ. of Manchester, available on the internet.

Side-pairing transformations

Definition 5

- Let $\Gamma < PSL_2(\mathbb{R})$ be a uniform lattice, and $D = D(p)$ a Dirichlet region.
- D is a hyperbolic polygon with finitely many sides.
- By a *side* we mean an edge of D with an orientation.
- If s is a side of D and $\gamma(s)$ is another side of D for $\gamma \in \Gamma$, then γ is called the *side-pairing transformation*.
- The sides s and $\gamma(s)$ are called *paired*; it is possible that $s = \gamma(s)$.

Lemma 6

Under the above assumptions, every side of $D(p)$ is paired with exactly one side (possibly with itself).

Side-pairing transformations (cont.)

First we need an auxiliary

Lemma 7

$\gamma D(p) = D(\gamma p)$ *for* $\gamma \in \Gamma$.

Proof.

We have $\forall\, g \in \Gamma \setminus \{e\}$:

$$\zeta \in D(\gamma p) \iff \mathrm{d}(\zeta, \gamma p) = \mathrm{d}(\gamma^{-1}\zeta, p) < \mathrm{d}(\zeta, g\gamma p);$$

$$\begin{aligned}\zeta \in \gamma D(p) \iff \gamma^{-1}\zeta \in D(p) \iff \mathrm{d}(\gamma^{-1}\zeta, p) &< \mathrm{d}(\gamma^{-1}\zeta, gp) \\ &= \mathrm{d}(\zeta, \gamma g p).\end{aligned}$$

But $\{\gamma g : g \in \Gamma \setminus \{e\}\} = \{g\gamma : g \in \Gamma \setminus \{e\}\}$. □

Side-pairing transformations (cont.)

Proof.

- Let s be a side of $D(p)$. Then $s \subseteq L_\gamma$ for some $\gamma \in \Gamma$, where L_γ is the perpendicular bisector of the geodesic segment $[p, \gamma p]$.
- Let $\{z\} = [p, \gamma p] \cap L_\gamma$, then

$$z \in \partial D(p) \cap \partial(\gamma D(p)) = \partial D(p) \cap \partial D(\gamma p).$$

- Then $\gamma^{-1} z \in \partial D(\gamma^{-1} p) \cap \partial D(p) \implies \gamma^{-1}(s)$ is also a side of $D(p)$.
- Thus s is paired with $\gamma^{-1}(s)$.
- Is it possible that s is paired with another side? No! If $\gamma'(s)$ is a side of $D(p)$, then, since s is a side of both $D(p)$ and $D(\gamma p)$, we obtain that $\gamma'(s)$ is a side of both $D(\gamma' p)$ and $D(\gamma' \gamma p)$, and also of $D(p)$!
- Hence either $\boxed{\gamma' p = p \implies \gamma' = e}$ or $\boxed{\gamma' \gamma p = p \implies \gamma' = \gamma^{-1}}$. □

Three types of elements of $PSL_2(\mathbb{R})$

Definition 8

Let $g(z) = \frac{az+b}{cz+d}$ be a Möbius transformation, with $g \in PSL_2(\mathbb{R})$.

The *trace* of g is $\mathrm{Tr}(g) = |a+d|$.

- If $\mathrm{Tr}(g) < 2$, g is called *elliptic*.
- If $\mathrm{Tr}(g) = 2$, g is called *parabolic*.
- If $\mathrm{Tr}(g) > 2$, g is called *hyperbolic*.

Lemma 9 (exercise)

1. *g is elliptic $\iff$ it is conjugate to* $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO_2(\mathbb{R})$.
2. *g is parabolic $\iff$ it is conjugate to* $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in U^-$.
3. *g is hyperbolic $\iff$ it is conjugate to* $\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \in A$.

