COMPLEXITY OF PLANE AND SPHERICAL CURVES

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Abstract

We show that the maximal number of singular moves required to pass between any two regularly homotopic plane or spherical curves with at most n crossings grows quadratically with respect to n. Furthermore, for any two regularly homotopic curves with at most n crossings, there exists such a sequence of singular moves, satisfying the quadratic bound, for which all curves along the way have at most $n + 2$ crossings.

1. Introduction

Our subject of study is plane and spherical curves, by which we mean immersions of S^1 into $\mathbb{R}^2$ and S^2. This has been a topic of interest ever since Whitney and Graustein classified the regular homotopy classes of plane curves in [W]. Smale has generalized this to a study of curves in general smooth manifolds in [S2] and then continued, with Hirsch, to develop a general theory of immersions of manifolds in [S1], [S3], and [H]. New interest in plane and spherical curves was aroused when Arnold in [A] introduced his three basic invariants of plane curves. This has stimulated vast study of such invariants by many authors, such as in [G], [K], [L], [M], [N2], [N1], and [P].

Our interest is in computational questions related to regular homotopy of plane and spherical curves. We are interested in the number of singular occurrences which may be required for passing between any two regularly homotopic curves, and in the number of crossings the curves along the way may be required to have. More precisely, define the distance between two regularly homotopic curves to be the minimal number of singular occurrences required to pass between them. Define the diameter of a set of regularly homotopic curves to be the maximal distance between its members. We show that the diameter of the set of all curves in a given regular homotopy class which have at most n crossings grows quadratically with respect to n. A quadratic upper bound is established by presenting an explicit algorithm transforming any curve to any other regularly homotopic curve. A quadratic lower bound is established using an invariant of curves introduced in [N2]. Furthermore, our explicit algorithm for passing between curves has the property that if the initial and final curves have at most n crossings, then all curves along the way have at most

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$n + 2$ crossings. It also implies a characterization of curves with the minimal number of crossings in their regular homotopy class.

The structure of the article is as follows. In Section 2, we present the necessary definitions and state our results. In Section 3, we present our explicit algorithm that proves the quadratic upper bound. In Section 4, we prove the quadratic lower bound. In Section 5, we compare our results to analogous results in other settings.

2. Definitions and statement of results

A plane or spherical curve is an immersion of S^1 into $\mathbb{R}^2$ or S^2, respectively. The *Whitney number* (or *rotation number*, or *winding number*) of a plane curve c takes its values in $\mathbb{Z}$, and it is defined as follows. The derivative of a curve $c : S^1 \to \mathbb{R}^2$ does not vanish by definition, and so it defines a map $S^1 \to \mathbb{R}^2 - \{0\} \simeq S^1$. The degree of this map is the Whitney number of c. For $\omega \in \mathbb{Z}$, we define the curve δ_ω to be the curve with Whitney number ω described in Figure 1.

For spherical curve c, we define the Whitney number of c as 0 or 1 according to the parity of the Whitney number of the plane curve obtained by deleting a point from the complement of c in S^2. The curves δ_0, δ_1, thought of as spherical curves, are our chosen representatives for spherical Whitney numbers 0 and 1, respectively.

Two plane or spherical curves are regularly homotopic if and only if they have the same Whitney number. This follows from [S2], and it first appeared for plane curves in [W]. The algorithm presented in Section 3 also shows that, indeed, any two curves with the same Whitney number are regularly homotopic.

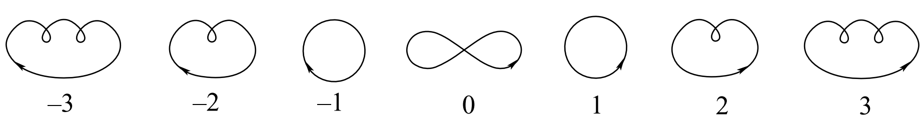

Figure 1. The curves δ_ω for $\omega = -3, \ldots, 3$

A curve is called *generic* if its only self-intersections are transverse double points. We denote the space of all generic plane or spherical curves by $\mathcal{C}^P$ and $\mathcal{C}^S$, respectively. The generic singularities a curve may have are either a tangency of first order between two strands, which is called a J-type singularity, or three strands meeting at a point, each two of which are transverse, which is called an S-type singularity. Singularities of type J and S appear in Figure 2. Motion through a J- or S-singularity as in Figure 2 is called a J- or S-move or, jointly, a *basic move*. An S-move preserves the number of crossings of the curve. On the other hand, the J-move that proceeds in Figure 2 from left to right (resp., from right to left), increases (resp., decreases) the number of crossings by 2. This is called an *increasing* J-move or a *decreasing* J-move, respectively.

