LECTURE NOTES: IMMERSIONS OF SURFACES IN 3-SPACE

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1. The Smale-Hirsch Theorem

This section is based on [T].

Let F, M be two smooth manifolds. An immersion $i : F \to M$ is a smooth map such that at each $p \in F$, the differential map $df : T_pF \to T_pM$ is a monomorphism. It follows that i is locally a smooth embedding, i.e. each $p \in F$ has a neighborhood U such that $i|_U : U \to M$ is a smooth embedding. Globally, though, i may have self intersection, i.e. globally i may not be 1-1.

In this section we will discuss the Smale-Hirsch Theorem, which is the fundamental theorem of immersion theory. It has been originally proved by Smale and Hirsch in [S1],[S2],[H]. We will follow the proof of Thurston sketched in [T]. Let F, M be two smooth manifolds. $Imm(F, M)$ will denote the space of all immersions of F into M, where the topology on $Imm(F, M)$ is the C^1 topology. A path in $Imm(F, M)$ is called a regular homotopy. A bundle monomorphism $b : TF \to TM$ is a map which is a linear monomorphism on each fiber of TF. $Mon(TF, TM)$ will denote the space of all bundle monomorphisms $b : TF \to TM$ where the topology on $Mon(TF, TM)$ is the compact-open topology. There is a natural map $d : Imm(F, M) \to Mon(TF, TM)$ given by $i \mapsto di$. This map is clearly 1-1 since a bundle map $b : TF \to TM$ in particular determines the map $\hat{b} : F \to M$ which it covers. Furthermore, the topology on $Imm(F, M)$ being the C^1 topology, precisely means that the map $d : Imm(F, M) \to Mon(TF, TM)$ is a topological embedding. So we may think of $Imm(F, M)$ as a subspace $I \subseteq Mon(TF, TM)$. Note that there are bundle monomorphisms $b \in Mon(TF, TM)$ whose underlying map $\hat{b} : F \to M$ is an immersion, but who are not members of the subspace I since b is not $d\hat{b}$.

When working in local coordinates $x_1, \dots, x_n$ for $U \subseteq F$ and $y_1, \dots, y_m$ for $V \subseteq M$, we may take $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}$ as bases, and then a bundle map may be written locally as a pair $(\hat{b}, b)$ where $\hat{b} : U \to V$ is the underlying map, and $b : U \to M_{m\times n}(\mathbb{R})$ (the

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space of $m \times n$ matrices), gives the matrix of each linear map with respect to the above bases.

We now state the Smale-Hirsch Theorem:

Theorem 1.1. *Let F, M be compact smooth manifolds with $\dim F < \dim M$.*
Then the map $d: Imm(F,M) \to Mon(TF,TM)$ is a homotopy equivalence.

This theorem extremely simplifies the computation of the homotopy groups of $Imm(F,M)$, in particular, we will be interested in π_0 and π_1. Before proving SHT (= the Smale-Hirsch Theorem), we give a first example of application, namely, that S^2 may be "turned inside out" by regular homotopy in $\mathbb{R}^3$. Such regular homotopy is called an eversion of the sphere. The question of whether this is indeed possible was open before Smale developed his theory, and was a major motivation for this development. Let $e: S^2 \to \mathbb{R}^3$ be the inclusion and let $e' = -e$. We need to show that there is a regular homotopy between e and e' i.e. a path in $Imm(S^2, \mathbb{R}^3)$ from e to e'. By SHT it is enough to show that there is a path in $Mon(TS^2, T\mathbb{R}^3)$ from de to de'. Such path b_t is easy to construct: As first step take $\hat{b}_t$ to be $(1-t)e + te'$ while b_t carries each tangent plane parallelly. $\hat{b}_1$ is indeed e' but b_1 is $-de'$. So as a second step, keeping $\hat{b}$ fixed, we rotate each tangent plane say in the clockwise direction (using an orientation on S^2), until they all arrive at de'.

We now prove SHT. We will prove it for F a surface and $M = \mathbb{R}^3$, and leave as exercise the extension of the proof to general F, M (Exercise 4, see discussion there). Denoting $Mon = Mon(TF, T\mathbb{R}^3)$ and $I = d(Imm(F, \mathbb{R}^3)) \subseteq Mon$ as before, we prove SHT by constructing a homotopy $h_t: Mon \to Mon$ $(0 \leq t \leq 1)$, such that $h_0 = \mathrm{Id}_{Mon}$, $h_t(I) \subseteq I$ for all t, and $h_1(Mon) \subseteq I$. We will construct h_t by describing $b_t = h_t(b)$ for given $b \in Mon$, and it will be clear that the construction can be made to vary continuously in Mon, since the constants that we will choose for given b, may be chosen as continuous functions of b.

So, given $b \in Mon$ we describe how to construct a path b_t with $b_0 = b$ and $b_1 \in I$ and such that if $b \in I$, then $b_t \in I$ for all t. We will perform all our constructions in one coordinate chart U of F. We will leave as an exercise how to diminish all matters toward the boundary of U and how to pass from one coordinate neighborhood to the next (Exercise 3). The path b_t will be constructed in three steps, i.e. in three segments which we will name 1,2,3, until indeed $b_t \in I$. If F has boundary then we will have a preliminary step, segment 0, where we slightly shrink a given collar neighborhood of ∂F within itself, leaving an "outside collar", and so all (finitely many) coordinate neighborhoods may be taken as open discs.

Segment 1 will smooth out the bundle map $b \in Mon$, i.e is a b_t as above such that b_1 is smooth in the sense that in local coordinates, when thinking of b as a pair of maps $\hat{b} : U \to V$, $b : U \to M_{3\times 2}(\mathbb{R})$, then $\hat{b}, b$ are both smooth. We may indeed achieve this via the technique of convolution with a smooth "bump function" as appears e.g. in [F] Theorem 8.14. The point to notice is that differentiation commutes with convolution ([F] Proposition 8.10), and so if the same bump function is used for $\hat{b}$ and b, then whenever $\hat{b}$ is smooth and $b = d\hat{b}$, then this property will indeed continue to hold.

The general plan now is as follows. From the previous step, our initial b is assumed smooth. We will use b to construct $\phi_t : U \to \mathbb{R}^3$ $(0 \le t \le 1)$ which will satisfy:

1. $\phi_0 = 0$
2. $b + d\phi_t$ is a bundle monomorphism for all $0 \le t \le 1$.
3. $(1-t)b + td\hat{b} + d\phi_1$ is a bundle monomorphism for all $0 \le t \le 1$

Segment 2 of our path will then be $(\hat{b} + \phi_t \ , \ b + d\phi_t)$ and segment 3 will be $(\hat{b} + \phi_1 \ , \ (1-t)b + td\hat{b} + d\phi_1)$.

Exercise 1. Check that indeed we always end up in I, and that if we started in I we remain in I all along.

So the problem is to find such path ϕ_t. This itself will be done in two steps corresponding to the two coordinates x, y in our coordinate chart U. We name these two subsegments 2.1 and 2.2. We describe segment 2.1, corresponding to the first coordinate, x.

As above, we write b as a pair $\hat{b} : U \to \mathbb{R}^3$ and $b : U \to M_{3\times 2}(\mathbb{R})$ such that for each $p \in U$, $b(p)$ is of rank 2. We define $A : U \to GL_3^+(\mathbb{R})$ (The space of 3×3 matrices of positive determinant) as follows: The two first columns of $A(p)$ will be the two columns v_1, v_2 of $b(p)$, and the third column v_3 will be the unique vector which is

1. perpendicular to v_1, v_2.
2. $\|v_3\| = \|v_1\|$.
3. the matrix $A = (v_1, v_2, v_3)$ has positive determinant.

We define the "wave function" $W = (W_1, W_2) : \mathbb{R} \to \mathbb{R}^2$ to be a periodic smooth function which traces a figure 8 as in Figure 1a. The range of the derivative vector of W is depicted in Figure 1b. The property to be noticed is that there is a constant k such that if e_1 denotes the unit vector in the x direction then, $\|e_1 + t\frac{dW}{ds}(s)\| \ge k$, for any $s \in \mathbb{R}$ and $t \ge 0$.

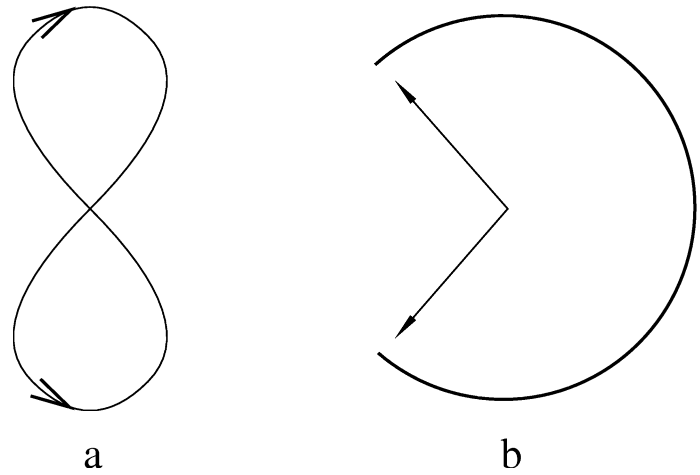

FIGURE 1. The wave function

Define $\bar{W}(s) = (W_1(s), 0, W_2(s))$ (written as a column). We then define $\varphi : U \to \mathbb{R}^3$ by $\varphi(x, y) = aA(x, y)\bar{W}(cx)$ where a is a small constant and c is a large constant, both to be determined.

We look at $b + td\varphi$ $(0 \leq t \leq 1)$. This is a matrix whose columns are the following pair of vectors:

$$\begin{aligned} v_1 + t\frac{\partial \varphi}{\partial x} &= v_1 + ta\frac{\partial A}{\partial x}\bar{W}(cx) + tacA\frac{d\bar{W}}{ds}(cx) \\ v_2 + t\frac{\partial \varphi}{\partial y} &= v_2 + ta\frac{\partial A}{\partial y}\bar{W}(cx). \end{aligned}$$

We choose a so small so that $\|a\frac{\partial A}{\partial x}\bar{W}\| < \epsilon$ and $\|a\frac{\partial A}{\partial y}\bar{W}\| < \epsilon$ on U, for ϵ defined in the following exercise:

Exercise 2. 1. Let $z_1, z_2 \in \mathbb{R}^3$ be independent vectors. Show that for any $u_1, u_2 \in \mathbb{R}^3$ satisfying $\|u_1\| < \frac{1}{2}dist(z_1, span(z_2))$ and $\|u_2\| < \frac{1}{2}dist(z_2, span(z_1))$, the pair $z_1 + u_1, z_2 + u_2$ is also independent.

2. For v_1, v_2, v_3, k appearing above, show that there is $\epsilon > 0$ such that for any u_1, u_2 with $\|u_1\| < \epsilon$, $\|u_2\| < \epsilon$ and for any $v \in span(v_1, v_3)$ with $\|v\| \geq k\|v_1\|$, the pair $v+u_1, v_2+u_2$ is everywhere independent.

