Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 206 (2006) 358–372
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Formulae for order one invariants of immersions of surfaces

Tahl Nowik [1]

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

Received 3 November 2004; accepted 13 September 2005
Available online 2 November 2005
Communicated by The Managing Editors

Abstract

The universal order 1 invariant f^U of immersions of a closed orientable surface into $\mathbb{R}^3$, whose existence has been established in [T. Nowik, Order one invariants of immersions of surfaces into 3-space, Math. Ann. 328 (2004) 261–283], is the direct sum

$$f^U = \bigoplus_{n\in\mathbb{Z}} f_n^H \oplus \bigoplus_{n\in\mathbb{Z}} f_n^T \oplus M \oplus Q$$

where each f_n^H, f_n^T is a $\mathbb{Z}$ valued invariant and M, Q are $\mathbb{Z}/2$ valued invariants. An explicit formula for the value of Q on any embedding has been given in [T. Nowik, Automorphisms and embeddings of surfaces and quadruple points of regular homotopies, J. Differential Geom. 58 (2001) 421–455]. In the present work we give explicit formulae for the value of each f_n^H, f_n^T on all immersions, and for the value of M on any embedding.

Keywords: Immersions of surfaces; Finite order invariants

E-mail address: tahl@math.biu.ac.il.
URL: http://www.math.biu.ac.il/~tahl.
[1] Partially supported by the Minerva Foundation.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2005.09.003

1. Introduction

Finite order invariants of stable immersions of a closed orientable surface into $\mathbb{R}^3$ have been defined in [4], where all order 1 invariants have been classified, and a universal order 1 invariant f^U has been introduced. In [5] all higher order invariants have been classified, and all shown to be explicit functions of f^U. However, the proof of the existence of f^U appearing in [4], provides no means for computing it.

The invariant f^U may be thought of as a direct sum $\bigoplus_{n\in\mathbb{Z}} f_n^H \oplus \bigoplus_{n\in\mathbb{Z}} f_n^T \oplus M \oplus Q$ where f_n^H and f_n^T are $\mathbb{Z}$ valued invariants and M and Q are $\mathbb{Z}/2$ valued invariants. The geometric meaning of each invariant will be explained in Section 3. In [3], an explicit formula has been given for $Q(i\circ h)-Q(i)$ where $i:F\to\mathbb{R}^3$ is any immersion and $h:F\to F$ any diffeomorphism such that i and $i\circ h$ are regularly homotopic, and for $Q(e')-Q(e)$ where $e,e':F\to\mathbb{R}^3$ are any two regularly homotopic embeddings.

In the present work we give explicit formulae for:

(1) the values of f_n^H and f_n^T on all immersions;
(2) $M(i\circ h)-M(i)$ where $h:F\to F$ is any diffeomorphism such that i and $i\circ h$ are regularly homotopic;
(3) $M(e')-M(e)$ for any two regularly homotopic embeddings.

Note that the invariant f^U is specified only up to an order 0 invariant, i.e., up to an additive constant in each regular homotopy class, and so the same is true for f_n^H, f_n^T, M, Q. For M and Q we will not have a specific choice of constants, and so as in (2), (3) above, we will speak only of the *difference* of the values of M or Q on regularly homotopic immersions.

The structure of the paper is as follows: In Section 2 we give the necessary background. Note that in the present work we deviate from [4] in our procedure for specifying order one invariants. This is of no consequence in the abstract setting of [4,5], but greatly simplifies the explicit formulae for f_n^H, f_n^T that we derive in the present work. In Section 3 we explain the geometric meaning of each invariant. In Section 4 we present the formulae that are proved in this paper. In Section 5 we prove the formulae for f_n^H, f_n^K. In Section 6 we give several applications. In Section 7 we prove the formulae for M. Appendix A relates the present work to previous work.

