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Finite order q-invariants of immersions of surfaces into 3-space

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Abstract. Given a surface F, we are interested in $\mathbb{Z}/2$ valued invariants of immersions of F into $\mathbb{R}^3$, which are constant on each connected component of the complement of the quadruple point discriminant in $Imm(F, \mathbb{R}^3)$. Such invariants will be called "q-invariants." Given a regular homotopy class $A \subseteq Imm(F, \mathbb{R}^3)$, we denote by $V_n(A)$ the space of all q-invariants on A of order $\leq n$. We show that if F is orientable, then for each regular homotopy class A and each n, $\dim(\ V_n(A)/V_{n-1}(A)\ ) \leq 1$.

1. Introduction

Let F be a closed surface. Let $Imm(F, \mathbb{R}^3)$ denote the space of all immersions of F into $\mathbb{R}^3$ and let $I_0 \subseteq Imm(F, \mathbb{R}^3)$ denote the space of all generic immersions.

Definition 1.1. *A function $f : I_0 \to \mathbb{Z}/2$ will be called a "q-invariant" if whenever $H_t : F \to \mathbb{R}^3$ ($0 \leq t \leq 1$) is a generic regular homotopy with no quadruple points, then $f(H_0) = f(H_1)$.*

Definition 1.2. *Let $I_n \subseteq Imm(F, \mathbb{R}^3)$ denote the space of all immersions whose unstable self intersection consists of precisely n generic quadruple points, and let $I = \bigcup_{n=0}^{\infty} I_n$.*

Definition 1.3. *Given a q-invariant $f : I_0 \to \mathbb{Z}/2$ we extend it to I as follows: For $i \in I_n$ let $i_1, ..., i_{2^n} \in I_0$ be the 2^n generic immersions that may be obtained by slightly deforming i. Define*

$$f(i) = \sum_{k=1}^{2^n} f(i_k).$$

For any q-invariant, we will always assume without mention that it is extended to the whole of I as in Definition 1.3.

The following relation clearly holds:

Proposition 1.4. *Let f be a q-invariant. Let $i \in I_n$, $n \geq 1$, and let $p \in \mathbb{R}^3$ be one of its n quadruple points. Then: $f(i) = f(i_1) + f(i_2)$ where $i_1, i_2 \in I_{n-1}$ are the two immersions that may be obtained by slightly deforming i in a small neighborhood of p.*

(Or equivalently, since we are in $\mathbb{Z}/2$, $f(i_2) = f(i_1) + f(i)$.)

Definition 1.5. *A q-invariant f will be called "of finite order" if $f|_{I_n} \equiv 0$ for some n.*

The "order" of a finite order q-invariant f is defined as the minimal n such that $f|_{I_{n+1}} \equiv 0$.

(Compare our Definitions 1.3 and 1.5 with 2.2 of [O].)

An example of a q-invariant of order 1 is the invariant Q which is defined by the property that if $H_t : F \to \mathbb{R}^3$ $(0 \leq t \leq 1)$ is a generic regular homotopy in which m quadruple points occur, then $Q(H_1) = Q(H_0) + m \bmod 2$. In other words Q is defined by the property that $Q|_{I_1} \equiv 1$. It was proved in [N] that Q indeed exists for any surface F.

As a side remark we mention the following: In addition to quadruple points, there are three other types of unstable self intersection that may occur during a generic regular homotopy $H_t : F \to \mathbb{R}^3$. For F orientable, "local" invariants arising from such occurrences have first been studied in [G] (see [G] for the definition.) The above invariant Q is such a local invariant and though the actual existence of Q has not been established in [G], certain facts relating to Q have been shown there. A rough statement of those may be as follows: 1. A local $\mathbb{Z}$-valued invariant analogous to Q does not exist. 2. Q is, in a sense, the only local invariant on immersions of F into $\mathbb{R}^3$, which is not a restriction of a local invariant on general maps of F into $\mathbb{R}^3$.