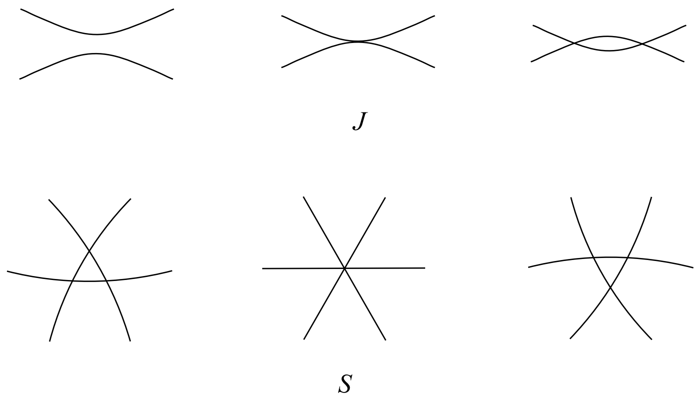

Figure 2. The basic moves

Definition 2.1

(1) For two curves c, c' in $\mathcal{C}^P$ or $\mathcal{C}^S$, the distance $d(c, c')$ is defined as the minimal number of basic moves needed to pass from c to c'. (If c, c' are not regularly homotopic, set $d(c, c') = \infty$.)

(2) For a subset A of $\mathcal{C}^P$ or $\mathcal{C}^S$, the diameter of A is defined as

$$\operatorname{diam} A = \sup_{c, c' \in A} d(c, c').$$

(3) Let B_n^ω be the set of all curves (either in $\mathcal{C}^P$ or $\mathcal{C}^S$) with Whitney number ω and with at most n crossings.

In this work, we prove the following two results.

THEOREM 2.2

There exist quadratic functions $a(n)$, $b(n)$ (with positive leading coefficients) such that for any ω and n,

$$a(n) \leq \operatorname{diam} B_n^\omega \leq b(n).$$

THEOREM 2.3

For any $c, c' \in B_n^\omega$, there exists a sequence of basic moves from c to c' (satisfying the bound of Theorem 2.2) such that all curves along the way are in B_{n+2}^ω.

3. Upper bound

We present an explicit algorithm for a regular homotopy starting with any $c \in B_n^\omega$ and ending with δ_ω.

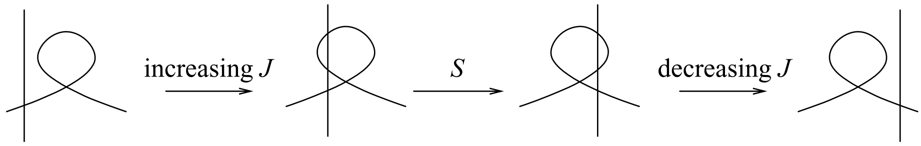

Figure 3. The Z-move

We define the following *composite* move (as opposed to the *basic* moves J and S), which we call a Z-move, appearing in Figure 3. This is a sequence of three basic moves: an increasing J-move, an S-move, and a decreasing J-move. So, during a Z-move, the number of crossings increases by 2, but it finally decreases back to the initial number. Our algorithm performs only the following moves:

(1) S-moves,

(2) decreasing J-moves,

(3) Z-moves.

It follows that the number of crossings of the curves along the way never exceeds $n + 2$, proving Theorem 2.3. We present the algorithm for plane curves, and we then present the slight modification required for spherical curves.