So since by definition of k and A, $v_1 + tacA\frac{d\bar{W}}{ds} \geq k\|v_1\|$ for any choice of positive t, a, c, the above choice of a guarantees that the pair $v_1 + tacA\frac{d\bar{W}}{ds} + ta\frac{\partial A}{\partial x}\bar{W}, v_2 + ta\frac{\partial A}{\partial y}\bar{W}$ will be independent for any $t, c \geq 0$.

What we have just done is to choose a so small so that the terms $ta\frac{\partial A}{\partial x}\bar{W}, ta\frac{\partial A}{\partial y}\bar{W}$ will not damage the independence of the pair $v_1 + tacA\frac{d\bar{W}}{ds}, v_2$ which is indeed independent since $0 \neq v_1 + tacA\frac{d\bar{W}}{ds} \in span(v_1, v_3)$. In what follows we will simply drop the terms $ta\frac{\partial A}{\partial x}\bar{W}, ta\frac{\partial A}{\partial y}\bar{W}$, so will approximate the pair $v_1+t\frac{\partial\varphi}{\partial x}, v_2+t\frac{\partial\varphi}{\partial y}$ by the pair $v_1+tacA\frac{d\bar{W}}{ds}, v_2$, with the understanding that a is taken small enough to allow this in all following considerations.

Now we will choose c to be so large so that the distance from $v_1 + acA\frac{d\bar{W}}{ds}$ to $span(v_2)$ will be large with respect to the vectors v_1 and $\frac{\partial\hat{b}}{\partial x}$. Note that since v_3 is perpendicular to v_1, v_2, the distance from v_2 to $span(v_1 + acA\frac{d\bar{W}}{ds})$ may only be larger than its distance to $span(v_1)$. This is important to note with reversed roles for segment 2.2, when we perform the same procedure on v_2 and want to know that we are not undoing our achievements for v_1. So indeed segment 2.1 will be defined by $(\hat{b} + t\varphi, b + td\varphi)$, $(0 \le t \le 1)$. Segment 2.2 will use the final b we approached, namely $(\hat{b} + \varphi, b + d\varphi)$ and begin the whole process with this bundle monomorphism and with the second variable y. We will have a new function $\psi(x,y) = a'A(x,y)\bar{W}(c'y)$, this time with $\bar{W} = (0, W_1, W_2)$ and A defined with $\|v_3\| = \|v_2\|$. We will then achieve that also the distance from $v_2 + a'c'A\frac{d\bar{W}}{ds}$ to $span(v_1)$ be large with respect to the vectors v_2 and $\frac{\partial\hat{b}}{\partial y}$, (where v_1 here denotes the new v_1 produced by segment 2.1). As noted, this will not damage the corresponding property already achieved in segment 2.1.

Segments 2.1, 2.2 may be written as: $(\hat{b}+t\varphi, b+td\varphi)$ $(0 \le t \le 1)$ and $(\hat{b}+\varphi+t\psi, b+d\varphi+td\psi)$ $(0 \le t \le 1)$. (So ϕ_t referred to at the outset is the concatenation of $t\varphi$ $(0 \le t \le 1)$ and $\varphi + t\psi$ $(0 \le t \le 1)$). We finally arrive at the bundle map $(\hat{b}+\phi, b+d\phi)$ where $\phi = \varphi + \psi$, and since $b + d\phi$ has the property that the distance of each of its columns to the span of the other column, is very large with respect to v_1, $\frac{\partial\hat{b}}{\partial x}$ and v_2, $\frac{\partial\hat{b}}{\partial y}$ respectively, the path $(1-t)b + td\hat{b} + d\phi$ is always a matrix with independent columns, which is what we were trying to achieve. This completes the proof of SHT, recalling that two tasks remain for you:

Exercise 3. Extend the construction from one coordinate neighborhood to the whole of F.

Exercise 4. Extend the proof to the general case stated in Theorem 1.1.

Remark: The substantial difficulty is the ability in step 2, to choose the one additional vector normal to the given frame, with respect to a Riemannian metric we choose for M. It is impossible in general that this choice will depend on the frame alone as may be seen for $Imm([0,1], \mathbb{R}^3)$. (That would imply a non-vanishing vector field on S^2.) Hint for solution: Let $U \subseteq F$ be the given coordinate neighborhood, and being after step 1, we assume our $b \in Mon(TU, TM)$ are all smooth. We would like to cover $Mon(TU, TM)$ with some finitely many contractible open subsets. Choose a basepoint $p \in U$, and for $b \in Mon(TU, TM)$ define $F(b) = (b(\frac{\partial}{\partial x_1}|_p), \ldots, b(\frac{\partial}{\partial x_n}|_p))$. This is an independent n-tuple of vectors in $TM|_{\hat{b}(p)}$. The space of all such n-tuples, at all points of M, is a manifold X. The map $F : Mon(TU, TM) \to$

X is a fiber bundle with contractible fiber (since U may be contracted to p). Now X is not compact, but itself is a fiber bundle with contractible fiber over a compact manifold (namely, the manifold of all orthonormal n-tuples at all points of M). So X has a finite cover by contractible open sets V_i, and then $G_i = F^{-1}(V_i)$ gives the desired cover of $Mon(TU,TM)$ by contractible open sets. For given G_i look at the space Y_i of all pairs (b,u) where $b \in G_i$ and $u : U \to TM$ is a smooth choice of normal for b (i.e. u is smooth, and for all $x \in U$, $0 \neq u(x) \in T|_{\hat{b}(x)}M$ and is normal to the image of $b(x)$). Then Y_i is a fiber bundle over G_i, and since G_i is contractible, there exists a section for this bundle, that is, on G_i there is a choice of normal with which to build the waves. So we build the waves for each G_i at a time (diminishing all matters toward $F^{-1}(\partial V_i)$). We use the exponential map of the Riemannian metric to transform the wave function from TM into M.

Exercise 5. Show that the assumption $\dim F < \dim M$ is needed, by showing that the statement of SHT is false for $Imm(S^1,S^1)$.

We concentrate from now on only on $Imm(F,\mathbb{R}^3)$, F a surface. We will further simplify matters by replacing $Mon(TF,T\mathbb{R}^3)$ by an even simpler space. We first note that for any F, the space $Imm(F,\mathbb{R}^3)$ is non-empty. Indeed if F is orientable or has boundary then it may even be embedded into $\mathbb{R}^3$, and any closed non-orientable surface may be immersed as a connect sum of Boy's surfaces. This implies that also $Mon(TF,T\mathbb{R}^3)$ is non-empty for any F. (For a clear description of Boy's surface, see [E].)

Since there is a natural identification $T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ we may think of a bundle map $b \in Mon(TF,T\mathbb{R}^3)$ as $(\hat{b},b) : TF \to \mathbb{R}^3 \times \mathbb{R}^3$. Via contracting the first factor $\mathbb{R}^3$ to a point, we get that $Mon(TF,T\mathbb{R}^3)$ is homotopy equivalent to the space of maps $b : TF \to \mathbb{R}^3$ which are a linear monomorphism on each fiber of TF. This is the same as $X = Mon(TF, pt \times \mathbb{R}^3)$ where $pt \times \mathbb{R}^3$ is the $\mathbb{R}^3$ vector bundle over a single point pt. Define $Y = Map(F, GL_3^+(\mathbb{R}))$, the space of all continuous maps from F to $GL_3^+(\mathbb{R})$. We will now show that X is homotopy equivalent to Y. Choose a base point $b_0 \in X$ once and for all ($X \neq \varnothing$), and define $f : X \to Y$ as follows: Let $b \in X$ then for any $p \in F$, $b(p)$ and $b_0(p)$ are two linear embeddings of the plane $P = T_pF$ into $\mathbb{R}^3$. Now $f(b)(p)$ is defined to be the unique $A \in Gl_3^+(\mathbb{R})$ such that $A \circ b_0(p) = b(p)$ and A maps a unit vector perpendicular to $b_0(p)(P)$ into a unit vector perpendicular to $b(p)(P)$. We define $g : Y \to X$ as follows: for $u \in Y$ and $p \in F$ let $g(u)(p) = u(p) \circ b_0(p)$.

Exercise 6. Show that $g \circ f = \mathrm{Id}_X$ and $f \circ g$ is homotopic to Id_Y.

Our last simplification is to notice that $Gl_3^+(\mathbb{R})$ is homotopy equivalent to SO_3 (hint: the Gram-Schmidt process). Adding all steps together, we get that $Imm(F,\mathbb{R}^3)$ is homotopy equivalent to $Map(F,SO_3)$. (The essential ingredients were SHT and the fact that $Mon(TF,T\mathbb{R}^3)\neq\varnothing$.) In what follows we will be interested in $\pi_0(Imm(F,\mathbb{R}^3))$ and $\pi_1(Imm(F,\mathbb{R}^3))$, so will compute $\pi_0(Map(F,SO_3))$ and $\pi_1(Map(F,SO_3))$. For this we need to know several homotopy groups of SO_3 itself, namely:

$$\pi_0(SO_3)=0,\quad \pi_1(SO_3)=\mathbb{Z}/2,\quad \pi_2(SO_3)=0,\quad \pi_3(SO_3)=\mathbb{Z}.$$

These are true since SO_3 is homeomorphic to $\mathbb{R}P^3$, which is double covered by S^3.

Exercise 7. Prove the following relative version of SHT. Let $S\subseteq F$ be a subsurface and let $i_0:F\to\mathbb{R}^3$ be an immersion. Let A be the space of all immersions $i:F\to\mathbb{R}^3$ such that $i|_S=i_0|_S$ and let B be the space of all bundle monomorphisms $b:TF\to T\mathbb{R}^3$ such that $b|_S=di_0|_S$. Then the map $d:A\to B$ is a homotopy equivalence.

Exercise 8. For the definitions appearing in the previous exercise, show that B is homotopy equivalent to the space of all maps $f:F\to SO_3$ such that $f(S)=I$ ($I\in SO_3$ the identity element).

Exercise 9. Let $i:F\to\mathbb{R}^3$ be an immersion and $S\subseteq F$ a subsurface. Let $h_t:S\to\mathbb{R}^3$ be a regular homotopy with $h_0=i|_S$. Show that h_t can be extended to F i.e. there is a regular homotopy $g_t:F\to\mathbb{R}^3$ with $g_0=i$ and $g_t|_S=h_t$ for all t. (Hint: First show that dh_t can be extended to a path of bundle monomorphisms $b_t\in Mon(TF,T\mathbb{R}^3)$.)

Exercise 10. Show that $Imm(S^1,\mathbb{R}^2)$ is homotopy equivalent to $Map(S^1,S^1)$ and compute all homotopy groups of this space (including π_0).

2. Regular Homotopy and Cobordism

This section is based on [P].