There are $M = 2^{2-\chi(F)}$ regular homotopy classes (i.e. connected components) in $Imm(F, \mathbb{R}^3)$. Given a regular homotopy class $A \subseteq Imm(F, \mathbb{R}^3)$, we may repeat all our definitions with A in place of $Imm(F, \mathbb{R}^3)$. Let then $V_n(A)$ (respectively V_n) denote the space of all q-invariants on A (respectively $Imm(F, \mathbb{R}^3)$) of order $\leq n$. $V_n(A)$ and V_n are vector spaces over $\mathbb{Z}/2$, and $V_n = \bigoplus_{\alpha=1}^{M} V_n(A_\alpha)$ where $A_1, ..., A_M$ are the regular homotopy classes in $Imm(F, \mathbb{R}^3)$. More precisely, a function $f : I_0 \to \mathbb{Z}/2$ is a q-invariant of order $\leq n$ iff for every $1 \leq \alpha \leq M$, $f|_{I_0 \cap A_\alpha}$ is a q-invariant of order $\leq n$. And so studying q-invariants on $Imm(F, \mathbb{R}^3)$ is the same as studying q-invariants on the various regular homotopy classes.

The purpose of this work is to prove the following:

Theorem 1.6. *If F is orientable then $\dim(\ V_n(A)/V_{n-1}(A)\ ) \leq 1$ for any A and n.*

By [N] $\dim(\, V_1(A)/V_0(A)\,) \geq 1$ for any A (for all surfaces, not necessarily orientable) and so we get:

Corollary 1.7. *If F is orientable then $\dim(\, V_1(A)/V_0(A)\,) = 1$ for any A.*

Since as mentioned, $V_n = \bigoplus_{\alpha=1}^{M} V_n(A_\alpha)$, we get:

Corollary 1.8. *If F is orientable of genus g then $\dim(V_n/V_{n-1}) \leq 2^{2g}$ for every n, and $\dim(V_1/V_0) = 2^{2g}$.*

2. General q-invariants

The results in this section will not assume that the q-invariant f is of finite order.

Theorem 2.1 (The 10 Term Relation). *Let $i : F \to \mathbb{R}^3$ be any immersion whose non-stable self intersection consists of one generic quintuple point, and some finite number of generic quadruple points. Let the quintuple point be located at $p \in \mathbb{R}^3$ and let $S_1, ..., S_5$ be the five sheets passing through p. Let i_k^1 and i_k^2 ($k = 1, ..., 5$) be the two immersions obtained from i by slightly pushing S_k away from p to either side. Then for any q-invariant f:*

$$\sum_{k=1}^{5} \sum_{l=1}^{2} f(i_k^l) = 0.$$

Proof. Starting with i, take S_1 and push it slightly to one side. Then take S_2 and push it away on a much smaller scale. What we now have is an immersion j where sheets $S_2, ..., S_5$ create a little tetrahedron, and S_1 passes outside this tetrahedron. We define the following regular homotopy $H_t : F \to \mathbb{R}^3$ beginning and ending with j, we describe it in four steps: (a) S_1 sweeps to the other side of the tetrahedron. In this step four quadruple points occur. (b) S_2 sweeps across the triple point of sheets S_3, S_4, S_5. This results in the vanishing of the tetrahedron and its inside-out reappearance. One quadruple point occurs here. (c) S_1 sweeps back to its place. Four more quadruple points occur. (d) S_2 sweeps back to its place. One more quadruple point occurs.

All together we have ten quadruple points, and say the mth quadruple point occurs at time t_m. It is easy to verify that the ten immersions $H_{t_1}, ..., H_{t_{10}}$ are precisely (equivalent to) the ten immersions i_k^l ($l = 1, 2$, $k = 1, ..., 5$.) Also, $f(H_{t_m}) = f(H_{t_m - \epsilon}) + f(H_{t_m + \epsilon})$ and so:

$$\sum_{kl} f(i_k^l) = \sum_{m=1}^{10} f(H_{t_m}) = \sum_{m=1}^{10} (f(H_{t_m - \epsilon}) + f(H_{t_m + \epsilon})).$$

But $f(H_{t_m + \epsilon}) = f(H_{t_{m+1} - \epsilon})$ (where $m + 1$ means $(m + 1)$ mod 10) and so this sum is 0. $\square$