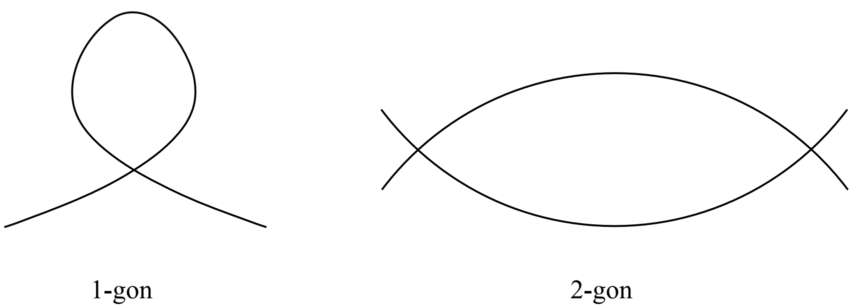

Figure 4. 1-gon and 2-gon

Let c be a *plane* curve. For $k = 1, 2$, a k-gon in c is a portion of c of the form appearing in Figure 4. The bounded region bounded by the k-gon is called its *interior*. The interior of a k-gon may intersect other portions of c. It is, however, assumed, as part of the definition, that the corners of a k-gon are *convex*, as appears in Figure 4. This means that the immediate continuations of the arcs of the k-gon lie outside its interior. We say a k-gon in c is *empty* if its interior is disjoint from c.

The algorithm is divided into two steps. Step 1 transforms the initial curve c into a curve c' which is a string of empty 1-gons. Step 2 transforms c' into δ_ω. The algorithm begins by defining a horizontal line L which crosses c at two points and is such that the portion of c below L is an embedded arc. This is obtained by starting with a horizontal line that is completely below c and pushing it upwards until slightly after it first touches c. The parts of c above and below L are called the *upper curve* and *lower curve*, respectively. Step 1 repeatedly uses two procedures, which we call Procedure A and Procedure B. Each application of Procedure A or B reduces the number of crossings of the upper curve by at least one, and Procedure A also adds an empty 1-gon at one of the ends of the lower curve. We repeat Procedures A and B until there are no crossings in the upper curve, and so c becomes a string of empty 1-gons.

We note the following.

LEMMA 3.1
If the upper curve has crossings, then it contains a 1*-gon.*

Proof
Parameterizing the upper curve by $[0, 1]$, let $t \in [0, 1]$ be minimal such that $c|_{[0,t]}$ is noninjective (i.e., t is the first time that c crosses itself), and let $s < t$ be such that $c(s) = c(t)$. Then we claim that $c|_{[s,t]}$ is a 1-gon. By definition of t, $c(s) = c(t)$ is the only crossing, and we need to show that the corner at $c(s) = c(t)$ is convex. Let $U \subseteq \mathbb{R}^2$ be the interior of the loop $c([s, t])$. If the corner is not convex, then $c(s - \epsilon) \in U$ for some small $\epsilon > 0$. So since $c(0) \notin U$, we must have some $0 < r < s - \epsilon$ and $s < r' < t$ with $c(r) = c(r')$, contradicting the definition of t. $\square$

If the upper curve contains an *empty* 1-gon, then we apply Procedure A, which is the following. We slide the 1-gon along the upper curve, passing all crossings on the way via Z-moves, until it passes L and joins the lower curve. There are two ways that this can be done; the empty 1-gon can be pushed either to the right or to the left. Let m denote the number of crossings in the upper curve when starting the procedure. Since there are $m - 1$ crossings that may be passed, and each may be passed twice, the sum of the number of Z-moves when the 1-gon is pushed to the right and to the left is $2(m - 1)$. So either to the right or to the left, we need no more than $m - 1$ Z-moves, which means no more than $3(m - 1)$ basic moves. This completes Procedure A. It reduces the number of crossings in the upper curve by one, and it adds an empty 1-gon to the lower curve.

If the upper curve contains no empty 1-gons, we apply Procedure B, which is the following. By Lemma 3.1, the upper curve does contain 1-gons. Take a minimal such 1-gon (i.e., one whose interior U contains no other 1-gon). By our assumption, this 1-gon is nonempty, and so U contains subarcs of c. If such an arc were not embedded, then by the same argument as in Lemma 3.1, there would be a 1-gon in U, so all such arcs are embedded. It follows that U contains a 2-gon, and by taking a minimal one in U, we know we have a 2-gon G whose interior V contains no 1-gon and no 2-gon. It follows that G satisfies the following properties.

(1) All arcs in V are embedded.

(2) All arcs in V pass from one edge of the 2-gon to the other edge.

(3) Any two arcs in V intersect at most once.