2. Background

In this section we summarize the background needed for this work. Given a closed oriented surface F, $\mathrm{Imm}(F,\mathbb{R}^3)$ denotes the space of all immersions of F into $\mathbb{R}^3$, with the C^1 topology. A CE point of an immersion $i:F\to\mathbb{R}^3$ is a point of self intersection of i for which the local stratum in $\mathrm{Imm}(F,\mathbb{R}^3)$ corresponding to the self intersection, has co-dimension one. We distinguish twelve types of CEs which we name $E^0, E^1, E^2, H^1, H^2, T^0, T^1, T^2, T^3, Q^2, Q^3, Q^4$. Their precise description appears in the proof of Proposition 5.1 below. A co-orientation for a CE is a choice of one of the two sides of the local stratum corresponding to the CE. All but two of the above CE types are non-symmetric in the sense that the two sides of the local stratum may be distinguished via the local configuration of the CE, and for those ten CE types, permanent co-orientations for the corresponding strata are chosen once and for all. The two exceptions are H^1 and Q^2 which are completely symmetric. In fact, there does not exist a consistent choice of co-orientation for H^1 and Q^2 CEs since the global strata corresponding to these CE types are one sided in $\mathrm{Imm}(F, \mathbb{R}^3)$ (see [4]).

We fix a closed oriented surface F and a regular homotopy class $\mathcal{A}$ of immersions of F into $\mathbb{R}^3$ (that is, $\mathcal{A}$ is a connected component of $\mathrm{Imm}(F, \mathbb{R}^3)$). We denote by $I_n \subseteq \mathcal{A}$ ($n \geqslant 0$) the space of all immersions in $\mathcal{A}$ which have precisely n CE points (the self intersection being elsewhere stable). In particular, I_0 is the space of all stable immersions in $\mathcal{A}$.

Given an immersion $i \in I_n$, a *temporary co-orientation* for i is a choice of co-orientation at each of the n CE points $p_1, \ldots, p_n$ of i. Given a temporary co-orientation $\mathfrak{T}$ for i and a subset $A \subseteq \{p_1, \ldots, p_n\}$, we define $i_{\mathfrak{T},A} \in I_0$ to be the immersion obtained from i by resolving all CEs of i at points of A into the positive side with respect to $\mathfrak{T}$, and all CEs not in A into the negative side. Now let $\mathbb{G}$ be any Abelian group and let $f : I_0 \to \mathbb{G}$ be an invariant, i.e., a function which is constant on each connected component of I_0. Given $i \in I_n$ and a temporary co-orientation $\mathfrak{T}$ for i, $f^{\mathfrak{T}}(i)$ is defined as follows:

$$f^{\mathfrak{T}}(i) = \sum_{A \subseteq \{p_1, \ldots, p_n\}} (-1)^{n-|A|} f(i_{\mathfrak{T},A})$$

where $|A|$ is the number of elements in A. If $\mathfrak{T}, \mathfrak{T}'$ are two temporary co-orientations for the same immersion i then $f^{\mathfrak{T}}(i) = \pm f^{\mathfrak{T}'}(i)$ and so having $f^{\mathfrak{T}}(i) = 0$ is independent of the temporary co-orientation $\mathfrak{T}$. An invariant $f : I_0 \to \mathbb{G}$ is called *of finite order* if there is an n such that $f^{\mathfrak{T}}(i) = 0$ for all $i \in I_{n+1}$. The minimal such n is called the *order* of f. The group of all invariants on I_0 of order at most n is denoted V_n.

We continue the discussion for order 1 invariants only. The more general setting may be found in [4,5]. For an immersion $i : F \to \mathbb{R}^3$ and any $p \in \mathbb{R}^3$, we define the degree $d_p(i) \in \mathbb{Z}$ of i at p as follows: If $p \notin i(F)$ then $d_p(i)$ is the (usual) degree of the map obtained from i by composing it with the projection onto a small sphere centered at p. If on the other hand $p \in i(F)$ then we first push each sheet of F which passes through p, a bit into its preferred side determined by the orientation of F, obtaining a new immersion i' which misses p, and we define $d_p(i) = d_p(i')$. If $i \in I_1$ and the unique CE of i is located at $p \in \mathbb{R}^3$, then we define $C(i)$ to be the expression R^a_m where R^a is the symbol describing the configuration of the CE of i at p (one of the twelve symbols above) and $m = d_p(i)$. We denote by $\mathcal{C}_1$ the set of all expressions R^a_m with R^a one of our twelve symbols and $m \in \mathbb{Z}$. The map $C : I_1 \to \mathcal{C}_1$ is surjective (given $R^a_m \in \mathcal{C}_1$, start with any $i \in \mathcal{A}$ and regularly homotope it to create a CE of configuration R^a, then if needed, continue by having some sheets of F pass the CE to obtain degree m).