2.1. **Regular homotopy.** Recall that a path in $Imm(F,\mathbb{R}^3)$ is called a regular homotopy. Accordingly, the path components are called regular homotopy classes, and two immersions in the same component are called regularly homotopic. For given surface F we would like to classify the regular homotopy classes in $Imm(F,\mathbb{R}^3)$. From the previous section we know that once we choose a base immersion i_0, the path components of $Imm(F,\mathbb{R}^3)$ correspond to the path components of $Map(F,SO_3)$, i.e. the homotopy classes of maps from F to SO_3. The set of homotopy classes is denoted $[F,SO_3]$. For

$h : F \to SO_3$ let h_* denote the homomorphism it induces on $H_1(\cdot, \mathbb{Z}/2)$. Then $h \mapsto h_*$ is a map $\phi : [F, SO_3] \to Hom(H_1(F, \mathbb{Z}/2), H_1(SO_3, \mathbb{Z}/2)) = Hom(H_1(F, \mathbb{Z}/2), \mathbb{Z}/2)$.

Exercise 11. Show that ϕ is bijective. (Hint: Use a CW structure on F, and the list of homotopy groups of SO_3 appearing above).

So the composed map $\pi_0(Imm(F, \mathbb{R}^3)) \to Hom(H_1(F, \mathbb{Z}/2), \mathbb{Z}/2)$ is a bijection. In particular we get that there are 2^n regular homotopy classes of immersions of F into $\mathbb{R}^3$ where $n = \dim H_1(F, \mathbb{Z}/2)$. (Note, for $F = S^2$ this gives that there is just one regular homotopy class, i.e. any two immersions of S^2 into $\mathbb{R}^3$ are regularly homotopic, in particular there exists an eversion of the sphere.) We see from the bijection $\pi_0(Imm(F, \mathbb{R}^3)) \to Hom(H_1(F, \mathbb{Z}/2), \mathbb{Z}/2)$ that the regular homotopy class of an immersion $i : F \to \mathbb{R}^3$ is determined by the relative twisting between i and the base immersion i_0 along loops in F. Instead of looking at the relative twisting between i and i_0 we will define a notion of "absolute" twisting for embedded loops in F, and the function h_* characterizing i will then be replaced by a new function on $H_1(F, \mathbb{Z}/2)$ which does not depend on a base immersion i_0. The new function will not be a homomorphism but will have other special algebraic properties. From now on we will denote $H_1(F, \mathbb{Z}/2)$ by H_1 and $Hom(H_1(F, \mathbb{Z}/2), \mathbb{Z}/2)$ by H_1^*. (H_1^* is naturally identified with the cohomology $H^1(F, \mathbb{Z}/2)$.)

(Abstractly, the situation is as follows: for any two elements of $\pi_0(Imm(F, \mathbb{R}^3))$ we have defined their "difference" h_*, an element in the linear space H_1^*. This gives a structure of affine space to $\pi_0(Imm(F, \mathbb{R}^3))$, and so once an origin for $\pi_0(Imm(F, \mathbb{R}^3))$ is chosen, $\pi_0(Imm(F, \mathbb{R}^3))$ is identified with H_1^*, which is the identification appearing above. What we would now like to do is to find some "absolute" model for this affine space, which does not depend on choice of origin.)

Let A be the annulus and M the Mobius band, then $[A, SO_3] = [S^1, SO_3] = [M, SO_3]$ has precisely two elements. (This explains the "belt trick".) Denote $\mathcal{H} = (\frac{1}{2}\mathbb{Z})/2$. For A, there is one regular homotopy class which includes an embedding contained in a plane in $\mathbb{R}^3$, and we assign the value $0 \in \mathcal{H}$ to this regular homotopy class and the value $1 \in \mathcal{H}$ to the other regular homotopy class. As for M we assign the value $\frac{1}{2} \in \mathcal{H}$ to one of the classes (once and for all), and $-\frac{1}{2} \in \mathcal{H}$ to the other regular homotopy class. Now given any surface F and immersion $i : F \to \mathbb{R}^3$ we define $g^i : H_1 \to \mathcal{H}$ as follows: For $x \in H_1$ take an embedded loop γ in F which represents x.

Exercise 12. Show that such embedded loop always exists.

Let U be a thin neighborhood of γ in F, then U is either an annulus or Mobius band. We defined $g^i(x) \in \mathcal{H}$ as the value in $\mathcal{H}$ assigned above to $i|_U$.

Exercise 13. Show that this value does not depend on the choice of identification of U with A or M and does not depend on the choice of embedded loop γ in F representing the homology class x.

We denote by $x \cdot y$ the intersection form on $H_1(F, \mathbb{Z}/2)$.

Exercise 14. Let $i : F \to \mathbb{R}^3$ be an immersion, then for any $x, y \in H_1$:

$$g^i(x+y) = g^i(x) + g^i(y) + x \cdot y.$$

(Note that $x \cdot y \in \mathbb{Z}/2 \subseteq \mathcal{H}$.)

A function $g : H_1 \to \mathcal{H}$ satisfying the above property will be called a quadratic form on H_1.

Exercise 15. Let $\mathcal{Q}(F)$ denote the set of all quadratic forms on H_1. Show that

1. If $g, g' \in \mathcal{Q}(F)$ then $g - g' \in H_1^*$.
2. If $g \in \mathcal{Q}(F), \varphi \in H_1^*$ then $g + \varphi \in \mathcal{Q}(F)$.

It follows that if an origin $g_0 \in \mathcal{Q}(F)$ is chosen then the map $\varphi \mapsto g_0 + \varphi$ is a bijection between H_1^* and $\mathcal{Q}(F)$. We now note that if $i_0 \in Imm(F, \mathbb{R}^3)$ is chosen as base immersion then for any $i \in Imm(F, \mathbb{R}^3)$, if $h_* \in H_1^*$ is the homomorphism we have originally attached to i using the base immersion i_0, then $g^i = g^{i_0} + h_*$. All put together we obtain:

Theorem 2.1. *The map $\pi_0(Imm(F, \mathbb{R}^3)) \to \mathcal{Q}(F)$ given by $i \mapsto g^i$, is a bijection.*

2.2. **Regular homotopy up to diffeomorphism of F.** Assume from now on that F is closed, and so the intersection form on H_1 is non-degenerate. For $a \in \mathcal{H}$ the number $e^{a\pi i} \in \mathbb{C}$ is well defined and $a \mapsto e^{a\pi i}$ maps sums to products. For $g \in \mathcal{Q}(F)$ we define

$$\mathrm{Arf}(g) = \frac{1}{\sqrt{2}^{\dim H_1}} \sum_{x \in H_1} e^{g(x)\pi i}$$

This gives a new invariant of immersions, namely $i \mapsto \mathrm{Arf}(g^i)$, which we will call the Arf invariant of the immersion i. An automorphism $u : H_1 \to H_1$ which preserves the intersection form will be called intersection preserving. Clearly, if $u : H_1 \to H_1$ is an intersection preserving automorphism, then $g \circ u$ is a quadratic form and $\mathrm{Arf}(g \circ u) = \mathrm{Arf}(g)$.

Exercise 16. 1. Show that the Arf invariant is multiplicative in the following sense: If g is a quadratic form on H_1 and $H_1 = V_1 \oplus V_2$ with $x \cdot y = 0$ for any $x \in V_1, y \in V_2$, then $\mathrm{Arf}(g) = \mathrm{Arf}(g|_{V_1})\mathrm{Arf}(g|_{V_2})$

2. Show that $\mathrm{Arf}(g)$ is always an eighth root of unity.
3. Show that if g_1, g_2 are two quadratic forms on H_1 with $\mathrm{Arf}(g_1) = \mathrm{Arf}(g_2)$ then there is an intersection preserving automorphism u of H_1, such that $g_1 = g_2 \circ u$. (Hint: Separate the cases of F orientable and non-orientable, and concentrate first on $\dim H_1 \leq 4$.)
4. Show that for any intersection preserving automorphism u of H_1 there is a diffeomorphism $h : F \to F$ with $h_* = u$.
5. Show that if $i : F \to \mathbb{R}^3$ is an immersion and $h : F \to F$ a diffeomorphism then $g^{i \circ h} = g^i \circ h_*$.
6. Conclude the following theorem:

Theorem 2.2. *Let $i, j : F \to \mathbb{R}^3$ be two immersions. Then $\mathrm{Arf}(g^i) = \mathrm{Arf}(g^j)$ iff there exists a diffeomorphism $h : F \to F$ such that i and $j \circ h$ are regularly homotopic.*

2.3. **Cobordism.** Let F_0, F_1 be two closed surfaces and $i_0 : F_0 \to \mathbb{R}^3$, $i_1 : F_1 \to \mathbb{R}^3$ be immersions. A cobordism between i_0 and i_1 is a 3-dimensional manifold M, a proper immersion $j : M \to \mathbb{R}^3 \times [0,1]$, and identification of the disjoint union $F_0 \cup F_1$ with ∂M such that $j|_{F_0} = i_0 \times \{0\}$ and $j|_{F_1} = i_1 \times \{1\}$.

Exercise 17. Show that if $i, j : F \to \mathbb{R}^3$ are immersions, $h : F \to F$ is a diffeomorphism, and i and $j \circ h$ are regularly homotopic, then i and j are cobordant.

If j is a cobordism between i_0, i_1, then by slightly perturbing j we may assume that the projection of j to the $[0,1]$ factor, is a Morse function. And so if we follow the sections $j^{-1}(\mathbb{R}^3 \times \{t\})$ then except for some finitely many times they are smooth closed (perhaps non-connected) surfaces moving around via regular homotopy, and at the special times, precisely four types of surgery may appear (see [Mi]), they are the disappearance of a small sphere, the pinching down of a thin tube, and the two reverse operations.

Exercise 18. Show that the Arf invariant is invariant under cobordism. (Hint: show that it is invariant under each of the above four surgeries.)

We will now prove the converse and thus obtain:

Theorem 2.3. *Two immersions $i_0 : F_0 \to \mathbb{R}^3$, $i_1 : F_1 \to \mathbb{R}^3$ have the same Arf invariant iff they are cobordant.*

We first show that if $i : F \to \mathbb{R}^3$ is an immersion and there is $0 \neq x \in H_1(F, \mathbb{Z}/2)$ with $g^i(x) = 0$, then i is cobordant to an immersion $j : S \to \mathbb{R}^3$ with $\dim H_1(S, \mathbb{Z}/2) < \dim H_1(F, \mathbb{Z}/2)$: Realizing x as an embedded loop γ, then γ is non-separating (since $x \neq 0$) and a neighborhood A of γ is an annulus (since $g^i(x) = 0$). Since γ is non-separating, there is an embedded loop γ' which intersects γ transversally at exactly one point. A neighborhood U of $\gamma \cup \gamma'$ is either a punctured torus or punctured Klein bottle. (The two cases occur when a neighborhood of γ' is an annulus or Mobius band, respectively). Let $c = \partial U$. Then since c is 0 in $H_1(F, \mathbb{Z}/2)$, then $g^i(c) = 0$ and so there is a regular homotopy of a neighborhood of c bringing it to the shape of a thin tube. By Exercise 9 this regular homotopy may be extended to the whole of F. We may pinch off this tube, disconnecting F into a torus or Klein bottle G, and another closed surface S with $\dim H_1(S, \mathbb{Z}/2) = \dim H_1(F, \mathbb{Z}/2) - 2$. Now the the loop γ which has $g^i(\gamma) = 0$ is contained in G, and the same procedure may be performed on it to pinch it off, Turning G into a sphere. Any immersion of a sphere is regularly homotopic to an embedded sphere, which may be made to shrink and disappear, leaving us only with the immersed surface S.