Proposition 2.2. *Let $B(1) \subseteq \mathbb{R}^3$ be the unit ball. Let $D_1(1), ..., D_4(1) \subseteq F$ be four disjoint discs which will each be parameterized as the unit disc, and let $D(1) = \bigcup_{k=1}^4 D_k(1)$. Let $i \in I$ and assume $i^{-1}(B(1)) = D(1)$ and $i|_{D(1)}$ maps each $D_k(1)$ linearly onto some $L_k \cap B(1)$ where L_k is a plane through the origin, and $L_1, ..., L_4$ are in general position. Let $i' : D(1) \to B(1)$ be an immersion of the same sort as $i|_{D(1)}$ but with planes $L'_1, ..., L'_4$.*

For $0 \leq r \leq 1$ let $B(r) \subseteq B(1)$ and $D_k(r) \subseteq D_k(1)$ be the ball and discs of radius r and let $D(r) = \bigcup_{k=1}^4 D_k(r)$.

Then: There exists an immersion $j : F \to \mathbb{R}^3$ satisfying:

1. *j is regularly homotopic to i via a regular homotopy that fixes $F - D(1)$.*
2. *$j^{-1}(B(\frac{1}{2})) = D(\frac{1}{2})$*
3. *$j|_{D(\frac{1}{2})} = i'|_{D(\frac{1}{2})}$*
4. *$f(j) = f(i)$ for any q-invariant f.*

Proof. Slightly perturb i if necessary so that the eight planes L_k, L'_k will all be in general position. We define a regular homotopy H_t from i to an immersion $\tilde{i}$ as follows: Say a is the point in $D_1(1)$ which is mapped to the origin. Keeping a and $F - D_1(1)$ fixed, we isotope $D_1(1)$ within $B(1)$ to get $\tilde{i}$ with $\tilde{i}^{-1}(B(\frac{7}{8})) = D(\frac{7}{8})$ and $\tilde{i}|_{D_1(\frac{7}{8})} = i'|_{D_1(\frac{7}{8})}$.

Let i^1, i^2 be the two immersions obtained from i by slightly pushing $D_1(1)$ off of the origin, and let $\tilde{i}^1, \tilde{i}^2$ be the corresponding slight deformations of $\tilde{i}$. H_t induces regular homotopies H_t^l ($l = 1, 2$) from i^l to $\tilde{i}^l$, and such that $H_t^l|_{D_1(1)}$ avoids the origin.

Now, the only triple point of $\{L_2, L_3, L_4\}$ is the origin, and $H_t^l|_{D_1(1)}$ is an isotopy which avoids the origin, and so H_t^l will have no quadruple point, and so $f(i^l) = f(\tilde{i}^l)$ ($l = 1, 2$). And so (By Proposition 1.4) $f(i) = f(i^1) + f(i^2) = f(\tilde{i}^1) + f(\tilde{i}^2) = f(\tilde{i})$.

We now repeat this process in the ball $B(\frac{7}{8})$ and with $D_2(\frac{7}{8})$, obtaining an immersion $\tilde{\tilde{i}}$ with $\tilde{\tilde{i}}|_{D_1(\frac{6}{8}) \cup D_2(\frac{6}{8})} = i'|_{D_1(\frac{6}{8}) \cup D_2(\frac{6}{8})}$. After four iterations we get the desired j. □

3. q-invariants of order n

We now prove the following theorem, which clearly implies Theorem 1.6 (our main theorem):

Theorem 3.1. *Assume F is orientable and let f be a q-invariant of order n.*

Then for any regular homotopy class $A \subseteq Imm(F, \mathbb{R}^3)$, f is constant on $I_n \cap A$.

Proof. Let $i \in I$ and $p \in \mathbb{R}^3$ a quadruple point of i. A ball $B \subseteq \mathbb{R}^3$ centered at p as in Proposition 2.2, i.e. such that $i^{-1}(B)$ is a union of four disjoint discs intersecting in B as four planes, will be called "a good neighborhood for i at p."