Let $f \in V_1$. For $i \in I_1$, if the CE of i is of type H^1 or Q^2 and $\mathfrak{T}$ is a temporary co-orientation for i, then $2f^{\mathfrak{T}}(i) = 0$ [4, Proposition 3.5], and so in this case $f^{\mathfrak{T}}(i)$ is independent of $\mathfrak{T}$. Using this fact, $f \in V_1$ will induce a function $\hat{f} : I_1 \to \mathbb{G}$ as follows: For any $i \in I_1$ we set $\hat{f}(i) = f^{\mathfrak{T}}(i)$, where if the CE of i is of type H^1 or Q^2 then $\mathfrak{T}$ is arbitrary, and if it is not of type H^1 or Q^2 then the permanent co-orientation is used for the CE of i. For $f \in V_1$ and $i, j \in I_1$, if $C(i) = C(j)$ then $\hat{f}(i) = \hat{f}(j)$ [4, Proposition 3.8], so any $f \in V_1$ induces a well-defined function $u(f): \mathcal{C}_1 \to \mathbb{G}$. The map $f \mapsto u(f)$ induces an injection $u: V_1/V_0 \to \mathcal{C}_1^*$ where $\mathcal{C}_1^*$ is the group of all functions from $\mathcal{C}_1$ to $\mathbb{G}$.

The main result of [4] is that the image of $u: V_1/V_0 \to \mathcal{C}_1^*$ is the subgroup $\Delta_1 = \Delta_1(\mathbb{G}) \subseteq \mathcal{C}_1^*$ which is defined as the set of all functions $g \in \mathcal{C}_1^*$ satisfying relations which we write as relations on the symbols R_m^a, e.g., $T_m^0 = T_m^3$ will stand for $g(T_m^0) = g(T_m^3)$. The relations defining Δ_1 are:

- $E_m^2 = -E_m^0 = H_m^2$, $E_m^1 = H_m^1$;
- $T_m^0 = T_m^3$, $T_m^1 = T_m^2$;
- $2H_m^1 = 0$, $H_m^1 = H_{m-1}^1$;
- $2Q_m^2 = 0$, $Q_m^2 = Q_{m-1}^2$;
- $T_m^3 - T_m^2 = H_m^2 - H_{m-1}^2$;
- $Q_m^4 - Q_m^3 = T_m^3 - T_{m-1}^3$, $Q_m^3 - Q_m^2 = T_m^2 - T_{m-1}^2$.

Let $\mathbb{B} \subseteq \mathbb{G}$ be the subgroup defined by $\mathbb{B} = \{x \in \mathbb{G}\colon 2x = 0\}$. To obtain a function $g \in \Delta_1$ one may assign arbitrary values in $\mathbb{G}$ for the symbols $\{T_m^2\}_{m\in\mathbb{Z}}$, $\{H_m^2\}_{m\in\mathbb{Z}}$ (here is where we deviate from [4]) and arbitrary values in $\mathbb{B}$ for the two symbols H_0^1, Q_0^2. Once this is done then value of g on all other symbols is uniquely determined, namely:

(1) $E_m^1 = H_m^1 = H_0^1$;
(2) $E_m^2 = -E_m^0 = H_m^2$;
(3) $T_m^3 = T_m^2 + H_m^2 - H_{m-1}^2$;
(4) $T_m^0 = T_m^3$, $T_m^1 = T_m^2$;
(5) $Q_m^2 = Q_0^2$;
(6) $Q_m^3\ (= Q_m^2 + T_m^2 - T_{m-1}^2) = Q_0^2 + T_m^2 - T_{m-1}^2$;
(7) $Q_m^4\ (= Q_m^3 + T_m^3 - T_{m-1}^3) = Q_0^2 + 2T_m^2 - 2T_{m-1}^2 + H_m^2 - 2H_{m-1}^2 + H_{m-2}^2$.

In the sequel we will refer to this procedure as the "7-step procedure."