Exercise 19. Show that if $\dim H_1 \geq 4$ and $i : F \to \mathbb{R}^3$ is an immersion then there is $0 \neq x \in H_1$ with $g^i(x) = 0$.

So for any F and immersion $i : F \to \mathbb{R}^3$, i is cobordant to an immersion $j : S \to \mathbb{R}^3$ with $\dim H_1(S, \mathbb{Z}/2) \leq 3$.

Exercise 20. Establish Theorem 2.3 for all F with $\dim H_1(F, \mathbb{Z}/2) \leq 3$.

All steps combined, Theorem 2.3 is proved.

We conclude this section with two remarks on convention.

1. In [P] the values of a quadratic form are not taken in $\mathcal{H} = (\frac{1}{2}\mathbb{Z})/2$ but rather in the isomorphic $\mathbb{Z}/4$. The defining relation then becomes $g(x+y) = g(x) + g(y) + 2x \cdot y$ and an extra $\frac{1}{2}$ appears in the exponent in the definition of the Arf invariant.
2. For an orientable surface, all terms in the sum defining the Arf invariant are ± 1, and so the Arf invariant itself is real and so is ± 1. In this case traditionally the Arf invariant is considered as additive values $0, 1 \in \mathbb{Z}/2$ in place of $1, -1 \in \mathbb{C}$ respectively.

3. The fundamental group and finite order invariants

In sections 3 and 4 we assume F is closed and orientable. The analogous results for non-orientable surfaces appear in [N6].

3.1. **The fundamental group.** This subsection is based on [N1].

We start by computing $\pi_1(Imm(F,\mathbb{R}^3), i_0)$ for $i_0 \in Imm(F,\mathbb{R}^3)$ a base point. Using the same i_0 as base immersion in the correspondence $Imm(F,\mathbb{R}^3) \to Map(F, SO_3)$ appearing in Section 1, we get that $\pi_1(Imm(F,\mathbb{R}^3), i_0) = \pi_1(Map(F, SO_3), f_0)$ where $f_0 \in Map(F, SO_3)$ is the map defined by $f_0(F) = \{I\}$ where $I \in SO_3$ is the identity element. But $\pi_1(Map(F, SO_3), f_0)$ is the same as the group of all maps $h : F \times [0,1] \to SO_3$ such that $h(p,0) = h(p,1) = I$ for all $p \in F$, and where the operation is concatenation along the $[0,1]$ factor.

Exercise 21. Show that this group (for F closed orientable) is $\mathbb{Z}/2 \oplus \mathbb{Z}$. Describe the order 2 element. Show that the generator of the $\mathbb{Z}$ factor may be realized by a regular homotopy which keeps all of F fixed except for a small disc.

We will now construct a generator of the $\mathbb{Z}$ factor in $\pi_1(Imm(F,\mathbb{R}^3))$, starting first with $F = S^2$. Let $D, U \subseteq S^2$ the unit discs in the two standard coordinate charts for S^2 as the Riemann sphere, so $D \cup U = S^2$, $D \cap U = \partial D = \partial U$, and on a neighborhood of ∂U, the change of coordinates function is $z \mapsto \frac{1}{z}$, that is, $\phi(x,y) = \frac{1}{x^2+y^2}(x,-y)$.

Exercise 22. Show that when moving from any $(x,y) \in \partial U$ along ∂U to $(-x,-y)$, $d\phi$ performs precisely one full rotation with respect to the fixed frame in D. In particular $d\phi(x,y) = d\phi(-x,-y)$ for any $(x,y) \in \partial U$.

(Note that when going all the way around ∂U, the number of rotations of $d\phi$ is $2 = \chi(S^2)$. This is a special case of a general phenomenon regarding vector fields on manifolds.)

Let $i_0 : S^2 \to \mathbb{R}^3$ be an immersion such that $i_0|_D$, in the coordinates of D, is the natural inclusion $(x,y) \mapsto (x,y,0)$. Let A be the space of all immersions of $i : S^2 \to \mathbb{R}^3$ such that $i|_D = i_0|_D$. Using Exercise 8, $\pi_1(A) = \pi_3(SO_3) = \mathbb{Z}$ which may be identified with the $\mathbb{Z}$ factor of $\pi_1(Imm(S^2,\mathbb{R}^3))$. So we are looking for the generator of $\pi_1(A)$. We define an involution $\tau : A \to A$ using the coordinates of U as follows: For $i \in A$, $\tau(i)(x,y) = -i(-x,-y)$.

Exercise 23.
1. Show that for $i \in A$, the formula above for $\tau(i)$ matches smoothly with $i_0|_D$ and so indeed $\tau(i) \in A$ is well defined.
2. Show that τ is a free involution, i.e. $\tau^2 = \mathrm{Id}_A$ and $\tau(i) \neq i$ for any $i \in A$.

Now let $h(t)$ be any path in A from i_0 to $\tau(i_0)$ (so if i_0 is an embedding then $h(t)$ is an eversion of the sphere) and let $g(t)$ be the loop $h * (\tau \circ h)$. We will show that the loop g is an odd power of the generator of $\pi_1(A)$. When denoting g as a regular homotopy $g_t : F \to \mathbb{R}^3$

$(0 \le t \le 1)$, then g_t fixes D and on U it satisfies $g_t(x,y) = -g_{(t+\frac{1}{2}) \bmod 1}(-x,-y)$ and so $dg_t(x,y) = dg_{(t+\frac{1}{2}) \bmod 1}(-x,-y)$. Since g_t fixes D, we have on ∂U that $dg_t = di_0$ for all t. Define S to be the space obtained from $U \times [0,1]$ by identifying $(p,0)$ with $(p,1)$ for any $p \in U$ and identifying $\{p\} \times [0,1]$ to a point for each $p \in \partial U$.

Exercise 24. Show that S is homeomorphic to S^3, and with a right choice of homeomorphism, the antipodal map on S^3 will correspond to the map $c : S \to S$ defined by $((x,y),t) \mapsto ((-x,-y),(t+\frac{1}{2}) \bmod 1)$.

Let $K : M_{3\times 2}(\mathbb{R}) \to SO_3$ be the map assigning to $A \in M_{3\times 2}(\mathbb{R})$ the unique matrix in SO_3 whose first two columns are obtained from the columns of A by the Gram-Schmidt process. By all the above, $K \circ dg_t$ induces a well defined function $\bar{g} : S \to SO_3$ satisfying $\bar{g} \circ c = \bar{g}$. So $\bar{g}$ defines a map $G : S/c = \mathbb{R}P^3 \to SO_3 = \mathbb{R}P^3$. The degree of the lift of $\bar{g}$ to the double cover S^3 of SO_3, determines the element that $\bar{g}$ represents in $\pi_3(SO_3)$, and so the element that g_t represents in $\pi_1(A)$. But this is the same as the degree of $G : \mathbb{R}P^3 \to \mathbb{R}P^3$. So it remains to show that the degree of G is odd.

Exercise 25. Show that $h : \mathbb{R}P^3 \to \mathbb{R}P^3$ has odd degree iff $h_* : \pi_1(\mathbb{R}P^3) \to \pi_1(\mathbb{R}P^3)$ is non-trivial.

Exercise 26. Use Exercise 22 to show that indeed G induces the non-trivial homomorphism $\pi_1(\mathbb{R}P^3) \to \pi_1(\mathbb{R}P^3)$.

This completes the proof that g_t represents an odd power of the generator in $\pi_1(A)$. As you shall see, this is all we will need in the sequel, nevertheless, we will now show that for a right choice of h_t, g_t represents an actual generator, not just an odd power. Let $A' = A/\tau$ then by Exercise 23 A is a double cover of A' and so $\mathbb{Z} = \pi_1(A)$ is a subgroup of index 2 in $H = \pi_1(A')$. Furthermore, h_t defines an element $h \in H$, and what we have shown above is that h^2 is an odd power of the generator of $\pi_1(A) \subseteq \pi_1(A')$.

Exercise 27. Let H be a group and $C \subseteq H$ an infinite cyclic subgroup of index 2. Assume there is $h \in H$ such that h^2 is an odd power of the generator of C. Then H is cyclic.

Exercise 28. Conclude from the previous exercise that there exists an h_t from i_0 to $\tau(i_0)$ such that the g_t as constructed above from h_t, represents the generator of $\pi_1(A)$.

We mention that in [MB] a similar construction on one specific eversion of the sphere is shown to produce a generator, via a lengthy analysis of a sequence of 19 drawings describing that eversion.

Exercise 29. Show that if the regular homotopy $g_t : S^2 \to \mathbb{R}^3$ of Exercise 28 is connect summed at the interior of D with any fixed immersion i of a closed orientable surface F, then the loop in $Imm(F, \mathbb{R}^3)$ thus obtained is a generator of the infinite cyclic factor of $\pi_1(Imm(F, \mathbb{R}^3))$.

3.2. **Finite order invariants.** This subsection is based on [N3]

A CE point of an immersion $i : F \to \mathbb{R}^3$ is a point of self intersection of i for which the local stratum in $Imm(F, \mathbb{R}^3)$ corresponding to the self intersection, has codimension one. The codim 1 strata are divided into four types which we call: E, H, T, Q. The four types may be demonstrated by the following local representatives, where formulae in 3-space defining the different sheets involved in the self intersection, are given. Representatives of the codim 1 strata are obtained from the formulae below by setting $\lambda = 0$. Letting λ vary, we obtain a 1-parameter family of immersions which is transverse to the given codim 1 stratum.

E: $z = 0$, $z = x^2 + y^2 + \lambda$. See Figure 4, ignoring the vertical plane.

H: $z = 0$, $z = x^2 - y^2 + \lambda$. See Figure 5, ignoring the vertical plane.

T: $z = 0$, $y = 0$, $z = y + x^2 + \lambda$. See Figure 6, ignoring the vertical plane $x = 0$.

Q: $z = 0$, $y = 0$, $x = 0$, $z = x + y + \lambda$. This is simply four planes passing through one point, any three of which are in general position.

The four types of CEs are further divided into twelve types, according to the relative orientations of the sheets involved. We name them: $E^0, E^1, E^2, H^1, H^2, T^0, T^1, T^2, T^3, Q^2, Q^3, Q^4$. This set of twelve symbols is denoted $\mathcal{C}$. A co-orientation for a CE is a choice of one of the two sides of the local stratum corresponding to the CE. All but two of the above CE types are non-symmetric in the sense that the two sides of the local stratum may be distinguished via the local configuration of the CE, and for those ten CE types, permanent co-orientations for the corresponding strata are chosen once and for all. The two exceptions are H^1 and Q^2 which are completely symmetric. In fact, there does not exist a consistent choice of co-orientation for H^1 and Q^2 CEs since as we shall see the global strata corresponding to these CE types are one sided in $Imm(F, \mathbb{R}^3)$.