We may think of V with its boundary as the square $[0, 1] \times [0, 1]$ with the left and right edges $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$ each identified to a point. So V itself is parameterized by $(0, 1) \times (0, 1)$, and so we have a notion of *horizontal levels* in V, namely, the lines $(0, 1) \times \{t\}$. An arc in V is called *monotonic* if it passes each horizontal level once. We prove the following.

LEMMA 3.2

If G is a 2-gon in c with interior V, satisfying properties (1)–(3) above, then there is an isotopy of V (keeping a neighborhood of the boundary fixed) after which all arcs in V become monotonic.

Proof

Say that there are k arcs; then by induction, we may perform an isotopy making the first $k - 1$ arcs monotonic. Denote the kth arc by A. By a slight isotopy, we may assume that there are only finitely many tangencies between A and the horizontal levels, and that they all occur at different horizontal levels. If A is not monotonic, then there are some extrema along A. Let $x \in A$ be the highest minimum. (By property (2), the extrema cannot be all maxima.) If we move from x along A to the right and to the left, then by property (2), we must reach some maximum. If we reach such a maximum only on one side (and reach the top edge on the other side), then let y be that maximum. If we reach a maximum on both sides, let y be the lower one of these two maxima. Let M be the horizontal line tangent to A at y. It bounds, together with the two subarcs ascending from x, a subregion W in V. The subarc of A connecting x and y may not reach x and y from the same side since if, say, it reaches both x and y from the left, then moving from y slightly to the right brings us into W, and in order to leave W, an additional minimum is required which is necessarily higher than x. So, assume that when traveling from x to the left, we arrive at y from the right. By a level-preserving isotopy, we may assume the configuration is as in Figure 5(a). Indeed, by property (3), the upper end of each arc in W must be in M. (The portion above M appearing in Figure 5(a), though drawn wide, represents a thin neighborhood above M.) We may now deform a neighborhood of W as in Figure 5(b) to cancel the minimum x with the maximum y, keeping the property that the first $k - 1$ arcs are monotonic. We repeat this process until all extrema of A are canceled, and so A is monotonic as well. □

So, we now have all arcs in V monotonic, and by an additional slight isotopy, we may assume that each crossing in V appears in a different horizontal level. We now move the top edge of the 2-gon G down via the horizontal levels until we pass all crossings.

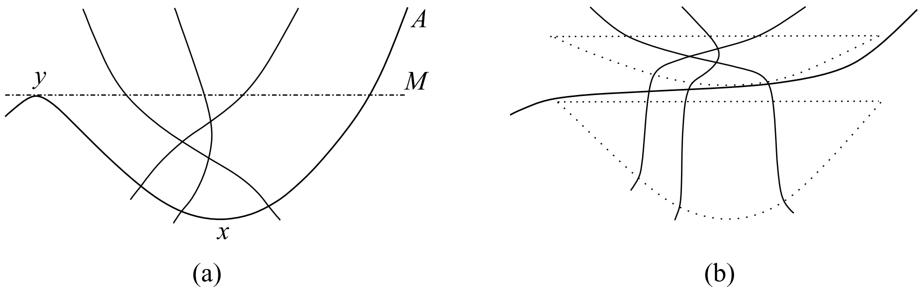

Figure 5. Canceling extrema

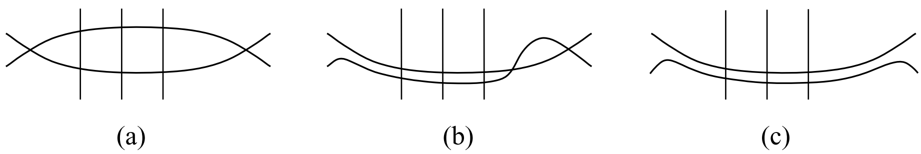

Figure 6. Concluding moves of Procedure B

Whenever we pass a crossing, an S-move occurs. After this sequence of S-moves, there are no more crossings in V. So the configuration in V is as in Figure 6(a). Some additional S-moves bring us to Figure 6(b), and then we perform a single decreasing J-move arriving at Figure 6(c), by this reducing the number of crossings in the upper curve by 2 (and leaving the lower curve unchanged). Let m denote the number of crossings in the upper curve when starting the procedure. The number of all S-moves we have performed is at most $m-2$, and so the total number of basic moves is at most $m-1 \leq 3(m-1)$. This completes the description of Procedure B.

