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COMPLEMENTARY REGIONS FOR MAPS OF SURFACES

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ABSTRACT

Let F be a closed connected surface, M a closed connected 3-manifold with $H_1(M, \mathbb{Z}/2) = 0$, and $i : F \to M$ a generic map. Then $M - i(F)$ is a union of connected regions, which may be colored black and white by a checkerboard coloring. This coloring induces a color black or white to each cross-cap of i, namely, the color of the majority of the three local regions in its neighborhood. For $k \geq 0$, let a_k and b_k respectively, be the number of black and white components U, with $\chi(U) = 1 - k$. Let C_a, C_b respectively be the number of black and white cross-caps of i. Two more integers attached to i are the number N of triple points of i, and $\chi = \chi(F)$. In this work, we determine what sets of data $(\{a_k\}, \{b_k\}, \chi, N, C_a, C_b)$ may appear in this way.

Keyword: Maps of surfaces.

Mathematics Subject Classification 2000: 57M99, 57R35

1. The Setting and Statement of Result

Let F be a closed connected surface and M a closed connected 3-manifold with $H_1(M, \mathbb{Z}/2) = 0$. For $i : F \to M$ a generic map, we will be interested in the connected components of $M - i(F)$. They will be called the *complementary regions* of i, or simply the *regions* of i. Choose one point $p_0 \in M - i(F)$ and color it black. This determines a color black or white for any point in $M - i(F)$ according to the following prescription: If $p \in M - i(F)$, we connect p to p_0 with a curve γ in general position with respect to $i(F)$, and we color p black or white according to whether γ intersects $i(F)$ an even or odd number of times, respectively. This is indeed well defined since $H_1(M, \mathbb{Z}/2) = 0$. We will be interested in the collection of Euler characteristics that may appear for the set of regions, so we first prove:

Lemma 1.1. *If U is a complementary region of $i : F \to M$, then $\chi(U) \leq 1$.*

Proof. Let ∂U be the natural notion of a boundary for U. It is enough to show that ∂U is connected, since then $\chi(\partial U) \leq 2$ and so $\chi(U) = \frac{1}{2}\chi(\partial U) \leq 1$. So assume ∂U has at least two connected components S_1, S_2, and let $T \subseteq U$ be a surface parallel to S_1. There is a path in U from S_1 to S_2 crossing T precisely once, and since $i(F)$ is connected and disjoint from T, this path can be completed to a loop in M crossing T precisely once, contradicting $H_1(M, \mathbb{Z}/2) = 0$. □

Given a generic map $i : F \to M$, color $M - i(F)$ as above, and we define two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ of non-negative integers as follows: Let $a_k = a_k(i)$ be the number of black regions U with $\chi(U) = 1 - k$ and let $b_k = b_k(i)$ be the number of white regions U with $\chi(U) = 1 - k$. Given a cross-cap of i located at $p \in M$, a little neighborhood of p in M is divided by $i(F)$ into three regions, two regions of the same color, and the third region of the opposite color. A cross-cap will be called black or white, according to the color of the majority of these three local regions. We denote by $C_a = C_a(i)$ the number of black cross-caps, and by $C_b = C_b(i)$ the number of white cross-caps. Note that the total number $C_a + C_b$ of cross-caps must be even, since they are the boundary points of the intersection curve in M. We attach two more integers to such a map, the number $N = N(i)$ of triple points of i, and $\chi = \chi(F)$. Our goal in this work is to determine what sets of data $(\{a_k\}, \{b_k\}, \chi, N, C_a, C_b)$ may arise in this way. We will prove:

Theorem 1.2. *Let M be a closed connected 3-manifold with $H_1(M, \mathbb{Z}/2) = 0$. Let $\{a_k\}_{k \geq 0}$, $\{b_k\}_{k \geq 0}$ be two sequences of non-negative integers which are not identically 0. Let χ, N, C_a, C_b be integers satisfying $\chi \leq 2$, $N, C_a, C_b \geq 0$ and $C_a + C_b$ is even. Then there is a closed connected surface F with $\chi(F) = \chi$ and a generic map $i : F \to M$ with N triple points, C_a black cross-caps and C_b white cross-caps which realizes the sequences $\{a_k\}, \{b_k\}$, if and only if the following equations hold*:

$$\sum_k (1-k)a_k = \frac{1}{2}(\chi + N + C_a), \quad \sum_k (1-k)b_k = \frac{1}{2}(\chi + N + C_b).$$

The "only if" part of Theorem 1.2 is already known, following from more general results in [5]. We will present a self contained proof in Sec. 2. The "if" part is the new result of this paper, and will be proved in Sec. 3.