For $i \in I_n$ let $p_1, ..., p_n \in \mathbb{R}^3$ be the n quadruple points of i in some order, and let $B_1, ..., B_n$ be disjoint good neighborhoods for i at $p_1, ..., p_n$. We define $\pi_k(i) : F \to \partial B_k$ as follows: Push each one of the four discs in B_k slightly away from p_k into the preferred side determined by the orientation of F. We now have a map that avoids p_k. Define $\pi_k(i)$ as the composition of this map with the radial projection $\mathbb{R}^3 - \{p_k\} \to \partial B_k$.

Let $d_k(i)$ denote the degree of the map $\pi_k(i)$.

Let the symmetric group S_n act on $\mathbb{Z}^n$ by $\sigma(a_1, ..., a_n) = (a_{\sigma(1)}, ..., a_{\sigma(n)})$, and let $\widetilde{\mathbb{Z}^n} = \mathbb{Z}^n / S_n$. Let the class of $(a_1, ..., a_n)$ in $\widetilde{\mathbb{Z}^n}$ be denoted by $[a_1, ..., a_n]$. For $i \in I_n$ we define $d(i) \in \widetilde{\mathbb{Z}^n}$ by $d(i) = [d_1(i), ..., d_n(i)]$.

We break our proof into two steps. *Step 1:* If $i, j \in I_n \cap A$ and $d(i) = d(j)$ then $f(i) = f(j)$. *Step 2:* For any $(a_1, ..., a_n) \in \mathbb{Z}^n$, there are immersions $i, j \in I_n \cap A$ with $d(i) = [a_1, a_2, ..., a_n]$, $d(j) = [a_1 + 1, a_2, ..., a_n]$ and $f(i) = f(j)$. The theorem clearly follows from these two claims.

Proof of Step 1: By composing i with an isotopy $U_t : \mathbb{R}^3 \to \mathbb{R}^3$ we may assume that $p_1, ..., p_n \in \mathbb{R}^3$ are the quadruple points of both i and j and that $d_k(i) = d_k(j)$ for each $1 \le k \le n$. Let $B_1, ..., B_n$ be disjoint good neighborhoods for both i and j at $p_1, ..., p_n$. By composing i with an isotopy $V_t : F \to F$ we may further assume that $i^{-1}(B_k) = j^{-1}(B_k)$ for every k. We name the four discs in F corresponding to p_k by D^{kl}, $l = 1, ..., 4$.

Using Proposition 2.2 we may now change i such that (for smaller B_k's) we will have $i|_{D^{kl}} = j|_{D^{kl}}$ for all $1 \le k \le n$, $1 \le l \le 4$. The process of Proposition 2.2 indeed does not change $d_k(i)$, since the slightly pushed discs appearing in the definition of $\pi_k(i)$ can follow the regular homotopy of Proposition 2.2 and this will induce a homotopy between the corresponding $\pi_k(i)$'s.

So we may assume $i|_{D^{kl}} = j|_{D^{kl}}$ for all $1 \le k \le n$, $1 \le l \le 4$. We will now show that there exists a regular homotopy from i to j such that each D^{kl} moves only within its image in $\mathbb{R}^3$, and $F - \bigcup_{kl} D^{kl}$ moves only within $\mathbb{R}^3 - \bigcup_k B_k$. We will then be done since such a regular homotopy cannot change $f(i)$. Indeed, no sheet will pass $p_1, ..., p_n$ and so the only singularities that might be relevant are the quadruple points occurring in $\mathbb{R}^3 - \bigcup_k B_k$. But whenever such a quadruple point occurs, then we will have $n + 1$ quadruple points all together, and so since f is of order n, $f(i)$ will not change. (Proposition 1.4.)

To show the existence of the above regular homotopy, we construct the following handle decomposition of F. Our discs D^{kl}, ($1 \le k \le n$, $1 \le l \le 4$) will be the 0-handles. If g is the genus of F we will have $2g + 4n - 1$

1-handles as follows: $2g$ 1-handles will have both ends glued to D^{11} such that D^{11} with these $2g$ handles will decompose F in the standard way. Then choose an ordering of the discs D^{kl} with D^{11} first, and connect each two consecutive discs with a 1-handle. The complement of the 0- and 1-handles is one disc which will be the unique 2-handle.