As mentioned, we repeat Procedures A and B until the upper curve is embedded, and so we have a curve that is a string of empty 1-gons. Each performance of Procedure A or B required at most $3(m-1)$ basic moves, and it reduced the number of crossings in the upper curve by at least one. And so the number of basic moves required to complete Step 1 of the algorithm is at most $3((n-1)+(n-2)+(n-3)+\cdots) = (3/2)(n^2-n)$.

We then begin Step 2 of the algorithm. If there are two consecutive 1-gons along our string of empty 1-gons, which face opposite sides, as on the left-hand side of Figure 7, then one Z-move and one decreasing J-move, as shown in Figure 7, cancel the two 1-gons. We call this Procedure C. We repeat Procedure C until all 1-gons are facing the same side. There are at most $(1/2)n$ such pairs of 1-gons, and each such pair was canceled by one Z-move and one J-move, which means four basic moves, and so this requires at most $2n$ basic moves.

If at this point all 1-gons are facing inward, or there is just one crossing, then we have reached our base curve δ_ω. Otherwise, all 1-gons are facing outward, and there

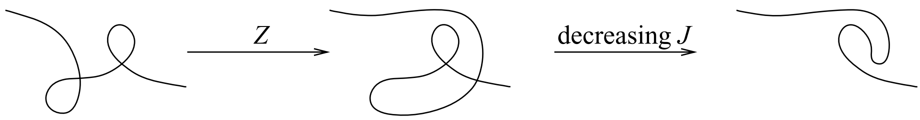

Figure 7. Procedure C: Canceling two consecutive empty 1-gons, facing opposite sides.

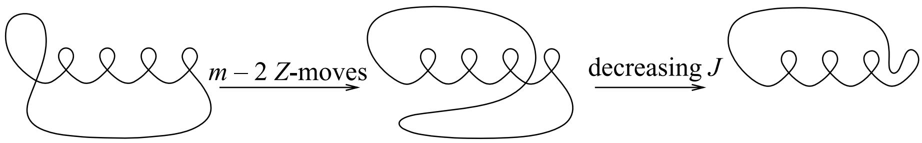

Figure 8. Procedure D: From all 1-gons facing outward to all 1-gons facing inward

are at least two of them. Let m denote the number of crossings at this time. In Figure 8, we describe Procedure D, which is a sequence of $m-2$ Z-moves and one decreasing J-move which transforms our curve into one with all 1-gons facing inward. So this requires $3m-5 \leq 3n-5$ basic moves.

To sum up, our algorithm required at most $(3/2)n^2 + (7/2)n - 5$ basic moves. It follows that the diameter of B_n^ω for plane curves is at most $3n^2 + 7n - 10$.

For spherical curves, we perform the exact same algorithm as above until obtaining a curve that is a string of empty 1-gons with all 1-gons facing the same side. Since we are now in S^2, we can always think of the 1-gons as facing "outward," and so if there are at least two 1-gons, we can apply Procedure D, which requires $3m-5 \leq 3(m-1)$ basic moves and reduces the number of crossings by 2. We repeat Procedure D until we have just one or zero crossings, and so we have reached δ_0 or δ_1. This requires at most $3((n-1)+(n-3)+(n-5)+\cdots) \leq (3/4)n^2$ basic moves. So, all together in the case of spherical curves, we need at most $(9/4)n^2 + (1/2)n$ basic moves, and so the diameter of B_n^ω is at most $(9/2)n^2 + n$.

Our algorithm also provides the following.

COROLLARY 3.3

The curve δ_ω has the minimal number of crossings in its regular homotopy class, this minimum being $||\omega|-1|$. Furthermore, any other curve with the minimal number of crossings can be obtained from δ_ω by a sequence of Z-moves.