2. Any Map Satisfies the Equations

In this section, we show that any generic map $i : F \to M$ satisfies the equalities of Theorem 1.2. This fact follows from [5], namely, from Theorem 3.6 and the first five rows of Table 1, observing that their $\chi_B(f)$ and $\chi_R(f)$ are twice our $\sum_k (1-k)a_k$ and $\sum_k (1-k)b_k$ respectively, since their $f(X)_B$ and $f(X)_R$ are the boundaries of the unions of our black and white regions respectively. We present the following self contained proof.

For a generic map $i : F \to M$, let $A = A(i)$ be the union of all black regions and $B = B(i)$ the union of all white regions of i, then $\chi(A) = \sum_k (1-k)a_k$ and $\chi(B) = \sum_k (1-k)b_k$. So we must show $\chi(A) = \frac{1}{2}(\chi + N + C_a)$ and $\chi(B) = \frac{1}{2}(\chi + N + C_b)$. We show this by induction on $N + C_a + C_b$. If $N = C_a = C_b = 0$,

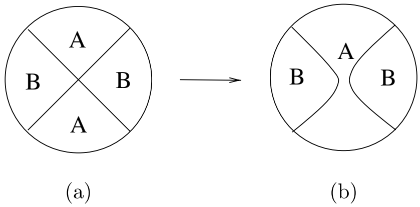


Fig. 1.

then we have an immersion with no triple points, so the intersection set is a disjoint union of embedded circles. A transverse cross section of a regular neighborhood of such a circle appears as in Fig. 1(a). We perform a change in F and i as prescribed in Fig. 1(b). This cuts and glues the surface along circles, and so $\chi(F)$ does not change. It glues A to itself along an annulus or Möbius band, and so also $\chi(A)$ does not change. So, we may continue removing all intersection circles until we obtain an embedding, for which our equalities indeed hold since for an embedding $\partial A = \partial B = F$ and so $\chi(A) = \chi(B) = \frac{1}{2}\chi(F)$.

If $C_a + C_b > 0$, assume we have a pair of cross-caps connected by an embedded isolated arc of intersection. These are of two types, either connecting two cross-caps of the same color, say black, as in Fig. 2 or of opposite color as in Fig. 3. In both cases, we perform an operation as shown in the figure, removing this arc of intersection. The first case does not change $\chi(F)$, but reduces C_a by 2, and glues A to itself along a disc, and so reduces $\chi(A)$ by 1. So if the new map satisfies the equalities, then so does the old one, so we are done by induction. For the second case $\chi(F)$ increases by 1, and C_a, C_b each decreases by 1, whereas $\chi(A), \chi(B)$ are unchanged, and so again we are done by induction.

So now assume there are cross-caps but the arcs connecting them are not embedded and isolated, and so there are triple points along any such arc. Look at the arc of intersection beginning at a cross-cap, say black, up to the first triple point, as in Fig. 4(a). We change the map as in Fig. 4(b), that is, a small disc in the center of

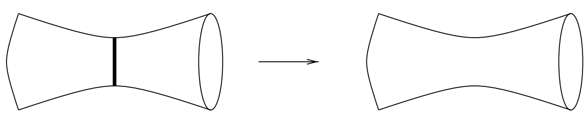


Fig. 2.

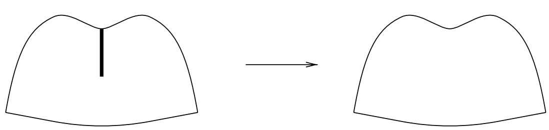
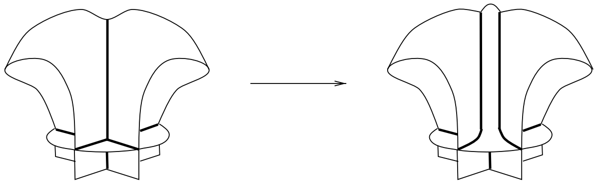


Fig. 3.