We fix a closed oriented surface F and a regular homotopy class $\mathcal{A}$ of immersions of F into $\mathbb{R}^3$ (that is, $\mathcal{A}$ is a connected component of $Imm(F, \mathbb{R}^3)$). We denote by $I_n \subseteq \mathcal{A}$ ($n \geq 0$) the space of all immersions in $\mathcal{A}$ which have precisely n CE points (the self intersection being elsewhere stable). In particular, I_0 is the space of all stable immersions in $\mathcal{A}$.

Given an immersion $i \in I_n$, a *temporary co-orientation* for i is a choice of co-orientation at each of the n CE points $p_1, \dots, p_n$ of i. Given a temporary co-orientation $\mathfrak{T}$ for i and a subset $A \subseteq \{p_1, \dots, p_n\}$, we define $i_{\mathfrak{T},A} \in I_0$ to be the immersion obtained from i by resolving all CEs of i at points of A into the positive side with respect to $\mathfrak{T}$, and all CEs not in A into the negative side. Now let $\mathbb{G}$ be any Abelian group and let $f : I_0 \to \mathbb{G}$ be an invariant, i.e. a function which is constant on each connected component of I_0. Given $i \in I_n$ and a temporary co-orientation $\mathfrak{T}$ for i, $f^{\mathfrak{T}}(i)$ is defined as follows:

$$f^{\mathfrak{T}}(i) = \sum_{A \subseteq \{p_1,\dots,p_n\}} (-1)^{n-|A|} f(i_{\mathfrak{T},A})$$

where $|A|$ is the number of elements in A. The statement $f^{\mathfrak{T}}(i) = 0$ is independent of the temporary co-orientation $\mathfrak{T}$ so we simply write $f(i) = 0$. An invariant $f : I_0 \to \mathbb{G}$ is called *of finite order* if there is an n such that $f(i) = 0$ for all $i \in I_{n+1}$. The minimal such n is called the *order* of f. The group of all invariants on I_0 of order at most n is denoted $V_n = V_n(\mathbb{G})$.

For an immersion $i : F \to \mathbb{R}^3$ and any $p \in \mathbb{R}^3$, we define the degree $d_p(i) \in \mathbb{Z}$ of i at p as follows: If $p \notin i(F)$ then $d_p(i)$ is the (usual) degree of the map obtained from i by composing it with the projection onto a small sphere centered at p. If on the other hand $p \in i(F)$ then we first push each sheet of F which passes through p, a bit into its preferred side determined by the orientation of F, obtaining a new immersion i' which misses p, and we define $d_p(i) = d_p(i')$.

For p a CE point of $i : F \to \mathbb{R}^3$ we define $C_p(i)$ to be the expression R^a_m where $R^a \in \mathcal{C}$ is the symbol describing the configuration of the CE of i at p (one of the twelve symbols above) and $m = d_p(i)$. $\mathcal{C}_n$ denotes the set of all *un-ordered* n-tuples of expressions R^a_m with $R^a \in \mathcal{C}, m \in \mathbb{Z}$. (So $\mathcal{C}_n$ is the set of un-ordered n-tuples of elements of $\mathcal{C}_1$.) A map $C : I_n \to \mathcal{C}_n$ is defined by $C(i) = [C_{p_1}(i), \dots, C_{p_n}(i)] \in \mathcal{C}_n$ where $p_1, \dots, p_n$ are the n CE points of i.

Exercise 30. Show that the map $C : I_n \to \mathcal{C}_n$ is surjective, that is, any n-tuple may be realized by some immersion, in any regular homotopy class.

A regular homotopy between two immersions in I_n is called an AB equivalence if it is alternatingly of type A and B, where $J_t : F \to \mathbb{R}^3$ $(0 \le t \le 1)$ is of type A if it is of the form $J_t = U_t \circ i \circ V_t$ where $i : F \to \mathbb{R}^3$ is an immersion and $U_t : \mathbb{R}^3 \to \mathbb{R}^3$, $V_t : F \to F$ are isotopies, and $J_t : F \to \mathbb{R}^3$ $(0 \le t \le 1)$ is of type B if $J_0 \in I_n$ and there are little balls $B_1, \dots, B_n \subseteq \mathbb{R}^3$ centered at the n CE points of J_0 such that J_t fixes $U = (J_0)^{-1}(\bigcup_k B_k)$ and moves $F - U$ within $\mathbb{R}^3 - \bigcup_k B_k$.

We will now prove:

Theorem 3.1. *Let $i, j \in I_n$, then i and j are AB equivalent iff $C(i) = C(j)$.*

If i and j are AB equivalent then clearly $C(i) = C(j)$. For the converse, assume $C(i) = C(j)$. One can order the CEs of i and j, respectively $p'_1, \dots, p'_n$ and $p_1, \dots, p_n$, such that $C_{p'_k}(i) = C_{p_k}(j)$, $k = 1, \dots, n$. This means in particular, that if $B'_1, \dots, B'_n$ and $B_1, \dots, B_n$ are neighborhoods of the p'_ks and p_ks respectively, then for each k there is an orientation preserving diffeomorphism from B'_k to B_k which takes each sheet of $i(F) \cap B'_k$ orientation preservingly onto the corresponding sheet of $j(F) \cap B_k$. These diffeomorphisms may all be realized by one ambient isotopy $U_t : \mathbb{R}^3 \to \mathbb{R}^3$. There is then an isotopy $V_t : F \to F$ such that the final immersion i' of the regular homotopy $U_t \circ i \circ V_t$ satisfies that i' and j have the same n CE points $p_1, \dots, p_n \in \mathbb{R}^3$, $i'^{-1}(\bigcup_k B_k) = j^{-1}(\bigcup_k B_k)$ which we name U, and $i'|_U = j|_U$. Also $d_{p_k}(i') = d_{p_k}(j)$ for $k = 1, \dots, n$. Now U is a union of some discs $D_1, \dots, D_r$. We construct the following handle decomposition of F. $D_1, \dots, D_r$ will be the 0-handles. If g is the genus of F we will have 1-handles $h_1, \dots, h_{2g+r-1}$ as follows: $h_1, \dots, h_{2g}$ will each have both ends glued to D_1 such that $F - (D_1 \cup h_1 \cup \dots \cup h_{2g})$ will be a disc containing $D_2, \dots, D_r$. Then for $k = 1, \dots r - 1$, h_{2g+k} will have one end glued to D_k and the other to D_{k+1}. The complement of all 0- and 1-handles is again one disc, which will be the unique 2-handle. We will now construct a regular homotopy of the form $i' \circ V'_t$ ($V'_t : F \to F$ an isotopy) from i' to an immersion i'' which will have the property that the restrictions of i'' and j to all 1-handles, are regularly homotopic keeping all 0-handles fixed. Since i' and j are regularly homotopic (recall $i, j \in I_n \subseteq \mathcal{A}$), this is already true for $h_1, \dots, h_{2g}$. Now take h_{2g+1}. If $i'|_{h_{2g+1}}$ and $j|_{h_{2g+1}}$ are not regularly homotopic keeping D_1 and D_2 fixed, then V'_t performs one full rotation of D_2, creating one full twist in a thin annulus around D_2 in F. h_{2g+1} will now satisfy the needed property. Note also that this rotation of D_2 moves only h_{2g+1} and h_{2g+2}, keeping all other 0- and 1-handles fixed. We continue this way along the chain of 1-handles, rotating D_{k+1} if necessary for the sake of h_{2g+k}. For $k < r - 1$ this will also move h_{2g+k+1}, but we never need to move 1-handles that have previously been taken care of. Also $d_{p_k}(i'') = d_{p_k}(i) = d_{p_k}(j)$ for all $k = 1, \dots, n$. We now perform a regular homotopy H_t on the union of 0- and 1-handles which fixes the 0-handles, and regularly homotopes each 1-handle h, from $i''|_h$ to $j|_h$, avoiding $\bigcup_k B_k$. This is possible by the construction of i''. Denote our 2-handle by D. So far we have constructed H_t only on $F - D$. By Exercise 9 H_t may be extended to D, still avoiding $\bigcup_k B_k$, arriving at an immersion i'''. And so, we are left with regularly homotoping $i'''|_D$ to $j|_D$ (relative ∂D). Since $d_{p_k}(i''') = d_{p_k}(j)$ for all

$k = 1, \ldots, n$, these maps are homotopic in $\mathbb{R}^3 - \bigcup_k B_k$. It then follows from SHT that they are also *regularly* homotopic in $\mathbb{R}^3 - \bigcup_k B_k$ (since $\pi_2(SO_3) = 0$).

The regular homotopy from i to i'' was of type A, and that from i'' to j was of type B. This completes the proof of Theorem 3.1.

Exercise 31. 1. Use the construction appearing in the proof of Theorem 3.1 to show that for a CE of type H^1 or Q^2 of $i \in I_n$ located at p, there is an AB equivalence from i to itself such that when carrying the co-orientations along, it will come back to itself at p with the reversed co-orientation, and will come back at all other $n-1$ CEs with the original co-orientation.

2. Conclude that if $f \in V_n$ and $i \in I_n$ has at least one CE of type H^1 or Q^2 and $\mathfrak{T}$ is a temporary co-orientation for i, then $2f^{\mathfrak{T}}(i) = 0$, and so in this case $f^{\mathfrak{T}}(i)$ is independent of $\mathfrak{T}$.

This fact is used to extend any $f \in V_n$ to I_n by setting for any $i \in I_n$, $f(i) = f^{\mathfrak{T}}(i)$, where if i includes at least one CE of type H^1 or Q^2 then $\mathfrak{T}$ is arbitrary, and if all CEs of i are not of type H^1 or Q^2 then the permanent co-orientation is used for all CEs of i. We will always assume without mention that any $f \in V_n$ is extended to I_n in this way. (If $f \in V_n$ then we are not extending f to I_k for $0 < k < n$).

Exercise 32. For $f \in V_n$ and $i, j \in I_n$, if $C(i) = C(j)$ then $f(i) = f(j)$. (Hint: Theorem 3.1.)

So any $f \in V_n$ induces a well defined function $u(f) : \mathcal{C}_n \to \mathbb{G}$. The map $f \mapsto u(f)$ induces an injection $u : V_n / V_{n-1} \to \mathcal{C}_n^*$ where $\mathcal{C}_n^*$ is the group of all functions from $\mathcal{C}_n$ to $\mathbb{G}$. Finding the image of u for all n gives a classification of all finite order invariants.

A subgroup $\Delta_n = \Delta_n(\mathbb{G}) \subseteq \mathcal{C}_n^*$ which contains the image of u is defined as the set of functions in $\mathcal{C}_n^*$ satisfying relations which we write as relations on the symbols R^a_m, e.g. $T^0_m = T^3_m$ will stand for the set of all relations of the form $g([T^0_m, {R_2}^{a_2}_{d_2}, \ldots, {R_n}^{a_n}_{d_n}]) = g([T^3_m, {R_2}^{a_2}_{d_2}, \ldots, {R_n}^{a_n}_{d_n}])$ with arbitrary ${R_2}^{a_2}_{d_2}, \ldots, {R_n}^{a_n}_{d_n} \in \mathcal{C}_1$. The relations are obtained from examining the CEs occurring when going in $Imm(F, \mathbb{R}^3)$ around strata of codimension 2. There are six types of such strata as presented below. The classification of the possible codimension 2 strata relies on [HK].