The Abelian group $\mathbb{G}_U$ is defined by the following Abelian group presentation (again note the difference from [4]):

$$\mathbb{G}_U = \bigl\langle \{t_m\}_{m\in\mathbb{Z}}, \{h_m\}_{m\in\mathbb{Z}}, h, q \mid 2h = 2q = 0 \bigr\rangle.$$

The universal element $g^U \in \Delta_1(\mathbb{G}_U)$ is defined by $g^U(T_m^2) = t_m$, $g^U(H_m^2) = h_m$, $g^U(H_0^1) = h$, $g^U(Q_0^2) = q$ and the values of g^U on all other symbols of $\mathcal{C}_1$ are determined by the 7-step procedure. In [4] the existence of an order 1 invariant $f^U: I_0 \to \mathbb{G}_U$ with $u(f^U) = g^U$ is proven. (Note that this is the same g^U as in [4] only presented via different generators.) The invariant f^U is a *universal* order 1 invariant, in the sense that any order 1 invariant with values in an arbitrary $\mathbb{G}$ is of the form $\varphi \circ f^U + c$ where $\varphi: \mathbb{G}_U \to \mathbb{G}$ is a (unique) homomorphism and $c \in \mathbb{G}$ is a (unique) constant element.

In [5] all higher order invariants are classified, and it is shown that any order n invariant with values in an arbitrary $\mathbb{G}$ is of the form $s \circ f^U$ for some function (not homomorphism) $s: \mathbb{G}_U \to \mathbb{G}$. This fact reverts our attention back to order 1 invariants.

3. The invariants

In this section we introduce the invariants f_n^H, f_n^T, M, Q which compose f^U, and explain their geometric meaning. We denote by f_n^H, f_n^T the $\mathbb{Z}$ valued invariants obtained by composing $f^U : I_0 \to \mathbb{G}_U$ with the projections $\mathbb{G}_U \to \mathbb{Z}$ corresponding to the generators h_n, t_n of $\mathbb{G}_U$, respectively. Similarly M, Q are the $\mathbb{Z}/2$ valued invariants obtained by composing f^U with the projections $\mathbb{G}_U \to \mathbb{Z}/2$ corresponding to the generators h, q, respectively. Equivalently, if Y denotes the set of symbols

$$\{T_m^2\}_{m\in\mathbb{Z}} \cup \{H_m^2\}_{m\in\mathbb{Z}} \cup \{H_0^1, Q_0^2\},$$

i.e., the set on which we impose the initial data for the 7-step procedure, then f_n^H (respectively f_n^T) is the $\mathbb{Z}$ valued order 1 invariant satisfying $u(f_n^H)(H_n^2) = 1$ (respectively $u(f_n^T)(T_n^2) = 1$) and $u(f_n^H)$ (respectively $u(f_n^T)$) has value 0 on all other symbols in Y. Similarly M (respectively Q) is the $\mathbb{Z}/2$ valued order 1 invariant for which $u(M)(H_0^1) = 1 \in \mathbb{Z}/2$ (respectively $u(Q)(Q_0^2) = 1$) and $u(M)$ (respectively $u(Q)$) has value 0 on all other symbols in Y. Note that f^U is defined only up to an additive constant in each regular homotopy class, and so the same is true for f_n^H, f_n^T, M, Q.

Applying the 7-step procedure to the above initial data we obtain for $M : I_0 \to \mathbb{Z}/2$ that $u(M)$ attains value $1 \in \mathbb{Z}/2$ on the symbols H_m^1, E_m^1 for all m, and value $0 \in \mathbb{Z}/2$ on all other symbols in $\mathcal{C}_1$. That is, if $i_+, i_- \in I_0$ are the two immersions obtained from $i \in I_1$ by resolving its CE, then $M(i_+) - M(i_-) = 1 \in \mathbb{Z}/2$ iff the CE of i is a "matching tangency," i.e., tangency of two sheets of the surface where the orientations of the two sheets match at time of tangency. (Thus the name M for this invariant.) And so for any $i, j \in I_0$, $M(j) - M(i) \in \mathbb{Z}/2$ is the number mod 2 of matching tangencies occurring in any generic regular homotopy between i and j.