(a) (b)

Fig. 4.

the horizontal sheet, is stretched upward along the vertical intersection line, until it passes the cross-cap. This reduces N by 1, and changes the color of the the cross-cap from black to white, thus decreasing C_a by 1 and increasing C_b by 1. It also adds a 1-handle to A (the thin 1-handle that is seen wrapped around the cross-cap in Fig. 4(b)), thus reducing $\chi(A)$ by 1. All together the equalities are preserved and so we are done by induction.

There remains the case $C_a = C_b = 0$ and $N > 0$. Here we follow the procedure of [1] for eliminating triple points. For $N \geq 2$, if we have two triple points in the same connected component of the intersection curve, then we apply the operation appearing in Fig. 5 to an intersection arc between two neighboring triple points. Here a tube is added between the left and right vertical sheets, which encloses the arc of intersection, and so the two triple points disappear. If we have at most one triple point in each connected component of the intersection curve, then we first perform the operation appearing in Fig. 6 to merge two of them. Here again a tube is added between the left and right vertical sheets, half way above and half way below the horizontal sheet. We may then use the operation of Fig. 5 as before, to eliminate two triple points. The operation of Fig. 5 decreases $\chi(F)$ by 2, decreases N by 2, and glues A and B each to itself twice, along two discs, and so reduces $\chi(A)$ and $\chi(B)$ each by 2 so the equalities are preserved. The operation of Fig. 6 decreases $\chi(F)$ by 2 and glues A and B each to itself along one disc and so reduces $\chi(A)$ and $\chi(B)$ each by 1 so again the equalities are preserved.

If $N = 1$, then first take a connected sum of i with Boy's surface, where by connected sum of two generic maps $i : F \to M$ and $i' : F' \to S^3$ we mean the

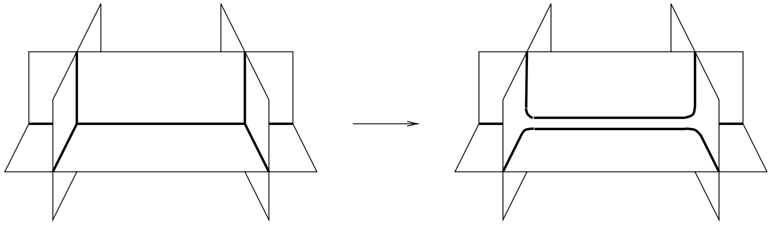


Fig. 5.

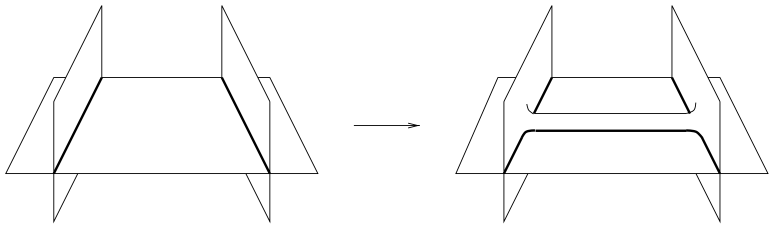


Fig. 6.

following. For $p \in i(F)$ a point not in the intersection set of i, let E be a small 3-ball neighborhood of p disjoint from the intersection set of i and delete E from M, obtaining a map $i|_{F-D} : F - D \to M - E$ where $D = i^{-1}(E)$ is a disc in F. For $i' : F' \to S^3$, do the same with some E', D'. Now glue $M - E$ to $S^3 - E'$ along their boundaries, so that the boundaries of $F - D$ and $F' - D'$ will match, obtaining a map $F\#F' \to M\#S^3 = M$ which we call a connected sum of i, i'. When such connected sum operation is performed, the two complementary regions of i on the two sides of $i(D)$, merge with the corresponding complementary regions of i' on the two sides of $i'(D')$. Each such merger is along a disc, namely, a hemisphere of ∂E.

In [7], Boy's surface is depicted with a small window removed, so one can peep inside and convince oneself that the complementary regions of Boy's surface in S^3 are two 3-cells, one of each color. Also, Boy's surface has one triple point. And so a connected sum of Boy's surface with any map $i : F \to M$, leaves the family of complementary regions unchanged, but adds a single triple point and reduces $\chi(F)$ by 1. So, when taking connected sum of Boy's surface with our map having $N = 1$, the equalities are preserved, and we now have $N = 2$, which may be reduced to $N = 0$ as before. This completes the proof of the "only if" part of Theorem 1.2.