For each of the first five types we give the following:

1. Formula for a local representative.
2. Sketch of the configuration for some value (λ_1, λ_2) of the parameters.

3. Diagram of the 2 dimensional parameter space, where intersection with the codim 1 strata is depicted, including their co-orientations (this is called a bifurcation diagram).
4. The relation arising.

For these five types, the bifurcation diagram is obtained from the sketch and formula in a straight forward manner. Whenever the plane $x = 0$ appears in a configuration below, we assume (by rotating the configuration if necessary) that its preferred side is $x > 0$. The integer m in terms of which the degrees of the CEs are given, is the degree of the central codim 2 immersion at its codim 2 self intersection. The sixth type is named QQ; it requires special analysis which appears in [N3].

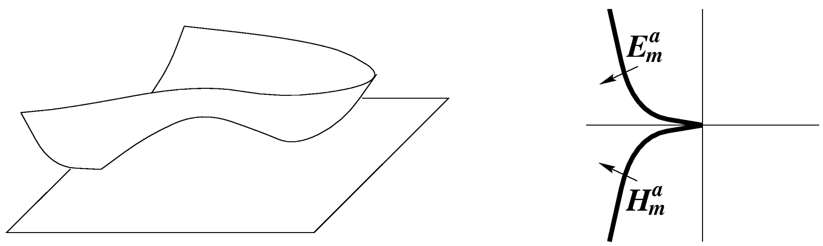

FIGURE 2. EH configuration

EH: $z = 0, \quad z = y^2 + x^3 + \lambda_1 x + \lambda_2.$

(1) $$0 = E^a_m - H^a_m$$

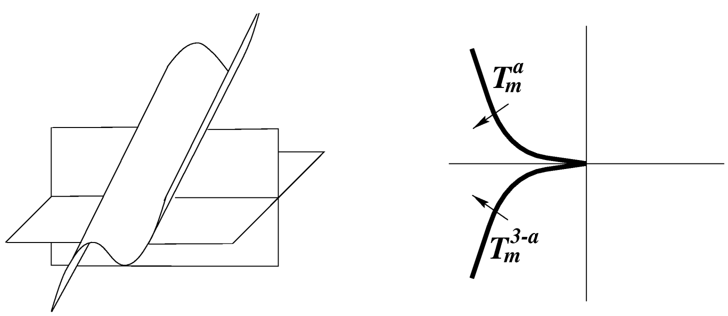

FIGURE 3. TT configuration

TT: $z = 0, \quad y = 0, \quad z = y + x^3 + \lambda_1 x + \lambda_2.$

(2) $$0 = T^a_m - T^{3-a}_m$$

ET: $z = 0, \quad x = 0, \quad z = (x - \lambda_1)^2 + y^2 + \lambda_2.$

(3) $$0 = T^a_m - T^{a+1}_m - E^a_{m-1} + E^a_m$$

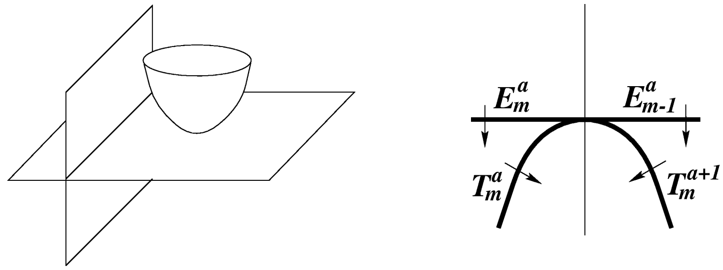

FIGURE 4. ET configuration

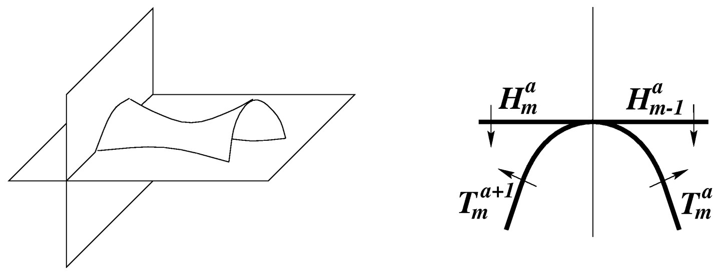

FIGURE 5. HT configuration

HT: $z = 0, \quad x = 0, \quad z = (x-\lambda_1)^2 - y^2 + \lambda_2.$

$$0 = -T^{a+1}_m + T^a_m - H^a_{m-1} + H^a_m \tag{4}$$

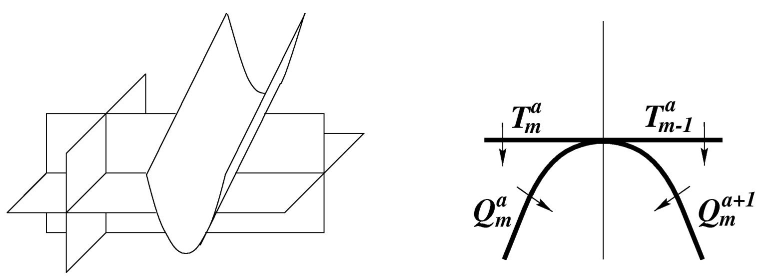

FIGURE 6. TQ configuration

TQ: $z = 0, \quad y = 0, \quad x = 0, \quad z = y + (x-\lambda_1)^2 + \lambda_2.$

$$0 = Q^a_m - Q^{a+1}_m - T^a_{m-1} + T^a_m \tag{5}$$

The sixth type QQ, whose analysis appears in [N3], leads the to the following relation:

$$Q^2_m = Q^2_{m-1} \tag{6}$$

These six relations, together with the relations coming from Exercise 31, are the relations with which we define Δ_n. They may be summed up as follows:

- $E^2_m = -E^0_m = H^2_m, \quad E^1_m = H^1_m.$

- $T^0_m = T^3_m, \quad T^1_m = T^2_m$.
- $2H^1_m = 0, \quad H^1_m = H^1_{m-1}$.
- $2Q^2_m = 0, \quad Q^2_m = Q^2_{m-1}$.
- $H^2_m - H^2_{m-1} = T^3_m - T^2_m$.
- $Q^4_m - Q^3_m = T^3_m - T^3_{m-1}, \quad Q^3_m - Q^2_m = T^2_m - T^2_{m-1}$.

Let $\mathbb{B} \subseteq \mathbb{G}$ be the subgroup defined by $\mathbb{B} = \{x \in \mathbb{G} : 2x = 0\}$. To obtain a function $g \in \Delta_1$ one may assign arbitrary values in $\mathbb{G}$ for the symbols $\{T^2_m\}_{m\in\mathbb{Z}}$, $\{H^2_m\}_{m\in\mathbb{Z}}$ (here we use the convention of [N5] not [N3],[N4]) and arbitrary values in $\mathbb{B}$ for the two symbols H^1_0, Q^2_0. Once this is done then the value of g on all other symbols is uniquely determined, namely:

1. $E^1_m = H^1_m = H^1_0$ for all m.
2. $E^2_m = -E^0_m = H^2_m$ for all m.
3. $T^3_m = T^2_m + H^2_m - H^2_{m-1}$
4. $T^0_m = T^3_m, \quad T^1_m = T^2_m$ for all m.
5. $Q^2_m = Q^2_0$ for all m.
6. $Q^3_m (= Q^2_m + T^2_m - T^2_{m-1}) = Q^0_m + T^2_m - T^2_{m-1}$ for all m.
7. $Q^4_m (= Q^3_m + T^3_m - T^3_{m-1}) = Q^0_m + 2T^2_m - 2T^2_{m-1} + H^2_m - 2H^2_{m-1} + H^2_{m-2}$ for all m.

We will refer to this procedure as the "7-step procedure".

Let $g_Q \in \Delta_1(\mathbb{Z}/2)$ be defined by $g_Q(Q^2_0) = 1 \in \mathbb{Z}/2$ and $g_Q(H^1_0) = g_Q(T^2_m) = g_Q(H^2_m) = 0$ for all m. By the 7-step procedure this extends to $g_Q(Q^a_m) = 1 \in \mathbb{Z}/2$ for all a, m and $g_Q(E^a_m) = g_Q(H^a_m) = g_Q(T^a_m) = 0$ for all a, m. We would like to establish the existence of an order 1 invariant $Q : I_0 \to \mathcal{A}$ satisfying $u(Q) = g_Q$. Along a regular homotopy in $\mathcal{A}$, such Q will change by $1 \in \mathbb{Z}/2$ whenever we pass a quadruple point, and will remain unchanged whenever we pass a CE of other type. So we may try to construct such Q by choosing a base immersion $i \in I_0$ and for any $j \in I_0$ define $Q(j)$ to be the number mod 2 of quadruple point occurring in a regular homotopy from i to j. This is well defined iff the number of quadruple points along any closed loop in $\mathcal{A}$ is 0 mod 2.

Exercise 33. Show that since $g_Q \in \Delta_1(\mathbb{Z}/2)$, the number mod 2 of quadruple points occurring in any null-homotopic loop in $\mathcal{A}$ is $0 \in \mathbb{Z}/2$.

So the number mod 2 of quadruple points is well defined on $\pi_1(\mathcal{A})$ and so it is enough to verify that it is 0 on generators of $\pi_1(\mathcal{A})$. The order 2 generator corresponds to rigid rotation of an immersion by one full rotation. This produces no quadruple points at all. As to the generator of the infinite cyclic factor C, let us begin with the case of $F = S^2$. By

Exercise 28 there exists a generating loop g such that the second half of g is obtained from the first half by the involution τ. But then it is clear that the number of quadruple points occurring in the first and second halves of g is equal, and so the total number is 0 mod 2. (Note that since we are interested in a homomorphism into $\mathbb{Z}/2$, g_t being some odd power of the generator would have sufficed.)

For general F, take the loop g we have for S^2, and choose a point x on a ray perpendicular to the fixed image of D in $\mathbb{R}^3$, x being far enough so that the image of g_t never passes x. Now change the constant embedding of D to be one with a thin "thorn" pulled out of D and embedded along this ray, reaching x. The new loop g'_t obtained in this way must also have 0 mod 2 quadruple points, and has the property that a given point in the fixed disc D (namely, the tip of the thorn) does not participate in any self intersections throughout g'_t. Now take a tiny immersion of F in I_0, located near the tip of our fixed thorn, and connect sum it with g'_t (for all t), obtaining a loop $f_t : F \to \mathbb{R}^3$. By Exercise 29 this loop $f_t : F \to \mathbb{R}^3$ represents a generator of the infinite cyclic factor C for F. Since the tip of the thorn was far away from any self intersections occurring in g'_t, the tiny fixed immersion of F will not contribute any additional quadruple points, and so this remains 0 mod 2. This completes the proof that Q is well defined for any surface F.