We first define our regular homotopy on the union of 0- and 1- handles. Take a 1-handle h of the first type. Since i and j are regularly homotopic, their restrictions to the annulus $D^{11} \cup h$ are also regularly homotopic. We can construct such a regular homotopy of $D^{11} \cup h$ fixing D^{11} and avoiding $\bigcup_k B_k$.

Next consider the 1-handles of the second type. Take the 1-handle h connecting D^{11} to the second disc in our ordering, call it D'. Then if $i|_h$ and $j|_h$ are not regularly homotopic relative the gluing of h to $D^{11} \cup D'$, then we perform one full rotation of D', as to make them regularly homotopic. (This will require a motion of the next 1-handle too.) Again we perform all regular homotopies while avoiding $\bigcup_k B_k$. We can now go along the chain of 1-handles of the second type, and regularly homotope them one by one as we did the first one. At each step we might need to move the next 0-handle and 1-handle, but we never need to change what we have already done.

Denote our 2-handle by D. So far we have constructed the desired regular homotopy on $F - D$. By means of [S], this regular homotopy may be extended to D (still avoiding $\bigcup_k B_k$.) And so, we are left with regularly homotoping $i|_D$ to $j|_D$ (relative ∂D.) Since $d_k(i) = d_k(j)$ for all k, these maps are homotopic in $\mathbb{R}^3 - \bigcup_k B_k$. It then follows from the Smale-Hirsch Theorem ([H],) that they are also *regularly* homotopic in $\mathbb{R}^3 - \bigcup_k B_k$, since the obstruction to that would lie in $\pi_2(SO_3) = 0$.

Proof of Step 2: Take any immersion $i' \in I_n \cap A$ with $d(i') = [a_1, ..., a_n]$ and let $p_1, ..., p_n \in \mathbb{R}^3$ be the quadruple points of i', ordered such that $d_k(i') = a_k$, $1 \leq k \leq n$. (Clearly any $[a_1, ..., a_n] \in \widetilde{\mathbb{Z}^n}$ may be realized within any regular homotopy class.) Take a disc in F which is away from the p_k's and start pushing it (i.e. regularly homotoping it) into its preferred side directing it towards p_1. Avoid any of the p_k's on the way, and so the immersion i we will get just before arriving at p_1, will still have $d_k(i) = a_k$ for all k. We then pass p_1 creating a quintuple point, and continue to the other side arriving at an immersion j which is again in I_n. Clearly $d_1(j) = a_1 + 1$ and $d_k(j) = a_k$ for $k \geq 2$. We will now use Step 1 and the 10 term relation (Theorem 2.1) to show that $f(i) = f(j)$. Indeed, let us name the five sheets of our quintuple point by $S_1, ..., S_5$ where S_1 is the sheet coming from the disc that we have pushed into p_1. Let i^1_m ($m = 1, ..., 5$) denote the immersion obtained by pushing S_m into its non-preferred side, and i^2_m the immersion obtained by pushing S_m into its preferred side. Then $i = i^1_1$ and $j = i^2_1$. Recall that $\pi_1(i^l_m)$ is constructed by pushing all four sheets involved in the quadruple point at p_1 into their preferred side. And so for each $1 \leq m \leq 5$, $\pi_1(i^1_m)$ has one sheet pushed into the non-preferred side and four sheets into the preferred side, and so $d_1(i^1_m)$ are all equal to each other. Similarly, for each $1 \leq m \leq 5$, $\pi_1(i^2_m)$ has all five sheets pushed into the preferred side and so also $d_1(i^2_m)$ are all equal to each other. Clearly all this has no effect on d_k for $k \geq 2$, and so we have $d(i^1_m) = d(i)$ and $d(i^2_m) = d(j)$ for all $1 \leq m \leq 5$. And so by step 1, $f(i^1_m) = f(i)$ and $f(i^2_m) = f(j)$ for all $1 \leq m \leq 5$. And so by the 10 term relation, $0 = \sum_{ml} f(i^l_m) = 5f(i) + 5f(j) = f(i) + f(j)$ i.e. $f(i) = f(j)$. $\square$

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