We make the following remark (compare [5]). The "only if" part of Theorem 1.2 is a refinement in this setting of the result of Izumiya and Marar [3, 4], $\chi(i(F)) = \chi(F) + N + \frac{1}{2}K$ where $K = C_a + C_b$ is the total number of cross-caps. Indeed, present M as the union of a regular neighborhood U of $i(F)$, and the union V of all complementary regions, slightly diminished, so $\chi(V) = \sum_k (1-k)a_k + \sum_k (1-k)b_k$. We get $\chi(M) = 0 = \chi(i(F)) + \chi(V) - \chi(\partial V) = \chi(i(F)) - \chi(V)$, or $\chi(i(F)) = \chi(V)$. This shows that the Izumiya–Marar equality is equivalent in this setting to $\chi(V) = \chi(F) + N + \frac{1}{2}K$, which is the sum of our two equalities $\sum_k (1-k)a_k = \frac{1}{2}(\chi + N + C_a)$ and $\sum_k (1-k)b_k = \frac{1}{2}(\chi + N + C_b)$.

Another consequence of the two equalities is that for any i, $\chi + N + C_a$ and $\chi + N + C_b$ are even, (compare [2, 6, 8]).

3. Realizing the Data by Maps

In this section, we show that any data $(\{a_k\}, \{b_k\}, \chi, N, C_a, C_b)$ satisfying the conditions of Theorem 1.2, may be realized by some F and $i : F \to M$. For convenience, we will construct all our maps into S^3. To realize the same data with general M, perform a connected sum of S^3 and M at a location in S^3 disjoint from $i(F)$. The proof is by induction on $N + C_a + C_b + \sum_k a_k + \sum_k b_k$.

Let $(\{a_k\}, \{b_k\}, \chi, N, C_a, C_b)$ be data satisfying the conditions of Theorem 1.2. Assume first $C_a \neq C_b$, say $C_a > C_b$. Since $C_a + C_b$ is assumed even, $C_a - C_b \geq 2$, in particular $C_a \geq 2$. If $a_0 \geq 2$, let $\{a'_k\}$ be the sequence defined by $a'_0 = a_0 - 1$ and $a'_k = a_k$ for all $k > 0$, then $\{a'_k\}$ is not identically 0. By induction there is a surface F and map $i : F \to S^3$ realizing $(\{a'_k\}, \{b_k\}, \chi, N, C_a - 2, C_b)$. Let U be any white region of i and change i in a disc on the boundary of U (which is disjoint from the intersection set of i) as in Fig. 7(a). Then there are two new black cross-caps and a new black 3-cell region appearing. The topology of all other regions is unchanged, and so the new map realizes $(\{a_k\}, \{b_k\}, \chi, N, C_a, C_b)$. If on the other hand $a_0 \leq 1$, then $\sum_k (1-k)a_k \leq 1$ and so $\sum_k (1-k)b_k \leq 0$. It follows that there is some index $p > 0$ with $b_p > 0$, and that $\chi \leq 0$. Let $\{b'_k\}$ be the sequence obtained from $\{b_k\}$ by subtracting 1 from b_p and adding 1 to b_{p-1}. By induction there is a surface F and map $i : F \to S^3$ realizing $(\{a_k\}, \{b'_k\}, \chi + 2, N, C_a - 2, C_b)$. We have $b'_{p-1} \geq 1$ and let U be a corresponding region, i.e. a white region with $\chi(U) = 1 - (p-1)$. Change F and i in a disc on the boundary of U as in Fig. 7(b). Then χ decreases by 2, C_a increases by 2, and $\chi(U)$ decreases by 1, and so the new map realizes $(\{a_k\}, \{b_k\}, \chi, N, C_a, C_b)$.