Fixed point properties

Corollary 10

1. *g is elliptic $\implies$ $g(z)$ has one neutral fixed point in $\mathbb{H}$ (and no fixed points in $\partial\mathbb{H}$).*
2. *g is parabolic $\implies$ $g(z)$ has one fixed point in $\partial\mathbb{H}$, which is attracting on one side and repelling on the other side.*
3. *g is hyperbolic $\implies$ $g(z)$ has two fixed points on $\partial\mathbb{H}$: one is attracting and one is repelling.*

Elliptic cycles

- Let D a convex hyperbolic polygon, equipped with *side-pairing transformations*. That is, for each side s of D we have $\gamma_s \in PSL_2(\mathbb{R})$ such that $s' = \gamma_s(s)$ is another side of D.
- Let v_0 be a vertex of D and s_0 a side of D such that $v_0 \in s_0$. Let $\gamma_1 = \gamma_s$ be such that $s_1 = \gamma_1(s)$ is another side of γ.
- Then $v_1 = \gamma_1(v_0)$ is a vertex of s_1, and (v_1, s_1) is another *vertex-side* pair. We have $(v_0, s_0) \xrightarrow{\gamma_1} (v_1, s_1)$.
- Now pass to the pair $(v_1, *s_1)$, where $*s_1$ is another side with the vertex v_1.
- Continue the procedure inductively. *Note that if we didn't switch to $*s_1$, we would simply go back $(v_1, s_1) \xrightarrow{\gamma_1^{-1}} (v_0, s_0)$.*
- Since there are finitely many pairs, the procedure must eventually return to the initial pair (v_0, s_0).

Elliptic cycles (cont.)

- Let n be minimal such that $(v_n, *s_n) = (v_0, s_0)$.

Definition 11

- The sequence of vertices $\mathcal{E} = v_0 \to v_1 \to \ldots \to v_{n-1}$ is called an *elliptic cycle*.
- The transformation $\gamma_n \gamma_{n-1} \cdots \gamma_1$ is called an *elliptic cycle transformation*.
- Denote by $\gamma_{v,s}$ the elliptic cycle transformation corresponding to the vertex-side pair (v, s).
- Denote by $\angle v$ the internal angle of D at the vertex v.
- Say that the *elliptic cycle condition* holds for $\mathcal{E}$ if there exists $m = m(\mathcal{E}) \geq 1$ such that

$$m \cdot \mathrm{sum}(\mathcal{E}) := m \cdot (\angle v_0 + \cdots + \angle v_{n-1}) = 2\pi.$$

Poincaré's Theorem

Theorem 12

- *Let $D \subseteq \mathbb{H}$ be a convex hyperbolic polygon, with finitely many sides.*
- *Suppose that D is equipped with a collection of side-pairing transformations $\mathcal{G}$ and no side is paired with itself.*
- *Let $\mathcal{E}_1, \ldots, \mathcal{E}_r$ be all the elliptic cycles, and suppose that every $\mathcal{E}$ satisfies the elliptic cycle condition: $m_j \cdot \mathrm{sum}(\mathcal{E}) = 2\pi$ for $m_j \geq 1$.*

Then

1. *the subgroup $\Gamma = \langle \mathcal{G} \rangle$ is a Fuchsian group;*
2. *D is a fundamental domain for Γ;*
3. *Γ can be written in terms of generators and relators as follows: let $\gamma_j = \gamma_{v,s}$ for some vertex on the cycle $\mathcal{E}_j$. Then*

$$\Gamma = \langle \gamma_s \in \mathcal{G} \mid \gamma_1^{m_1} = \ldots = \gamma_r^{m_r} = e \rangle.$$

Remarks.

1. In Poincaré's Theorem, if a side is paired in itself by γ, we can introduce an extra vertex with angle π in the middle, fixed by γ, and the theorem applies.

2. There is a version of Poincaré's Theorem for non-uniform lattices, when some of the vertices are in $\partial\mathbb{H}$.

3. We saw (implicitly) the necessity of the elliptic cycle condition in the proof of Siegel's Theorem, where we said that if $x_1, \ldots, x_n$ are vertices of $D(p)$ on one Γ-orbit, then

$$|\Gamma_k|(\delta_1 + \cdots + \delta_n) = 2\pi,$$

where $|\Gamma_k|$ is the order of the stabilizer of (any) x_k and δ_j are internal angles.

Recall that Γ_k is finite since $\Gamma_k < \Gamma$, which is discrete, and the stabilizer of a point is conjugate to $SO_2(\mathbb{R})$.