We now repeat this procedure for another $\mathbb{Z}/2$ valued order 1 invariant. Let $g_M \in \Delta_1(\mathbb{Z}/2)$ be defined by $g_M(H^1_0) = 1 \in \mathbb{Z}/2$ and $g_M(Q^2_0) = g_M(T^2_m) = g_M(H^2_m) = 0$ for all m. By the 7-step procedure this extends to $g_M(H^1_m) = g_M(E^1_m) = 1$ for all m, $g_M(H^a_m) = g_M(E^a_m) = 0$ for $a \neq 1$ and any m and $g_M(T^a_m) = g_M(Q^a_m) = 0$ for all a, m. As for Q, we would like to establish the existence of an order 1 invariant $M : I_0 \to \mathcal{A}$ satisfying $u(M) = g_M$. Along a regular homotopy in $\mathcal{A}$, such M will change by $1 \in \mathbb{Z}/2$ iff the CE we are passing is a "matching tangency", i.e. tangency of two sheets of the surface where the orientations of the two sheets match at time of tangency. (Thus the name M for this invariant). As before, M exists iff the number of matching tangencies along any closed loop in $\mathcal{A}$ is 0 mod 2. The same argument as for Q shows that this is also true for M.

The Abelian group $\mathbb{G}_U$ is defined as follows (again this is the convention of [N5] not [N3],[N4]):

$$\mathbb{G}_U = \left\langle \{t^2_m\}_{m\in\mathbb{Z}}, \{h^2_m\}_{m\in\mathbb{Z}}, h^1_0, q^2_0 \mid 2h^1_0 = 2q^2_0 = 0 \right\rangle .$$

The universal element $g^U \in \Delta_1(\mathbb{G}_U)$ is defined by $g^U(T^2_m) = t^2_m$, $g^U(H^2_m) = h^2_m$, $g^U(H^1_0) = h^1_0$, $g^U(Q^2_0) = q^2_0$ and the value of g^U on all other symbols of $\mathcal{C}_1$ is determined by the 7-step procedure. We will prove the existence of an order 1 invariant $f^U : I_0 \to \mathbb{G}_U$ with

$u(f^U) = g^U$. Our proof will somewhat differ from that appearing in [N3]. We have already proved the existence of the $\mathbb{Z}/2$ valued invariants Q and M, and so the existence of the projection of f^U to the subgroup of $\mathbb{G}_U$ generated by h_0^1, q_0^2. We will establish the existence of the complementary projection of f^U to the subgroup K generated by $\{t_m^2\}_{m\in\mathbb{Z}} \cup \{h_m^2\}_{m\in\mathbb{Z}}$, by giving an explicit formula for this invariant in the following section. The invariant f^U, if exists, is a *universal* order 1 invariant, meaning the following:

Definition 3.2. A pair $(\mathbb{G}, f)$ where $\mathbb{G}$ is an Abelian group and $f : I_0 \to \mathbb{G}$ is an order n invariant, will be called a *universal order n invariant* if for any Abelian group $\mathbb{G}'$ and any order n invariant $f' : I_0 \to \mathbb{G}'$ there exists a unique homomorphism $\varphi : \mathbb{G} \to \mathbb{G}'$ such that $f' - \varphi \circ f$ is an invariant of order at most $n - 1$.

In [N4] all higher order invariants are classified, and for every n a universal order n invariant is constructed as $\mathcal{F}_n \circ f^U$ where $\mathcal{F}_n : \mathbb{G}_U \to M_n$ is an explicit function (not homomorphism) into a certain Abelian group M_n.

4. Formulae for order one invariants

4.1. **Formula for f^K.** This subsection is based on [N5].

Recall $K \subseteq \mathbb{G}_U$ is the subgroup generated by $\{t_m^2\}_{m\in\mathbb{Z}} \cup \{h_m^2\}_{m\in\mathbb{Z}}$ and let $g^K \in \Delta_1(K)$ be the projection of g^U to K. We will now establish the existence of order 1 invariant $f^K : I_0 \to K$ satisfying $u(f^K) = g^K$. We will do this by explicitly constructing f^K. This will complete the proof of the existence of f^U.

Let $i \in I_0$. For every $m \in \mathbb{Z}$ let $U_m = U_m(i) = \{p \in \mathbb{R}^3 - i(F) \ : \ d_p(i) = m\}$. This is an open set in $\mathbb{R}^3$ which may be empty, and may be non-connected or unbounded, but in any case, the Euler characteristic $\chi(U_m)$ is defined. Denote by $N_m = N_m(i)$ the number of triple points $p \in \mathbb{R}^3$ of i having $d_p(i) = m$.

We define the group $\mathbb{O}$ to be the free Abelian group with generators $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$. For $i \in I_0$ we define $k(i) \in \mathbb{O}$ as follows (the sums are always finite):

$$k(i) = \sum_{m\in\mathbb{Z}} \chi(U_m)x_m + \sum_{m\in\mathbb{Z}} \frac{1}{2}N_m y_m.$$

Indeed this is an element of $\mathbb{O}$ since as we shall see below, N_m is always even. In the mean time say k attains values in the $\mathbb{Q}$ vector space with same basis.

Exercise 34. The invariant k is an order 1 invariant, with $u(k)$ given by:

- $u(k)(E_m^a) = u(k)(H_m^a) = x_{m+a-2} - x_{m-a}$

- $u(k)(T^a_m) = x_{m+a-3} + x_{m-a} + y_m$
- $u(k)(Q^a_m) = x_{m+a-4} - x_{m-a} + (a-2)y_m + (2-a)y_{m-1}$

We can now verify that indeed the values of k are in $\mathbb{O}$ i.e. no half integer coefficients appear (which means N_m is always even). From Exercise 34 we see that the change in the value of k is in $\mathbb{O}$ along any regular homotopy, and so it is enough to show that the value is in $\mathbb{O}$ for one immersion in any given regular homotopy class.

Exercise 35. Let g be the genus of F. Any immersion $i : F \to \mathbb{R}^3$ is regularly homotopic to an immersion j with $k(j) = (2-g)x_0 + (1-g)x_{-1}$.

We define a homomorphism $\varphi : \mathbb{G}_U \to \mathbb{O}$ on generators as follows:

- $\varphi(h^2_m) = x_m - x_{m-2}$
- $\varphi(t^2_m) = x_{m-1} + x_{m-2} + y_m$
- $\varphi(h^1_0) = \varphi(q^2_0) = 0$

By Exercise 34, $u(k) = \varphi \circ g^U$. We define the following homomorphism $F : \mathbb{O} \to K$ satisfying that $F \circ \varphi$ is the projection of $\mathbb{G}_U$ onto K, and so $u(F \circ k) = F \circ \varphi \circ g^U = g^K$. So $f^K = F \circ k$ is the invariant we are seeking. We define F on generators of $\mathbb{O}$ as follows:

$$F(x_m) = \sum_{-\frac{1}{2} < k < \lfloor \frac{m}{2} \rfloor + \frac{1}{2}} h^2_{m-2k} \qquad\qquad F(y_m) = t^2_m - \sum_{-\frac{1}{2} < k < m - \frac{1}{2}} h^2_k$$

where for $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the greatest integer $\leq a$, and for $a, b \in \mathbb{R}$ the sum $\sum_{a<k<b}$ means the following: If $a < b$ then it is the sum over all integers $a < k < b$, if $a = b$ then the sum is 0, and if $a > b$ then $\sum_{a<k<b} = -\sum_{b<k<a}$.

Exercise 36. Check that indeed $F \circ \varphi$ maps each generator of K to itself.

Composing the formula for F with the formula for k we obtain our formula for f^K:

$$f^K(i) = \sum_{m \in \mathbb{Z}} \chi(U_m) \Bigg(\sum_{-\frac{1}{2} < k < \lfloor \frac{m}{2} \rfloor + \frac{1}{2}} h^2_{m-2k} \Bigg) + \sum_{m \in \mathbb{Z}} \frac{1}{2} N_m \Bigg(t^2_m - \sum_{-\frac{1}{2} < k < m - \frac{1}{2}} h^2_k \Bigg).$$

The existence of f^K together with the existence of Q and M (which we have proved in a completely different way), establishes the existence of the invariant f^U satisfying $u(f^U) = g^U$.

Exercise 37. Use the existence of f^U to show that for any $\mathbb{G}$, the map $u : V_1(\mathbb{G}) \to \Delta_1(\mathbb{G})$ is surjective.

4.2. **A formula for Q.** This subsection is based on [N2]

A general explicit formula as we have for f^K, is not known for Q and M. We will present a formula for one specific cases, others may be found in [N2] (for Q) and [N5] (for M). (And as mentioned, all analogous results for non-orientable surfaces may be found in [N6].) We emphasize that as opposed to our formula for f^K, which itself proved the existence of f^K, in the case of Q and M the formulae rely on the existence. Another difference between f^K and Q, M is that, as may be seen from the formula for f^K, its value on an immersion i depends only on the oriented image of i. For Q and M this is not so. Their value does change when composing i with orientation preserving diffeomorphisms of F. Indeed we will now present the dependence of Q on composition with diffeomorphisms of F.

For two regularly homotopic generic immersions $i, j : F \to \mathbb{R}^3$ we denote $Q(i,j) = Q(i) - Q(j)$, that is $Q(i,j)$ is the number mod 2 of quadruple points in any regular homotopy between i and j. The following holds (for F closed orientable): For any generic immersion $i : F \to \mathbb{R}^3$ and any diffeomorphism $h : F \to F$ such that i and $i \circ h$ are regularly homotopic,

$$Q(i, i\circ h) = \Big(\mathrm{rank}(h_* - \mathrm{Id}) + (n+1)\epsilon(h)\Big) \bmod 2,$$

where h_* is the map induced by h on $H_1(F, \mathbb{Z}/2)$, n is the genus of F and $\epsilon(h)$ is 0 or 1 according to whether h is orientation preserving or reversing, respectively.

We will present a certain fragment of the proof. The rest may be found in [N2]. (Note that the Arf invariant there is taken as $0, 1 \in \mathbb{Z}/2$.) For given immersion $i : F \to \mathbb{R}^3$ we are considering the group of all diffeomorphisms $h : F \to F$ (up to isotopy in F) such that i and $i \circ h$ are regularly homotopic. We name this group M_i. By Exercise 16.5 this is the group of all h such that $h_* : H_1(F, \mathbb{Z}/2) \to H_1(F, \mathbb{Z}/2)$ preserves g^i.

Exercise 38. The map $M_i \to \mathbb{Z}/2$ given by $h \mapsto Q(i, i\circ h)$ is a homomorphism.

It is shown in [N2] that the map $M_i \to \mathbb{Z}/2$ given by $h \mapsto \mathrm{rank}(h_* - \mathrm{Id})$ also defines a homomorphism, and $h \mapsto \epsilon(h)$ is clearly a homomorphism, and so it is enough to verify the proposed formula for $Q(i, i \circ h)$ only on a set of generators of M_i.

In [N2] a set of generators for this group is found, and for each such generator h (except for one special case) there is an embedded loop c in F, which separates F into two subsurfaces F_1, F_2 of smaller genus, such that $h(F_1) = F_1$ and $h(F_2) = F_2$. This allows an inductive argument, and we will present the inductive step here.