So, we may assume from now on that $C_a = C_b$ which we will denote by C, and so from now on, we have $\sum_k (1-k)a_k = \sum_k (1-k)b_k$. If $a_0 \geq 1$ and there is $r \geq 1$ with $a_r \geq 1$, then let $\{a'_k\}$ be the sequence obtained from $\{a_k\}$ by subtracting 1 from a_0, subtracting 1 from a_r, and then adding 1 to a_{r-1}. (So if $r = 1$ then $a'_0 = a_0$.) By induction there is a surface F and map $i : F \to S^3$ realizing $(\{a'_k\}, \{b_k\}, \chi, N, C, C)$. We have $a'_{r-1} \geq 1$, so let U be a black complementary region for i with $\chi(U) = 1 - (r-1)$, and change i in a disc on the boundary of U as in Fig. 8. Then $\chi(U)$ decreases by 1, and there is a new black 3-cell region appearing. The topology of all other regions is unchanged, and so the new map realizes $(\{a_k\}, \{b_k\}, \chi, N, C, C)$. So we may assume that either (i) $a_0 = 0$, or (ii) $a_k = 0$ for all $k \geq 1$. By the same argument, the same may be assumed for $\{b_k\}$. If say $\{a_k\}$ satisfies (i) and $\{b_k\}$ satisfies (ii), then $\sum_k (1-k)a_k \leq 0$, and since the sequences are assumed not to be identically 0, $\sum_k (1-k)b_k > 0$, and so they cannot be equal. It follows that we may assume either both $\{a_k\}, \{b_k\}$ satisfy (i) or both satisfy (ii).

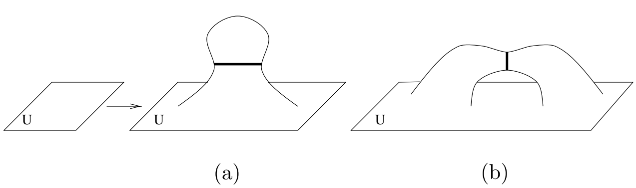


Fig. 7.

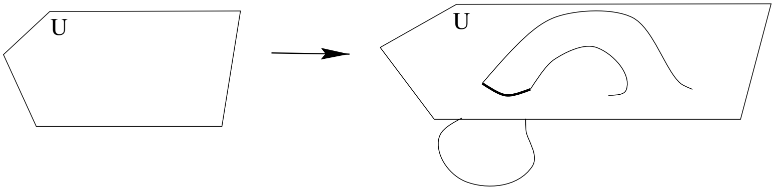
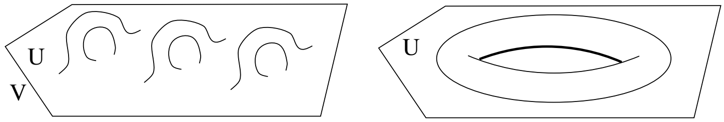


Fig. 8.

We will first assume both sequences satisfy (i), that is $a_0 = b_0 = 0$, and so $\sum_k (1-k)a_k \leq 0$, and so necessarily $\chi \leq 0$. If $N \geq 1$, then by induction we have a surface F and map i realizing $(\{a_k\}, \{b_k\}, \chi+1, N-1, C, C)$. Take the connected sum of i with Boy's surface, then as explained above, this leaves the family of regions unchanged, adds a triple point, and reduces χ by 1, and so the new map realizes $(\{a_k\}, \{b_k\}, \chi, N, C, C)$. So we may assume $N = 0$. If $C \geq 1$, we realize $(\{a_k\}, \{b_k\}, \chi+1, N, C-1, C-1)$, and then perform the inverse of the operation appearing in Fig. 3, in some arbitrary region. This adds one cross-cap of each color, and reduces χ by 1, and so the new map realizes $(\{a_k\}, \{b_k\}, \chi, N, C, C)$. So we may also assume $C = 0$.

Before dealing with this case, we introduce two more operations on generic maps. Let i be a map and let $D \subseteq i(F)$ be a disc disjoint from the intersection set of i. D is part of the boundary of two neighboring regions U, V (of opposite colors). For $g \geq 0$, a g-operation on D is an operation as in Fig. 9(a), adding g handles to F, by this reducing $\chi(F)$ by $2g$, and reducing $\chi(U), \chi(V)$ each by g. We will say that this g-operation was performed on the pair U, V. The second operation will be called a ring operation, which adds a *ring* on D, say on the side of U, as in Fig. 9(b). (Only the map changes here, not the topology of the surface.) The only effect of a ring operation is the creation of a new solid torus region which is a neighbor of U, and so of color opposite that of U. The topology of all previously existing regions remains unchanged. We will say that this ring operation was performed in the region U.