Proof

Starting with any curve c of Whitney number ω, our algorithm uses only S-moves, decreasing J-moves, and Z-moves, and it brings us to δ_ω, which has $||\omega|-1|$ crossings. Since these three moves do not increase the number of crossings, the first statement follows. For the second statement, note that of our four procedures, only Procedure A does not decrease the total number of crossings. So if c also has the minimal number of crossings, then our algorithm necessarily applies only Procedure A, which only uses Z-moves. □

4. Lower bound

Recall that $\mathcal{C}^P$ and $\mathcal{C}^S$ denote the spaces of all generic plane curves and spherical curves, respectively. Let $\mathbb{X}$ be the free abelian group whose basis is the set of all symbols of the form $X_{a,b}$ with $a, b \in \mathbb{Z}$. We construct the invariant f^X from $\mathcal{C}^P$ or $\mathcal{C}^S$ to $\mathbb{X}$ as follows.

Definition 4.1

Let E be an oriented surface with nonempty boundary, and let e be a generic arc in E (i.e., $e : [0, 1] \to E$ is an immersion with $e(0) \neq e(1) \in \partial E$, and its only self-intersections are transverse double points).

(1) For a double point v of e, we define $i(v) \in \{1, -1\}$, where $i(v) = 1$ if the orientation at v given by the two tangents to e at v, in the order they are visited, coincides with the orientation of E. Otherwise, $i(v) = -1$.

(2) We define the index of e, $i(e) \in \mathbb{Z}$, by $i(e) = \sum_v i(v)$, where the sum is over all double points v of e.

For c in $\mathcal{C}^P$ or $\mathcal{C}^S$, let v be a double point of c, and let u_1, u_2 be the two tangents at v ordered by the orientation of $\mathbb{R}^2$ (resp., S^2). Let U be a small neighborhood of v, and let $E = \mathbb{R}^2 - U$ (resp., $S^2 - U$). Now, $c|_{c^{-1}(E)}$ defines two arcs c_1, c_2 in E, ordered so that the tangent u_i leads to c_i, $i = 1, 2$. We denote $a(v) = i(c_1)$ and $b(v) = i(c_2)$, where the orientation on E is that restricted from $\mathbb{R}^2$ (resp., S^2). Then f^X from $\mathcal{C}^P$ or $\mathcal{C}^S$ to $\mathbb{X}$ is defined as

$$f^X(c) = \sum_v X_{a(v),b(v)},$$

where the sum is over all double points v of c.

We mention that for spherical curves, f^X has been introduced in [N2] as part of the construction of a universal order 1 invariant for spherical curves. For plane curves, f^X can be obtained from the universal order 1 invariant of plane curves introduced in [N1] by the reduction $X^{a_1,b_1}_{a_2,b_2} \mapsto X_{a_1,b_1}$.

Now, let g be the $\mathbb{Z}$-valued invariant (for both plane and spherical curves) defined by $g = \phi \circ f^X$, where $\phi : \mathbb{X} \to \mathbb{Z}$ is the homomorphism defined by $X_{a,b} \mapsto a - b$. It is shown in [N2] (and applies for plane curves in just the same way) that the change in the value of f^X due to a basic move is of one of the following forms:

- $X_{a,b} + X_{b,a}$,
- $X_{a,b+1} + X_{b,a+1}$,
- $X_{a-1,b} + X_{b-1,a}$,

- $-X_{a,b+c+2} - X_{b,c+a+2} - X_{c,a+b+2} + X_{a,b+c} + X_{b,c+a} + X_{c,a+b}$,
- $-X_{c,a+b+1} - X_{b+c-1,a} - X_{b,c+a+1} + X_{b+c+1,a} + X_{b,c+a-1} + X_{c,a+b-1}$,
- $-X_{c,a+b+1} - X_{c+a-1,b} - X_{b+c-1,a} + X_{b+c+1,a} + X_{c,a+b-1} + X_{c+a+1,b}$,
- $-X_{b+c-2,a} - X_{c+a-2,b} - X_{a+b-2,c} + X_{a+b,c} + X_{b+c,a} + X_{c+a,b}$.

Applying ϕ to these elements, we see that the change in the value of g due to a basic move is at most 6. This offers a way for bounding below the distance between two regularly homotopic curves c, c', namely, $d(c, c') \geq (1/6)|g(c) - g(c')|$. We use this to obtain a lower bound to the diameter of B_n^ω. By taking mirror images of all curves, it is clear that diam $B_n^\omega = \text{diam} B_n^{-\omega}$, and so it is enough to find lower bounds when $\omega \geq 0$. We note that the number of crossings of a curve is always of opposite parity to its Whitney number. So if ω, n are of opposite parity, then $B_{n+1}^\omega = B_n^\omega$. So it is enough to find lower bounds for diam B_n^ω for ω, n of opposite parity.