Let h be a generator and c, F_1, F_2 be as described above. Let A be a neighborhood of c in F. Slightly diminishing F_1, F_2 to be the components of $F - intA$, we may still assume

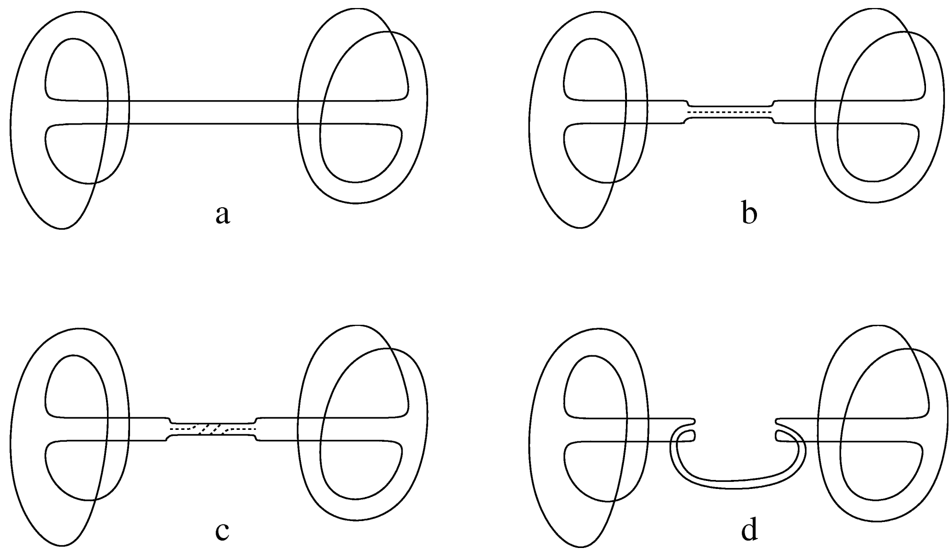

FIGURE 7

$h(F_k) = F_k$, $k = 1, 2$. Since c is separating in F, $[c] = 0$ in $H_1(F, \mathbb{Z}/2)$, so $g^i([c]) = 0$ and so $i|_A$ is regularly homotopic to a standard embedding of A, in the shape of a thin tube. By Exercise 9 we may extend such a regular homotopy of A to the whole of F. We now stretch this tube to be very long, at the same time pulling F_1 and F_2 rigidly away from each other until they are disjoint. See Fig. 7a. By taking a smaller A if necessary, we may assume $i(A)$ is disjoint from $i(F - A)$, Fig. 7b.

Exercise 39. If $h \in M_i$ and j is regularly homotopic to i then $h \in M_j$ and $Q(i, i \circ h) = Q(j, j \circ h)$.

By Exercise 39, we may assume that the new immersion that we obtained is in fact our immersion i. Let $\bar{F}_1, \bar{F}_2$ be the closed surfaces obtained by gluing a disc D_k to F_k and let $h_k : \bar{F}_k \to \bar{F}_k$ be an extension of $h|_{F_k} : F_k \to F_k$. If the tube $i(A)$ is very thin, then there is also a naturally defined extension $i_k : \bar{F}_k \to \mathbb{R}^3$ of $i|_{F_k}$. We may further assume that the thin ball B in $\mathbb{R}^3$ which is bounded by the sphere $i_1(D_1) \cup i(A) \cup i_2(D_2)$, is disjoint from $i(F - A)$.

Since $h|_{F_k}$ preserves $g^i|_{H_1(F_k, \mathbb{Z}/2)}$ then h_k preserves g^{i_k}. It follows that there is a regular homotopy H_t^k between i_k and $i_k \circ h_k$. We perform H_t^1 and H_t^2 inside disjoint balls, and we let the thin tube A be carried along. We can make sure that no quadruple point of H_t^k occurs in D_k $(k = 0, 1)$ and that the very thin tube A does not pass triple points. The regular homotopy H_t induced on F in this way will then have the sum of the numbers of quadruple points of H_t^1 and H_t^2.

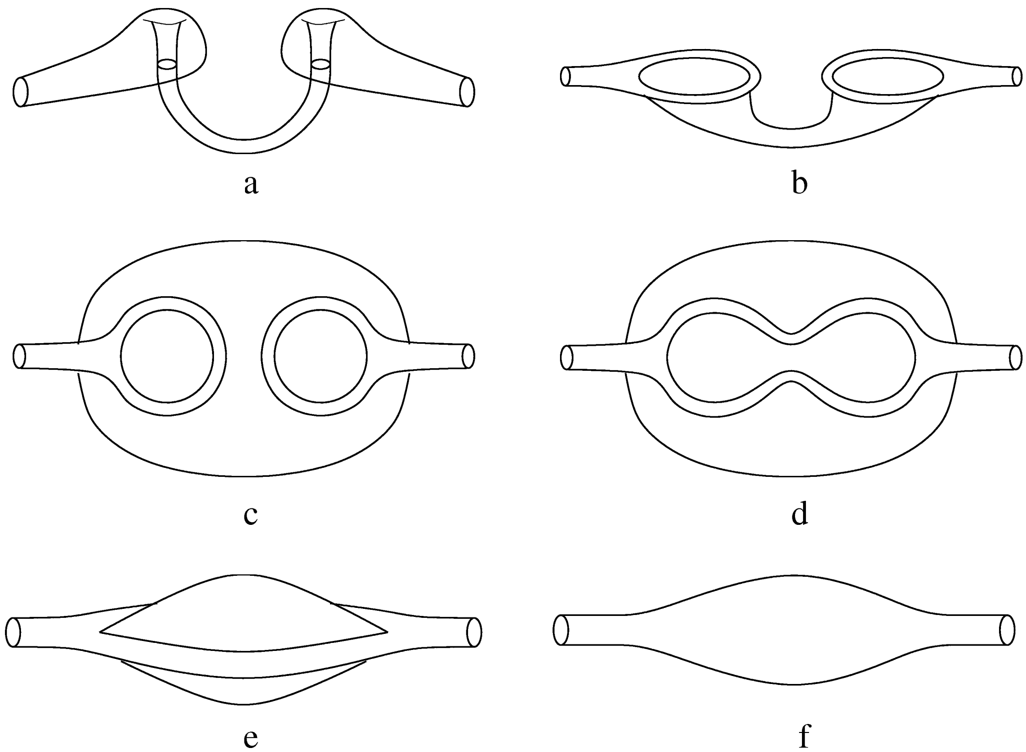

FIGURE 8

Now if h is orientation preserving then so are h_k, in particular $h_k|_{D_k}$ is orientation preserving. So if we had carried the thin ball B along with the tube A, then it would now approach the D_ks from the same side it had for i. And so we may continue the regular homotopy on the tube A, still not passing through triple points, and cancelling all knotting by having the thin tube pass itself, until it is back to its original place, and this will not contribute any quadruple points. However, the new embedding of A may differ from $i \circ h|_A$ by some number of twists as in Fig. 7c. We may resolve this by rigidly rotating say F_1 around the axis of the tube.

If on the other hand h is orientation reversing, then after applying H_t^1 and H_t^2 and carrying the tube along, the thin ball B will approach both D_ks from the wrong side. And so after we cancel all knotting, the tube A will be as in Fig 7d. Fig. 8 presents a regular homotopy that resolves this, and has 1 mod 2 quadruple points. Fig. 8a depicts the relevant part of Fig. 7d, where the regular homotopy will take place. Fig. 8a→b→c is a regular homotopy with no singular occurrences, or alternatively may be thought of as an ambient isotopy of $\mathbb{R}^3$. It shows that we may view the immersion of A as a sphere with two rings facing outward, each of which has a tube attached to it.

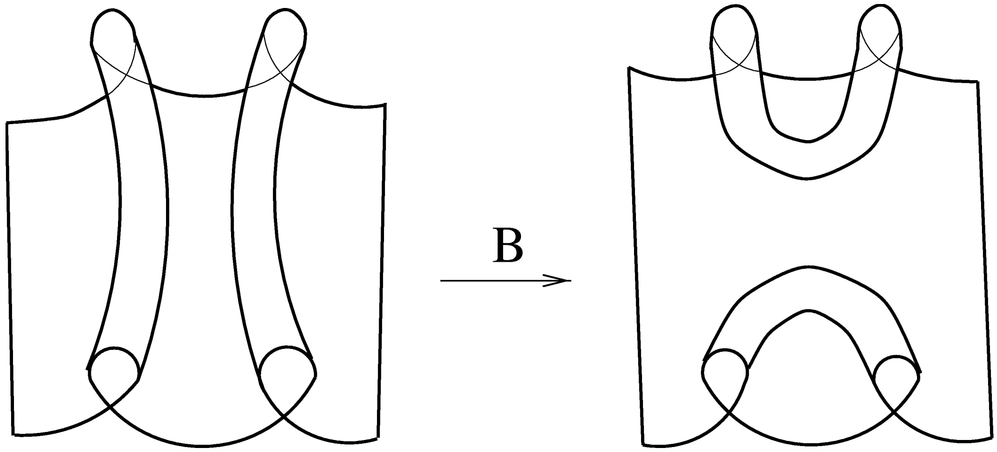

FIGURE 9

Exercise 40. Show that Fig. 8c is regularly homotopic to Fig. 8d by showing that the two immersions of a disc appearing in Fig. 9 are regularly homotopic while keeping a neighborhood of the boundary fixed. (Hint: $\pi_2(SO_3) = 0$.)

Then by ambient isotopy, the ring may be brought to the equator, Fig. 8d→e. Finally we exchange the northern and southern halves of the sphere, arriving at an embedding, Fig. 8e→f. We then continue to bring A back to place. As above, the new embedding of A may differ from $i \circ h|_A$ by some number of twists, and those may be cancelled by rigidly rotating F_1.

We will now show that the number of quadruple points occurring in the regular homotopy just described is necessarily 1 mod 2. Indeed, this number does not depend on the F_1, F_2 attached on the two sides, so imagine they are simply discs. Then this gives a certain regular homotopy of S^2 which we would now like to analyze.

Exercise 41. Show that this regular homotopy of S^2 that we have obtained, necessarily has the same number mod 2 of quadruple points as an eversion of the sphere.

So we must check what this number is for an eversion of the sphere. Since Q is well defined, indeed for all eversions this value is the same, and it is enough to look at one such eversion. In [Ma] an explicit eversion is described with precisely 1 quadruple point, and so the number is 1 mod 2 for any eversion.

Back to our F and h: We have constructed a regular homotopy between i and $i \circ h$ such that the number mod 2 of quadruple points, is the sum of the numbers occurring in the $\bar{F}_k$s in case h is orientation preserving, and the sum plus 1, in case h is orientation reversing. In other words $Q(i, i \circ h) = Q(i_1, i_1 \circ h_1) + Q(i_2, i_2 \circ h_2) + \epsilon(h)$.

Exercise 42. Assuming the truth of the formula for $Q(i_1, i_1 \circ h_1)$ and $Q(i_2, i_2 \circ h_2)$, deduce its truth for $Q(i, i \circ h)$.

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