Back to our sequences $\{a_k\}, \{b_k\}$, recall that we are now assuming $a_0 = b_0 = 0$, $N = C = 0$. Assume first that the following holds: There is a p such that $a_k = b_k = 0$ for all $k \neq p$ (and so $a_p \geq 1$, $b_p \geq 1$). If $p = 1$, then by our equalities, $\chi = 0$. Start with a standard embedding of a torus, and perform $a_p - 1$ ring operations in the white side and $b_p - 1$ ring operations in the black side, obtaining a map realizing



(a) (b)

Fig. 9.

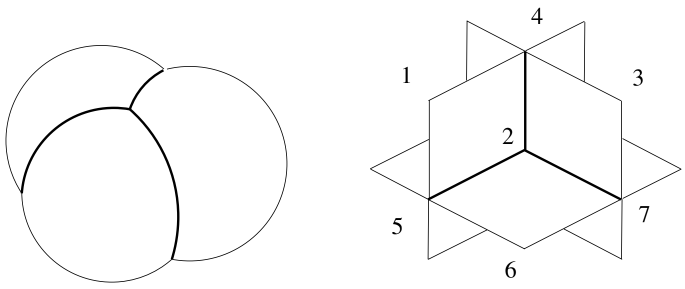
the data. So we may assume $p \geq 2$ and so necessarily $a_p = b_p$. If $a_p = b_p = 1$, then $\chi = 2(1-p)$ and we may realize our data by a standard embedding of an orientable surface F of genus p. Otherwise, $a_p = b_p \geq 2$. Let $\{a'_k\}$, $\{b'_k\}$ be the sequences with $a'_p = b'_p = a_p - 1$ and $a'_k = b'_k = 0$ for all $k \neq p$, and note that necessarily $\chi \leq 2(1-p)$. By induction we have a map i realizing $(\{a'_k\}, \{b'_k\}, \chi+2(p-1), 0, 0, 0)$. Now perform a ring operation on i in some arbitrary place, creating a new solid torus region U. In U perform another ring operation creating another solid torus region V which is a neighbor of U (and so of opposite color). Now perform a $(p-1)$-operation on the pair U, V to obtain a map realizing $(\{a_k\}, \{b_k\}, \chi, 0, 0, 0)$.

If there is no p as above, then there are some $r \neq s$ such that $a_r \geq 1$ and $b_s \geq 1$, and assume $r < s$. Let $\{a'_k\}, \{b'_k\}$ be the sequences obtained from $\{a_k\}, \{b_k\}$ in the following way. Subtract 1 from a_r, subtract 1 from b_s and then add 1 to b_{s-r+1} (so if $r = 1$, then the sequence $\{b_k\}$ remains unchanged). The sequence $\{a'_k\}$ is not identically 0 since that would imply that $\sum_k (1-k)a_k = 1 - r$ which could not equal $\sum_k (1-k)b_k \leq 1 - s$. Note also that necessarily $\chi \leq 2(1-r)$. By induction we may realize $(\{a'_k\}, \{b'_k\}, \chi + 2(r-1), 0, 0, 0)$. We have $b'_{s-r+1} \geq 1$ and let U be a corresponding region, i.e. U is a white region with $\chi(U) = r - s$. Now, first perform a ring operation in U, creating a new black solid torus region V which is a neighbor of U. Then perform an $(r-1)$-operation on the pair U, V of neighboring regions, by this decreasing $\chi(U)$ and $\chi(V)$ each by $r-1$, and decreasing $\chi(F)$ by $2(r-1)$, and so realizing our original data $(\{a_k\}, \{b_k\}, \chi, 0, 0, 0)$. (If $r = 1$, then the $(r-1)$-operation means doing nothing.)

We are left with the case where both sequences satisfy (ii), that is $a_k = b_k = 0$ for all $k \geq 1$. Since the sequences are not identically 0, $\sum_k (1-k)a_k \geq 1$ and so $\chi + N + C \geq 2$. If $\chi < 2$ then $N > 0$ or $C > 0$, and as before we may use a realization of $(\{a_k\}, \{b_k\}, \chi+1, N-1, C, C)$ or $(\{a_k\}, \{b_k\}, \chi+1, N, C-1, C-1)$, respectively, to produce a realization of $(\{a_k\}, \{b_k\}, \chi, N, C, C)$, so we may assume $\chi = 2$, i.e. the surface F is S^2. So now $\frac{1}{2}(2 + N + C)$ is a sum of integers and so $N + C$ is even. Our task then, is to construct for any N, C with $N + C$ even, a map $i : S^2 \to S^3$ with complementary regions which are $\frac{1}{2}(2 + N + C)$ 3-cells of each color. From the "only if" part of Theorem 1.2, we know that if $C_a = C_b$ and all complementary regions are 3-cells, then necessarily the number of black 3-cells is equal to the number of white 3-cells. And so when constructing our maps below, we need only verify that the *total* number of 3-cells is $2 + N + C$.