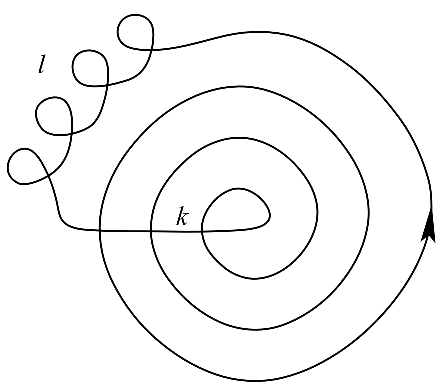

Figure 9. The curve $c_{\omega,n}$ for $\omega = 0$ and $n = 7$ ($k = 3, l = 4$)

For $n \geq \omega \geq 0$ of opposite parity, we construct the curve $c_{\omega,n}$ appearing in Figure 9, having n crossings and Whitney number ω. If, as indicated in Figure 9, there are k crossings along the middle horizontal line, and l empty 1-gons along the outer circle, then $n = k + l$ and $\omega = k + 1 - l$. Inverting these equations gives $k = (1/2)(n + \omega - 1)$ and $l = (1/2)(n - \omega + 1)$, so these are the values for k and l which we use to construct $c_{\omega,n}$.

Direct inspection of $c_{\omega,n}$ gives

$$f^X(c_{\omega,n}) = \sum_{i=1}^{k} X_{l+i-1,i-k} + lX_{0,l-1-k},$$

and so $g(c_{\omega,n}) = k(l + k - 1) + l(k - l + 1) = (1/2)n^2 + (\omega - 1)n + (1/2)(1 - \omega^2)$. On the other hand, $f^X(\delta_0) = X_{0,0}$, and for $\omega > 0$, $f^X(\delta_\omega) = (\omega - 1)X_{2-\omega,0}$. So $g(\delta_0) = 0$, and for $\omega > 0$, $g(\delta_\omega) = -\omega^2 + 3\omega - 2$. Since diam $B_n^\omega = $ diam $B_n^{-\omega}$, we obtain the following lower bound on $\mathrm{diam} B_n^\omega$ for any ω, and all $n \geq |\omega|$ of opposite parity, both in $\mathcal{C}^P$ and $\mathcal{C}^S$:

$$\mathrm{diam} B_{n+1}^\omega = \mathrm{diam} B_n^\omega \geq \frac{1}{12}n^2 + \frac{1}{6}(|\omega| - 1)n + k_\omega \geq \frac{1}{12}(n-1)^2,$$

where $k_0 = 1/12$ and for $\omega \neq 0$, $k_\omega = (1/12)(\omega^2 - 6|\omega| + 5)$. The last inequality uses $n \geq |\omega|$.

5. Concluding remarks

We first sum up our proofs of Theorems 2.2 and 2.3. The upper and lower bounds given in Sections 3 and 4 establish Theorem 2.2. The fact that our algorithm uses only S-moves, decreasing J-moves, and Z-moves implies Theorem 2.3 since for any $c, c' \in B_n^\omega$ we may use the algorithm to get from c and from c' to δ_ω.

Next, we remark that the statement of Theorem 2.3 cannot be improved. Indeed, it is clear that, for example, our curves $c_{\omega,n}$ do not contain a configuration that allows either an S-move or a decreasing J-move, and so any regular homotopy from $c_{\omega,n}$ to δ_ω must begin with an increasing J-move, and so the number of crossings increases by 2.

In [MY] and [V], a different complexity count is carried out for curves. The curves are polygonal, and the moves that are counted are translations of a vertex, together with its two neighboring edges. It is shown in [V] that the number of such moves required to pass between any two regularly homotopic curves (in an appropriate polygonal sense) is linear with respect to the number of edges of the curves.

Finally, we would like to compare our result to what is known for knot diagrams. One defines the distance between two knot diagrams in terms of the number of Reidemeister moves required for passing between them. In [HL], an upper bound for the diameter of the set of all diagrams of the unknot with at most n crossings is given which is exponential with respect to n. In [HN], a lower bound is given that is quadratic with respect to n.

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