For $N = C = 0$, take an embedding which indeed gives two 3-cells.

For $N = 2, C = 0$, we must construct an immersion $i : S^2 \to S^3$ with 2 triple points and 4 complementary 3-cells. Start with three spheres, each embedded, and intersecting each other with two triple points, as in Fig. 10(a). They divide S^3 into eight 3-cells. A small neighborhood of one of the two triple points appears as in Fig. 10(b). The 8 different regions appearing in this neighborhood, are included in the 8 different complementary regions. In the figure, one of these regions is hidden.



(a) (b)

Fig. 10.

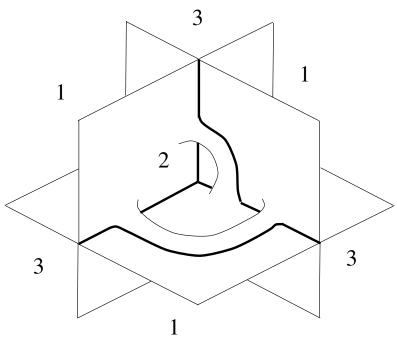


Fig. 11.

We now attach two tubes to the three spheres, merging them into one sphere. We add the tubes as in Fig. 11 which we now explain. The horizontal tube connects the two vertical sheets, and is half way above and half way below the horizontal sheet. The upper half of this tube creates a path merging regions 1 and 3 of Fig. 10(b), and its lower half merges regions 5 and 7 of that figure. The vertical tube connects the horizontal sheet with the left-hand vertical sheet and is half to the left and half to the right of the right-hand vertical sheet. It opens paths merging regions 1 and 6 of Fig. 10(b), and regions 4 and 7 of that figure. All together, regions 1, 3, 6 of Fig. 10(b) have merged into the one 3-cell region 1 of Fig. 11, and regions 4, 5, 7 of Fig. 10(b) have merged into the one 3-cell region 3 of Fig. 11. And so we have four complementary 3-cells as needed. This completes the case $N = 2, C = 0$.

For $N = 0, C = 2$, start with an embedding of a 2-sphere and perform the operation of Fig. 7(a) twice, once in the black side and once in the white side. This indeed creates two cross-caps of each color and four complementary 3-cells.

For $N = 1, C = 1$, we proceed as follows. Start with a map of a 2-sphere as in Fig. 12(a). Add to it an embedded 2-sphere intersecting with it as in Fig. 12(b). At this stage, we have one cross-cap of each color, one triple point, and six 3-cell regions. Now add one tube near the triple point, as in Fig. 12(c). This will merge

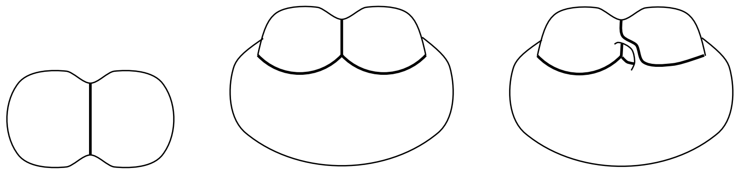


Fig. 12.

the two spheres into one sphere, and will reduce the number of complementary 3-cells from six to four, as needed.

For general N, C with $N+C$ even, take an appropriate connected sum of copies of the maps we have constructed above for the three cases when $N+C=2$. For each additional such map, two of its four 3-cell regions merge along discs with two of the existing ones, and so precisely two new 3-cell regions are added, as needed.

We have thus completed the construction of maps $i : F \to S^3$ realizing any data satisfying the conditions of Theorem 1.2. Recall that in order to realize any data with general 3-manifold M, we start with a map $i : F \to S^3$ realizing the same data, and then perform a connected sum of S^3 and M at a location in S^3 disjoint from $i(F)$.

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