Similarly, by the 7-step procedure applied to the initial data for Q, we see that $u(Q)$ attains value $1 \in \mathbb{Z}/2$ on all symbols of the form Q_m^a and value 0 on all other symbols in $\mathcal{C}_1$. That is, Q is an invariant such that for any $i, j \in I_0$, $Q(j) - Q(i) \in \mathbb{Z}/2$ is the number mod 2 of quadruple points occurring in any regular homotopy between i and j. This invariant has been studied in [2,3]. In [3] an explicit formula has been given for $Q(i \circ h) - Q(i)$ for any diffeomorphism $h : F \to F$ such that i and $i \circ h$ are regularly homotopic, and for $Q(e') - Q(e)$ for any two regularly homotopic embeddings. In the present work we will do the same for M, leaving open the interesting problem of finding explicit formulae for M and Q on *all* immersions. (For f_n^H, f_n^T however, we will indeed give formulae for all immersions.)

The 7-step procedure applied to the initial data for the $\mathbb{Z}$ valued invariant f_n^H produces for $g = u(f_n^H)$ the following:

$$g(H_n^2) = g(E_n^2) = 1, \qquad g(E_n^0) = -1, \qquad g(T_n^0) = g(T_n^3) = 1,$$
$$g(T_{n+1}^0) = g(T_{n+1}^3) = -1, \qquad g(Q_n^4) = 1, \qquad g(Q_{n+1}^4) = -2, \qquad g(Q_{n+2}^4) = 1,$$

and the value on all other symbols in $\mathcal{C}_1$ is 0. Given $i, j \in I_0$ and given a generic regular homotopy from i to j, then for a symbol $R_m^a \in \mathcal{C}_1$ we will denote by $N(R_m^a)$ the signed number of CEs of type R_m^a occurring during the given regular homotopy, each counted as ± 1 according to whether it has been passed in the direction of its chosen permanent co-orientation, or not. Using the values for $u(f_n^H)$ that we have found, the following geometric meaning for f_n^H analogous to the meaning we have attached to M, Q, may be given: For any $i, j \in I_0$, $f_n^H(j) - f_n^H(i)$ is equal to

$$N\big(H_n^2\big) + N\big(E_n^2\big) - N\big(E_n^0\big) + N\big(T_n^0\big) + N\big(T_n^3\big) - N\big(T_{n+1}^0\big)$$
$$- N\big(T_{n+1}^3\big) + N\big(Q_n^4\big) - 2N\big(Q_{n+1}^4\big) + N\big(Q_{n+2}^4\big)$$

for any regular homotopy from i to j. Note that indeed all CE types involved here possess a co-orientation.

Finally, for f_n^T the 7-step procedure applied to the above initial data produces for $g = u(f_n^T)$ the following:

$$g\big(T_n^0\big) = g\big(T_n^1\big) = g\big(T_n^2\big) = g\big(T_n^3\big) = 1, \qquad g\big(Q_n^3\big) = 1, \qquad g\big(Q_{n+1}^3\big) = -1,$$
$$g\big(Q_n^4\big) = 2, \qquad g\big(Q_{n+1}^4\big) = -2,$$

and the value of g on all other symbols of $\mathcal{C}_1$ is 0. and so as above, the geometric meaning we attach to f_n^T is that for any $i, j \in I_0$, $f_n^T(j) - f_n^T(i)$ is equal to

$$N(T_n) + N\big(Q_n^3\big) - N\big(Q_{n+1}^3\big) + 2N\big(Q_n^4\big) - 2N\big(Q_{n+1}^4\big)$$

for any regular homotopy from i to j, where $N(T_n)$ stands for $\sum_{a=0}^{3} N(T_n^a)$, and notice again that all CE types involved possess co-orientations.

4. Statement of results

Let $i \in I_0$. For every $m \in \mathbb{Z}$ let $U_m = U_m(i) = \{p \in \mathbb{R}^3 - i(F)\colon\ d_p(i) = m\}$. This is an open set in $\mathbb{R}^3$ which may be empty, and may be non-connected or unbounded, but in any case, the Euler characteristic $\chi(U_m)$ is defined. Denote by $N_m = N_m(i)$ the number of triple points $p \in \mathbb{R}^3$ of i having $d_p(i) = m$. We define $K \subseteq \mathbb{G}_U$ to be the subgroup generated by $\{t_m\}_{m\in\mathbb{Z}} \cup \{h_m\}_{m\in\mathbb{Z}}$ (this is the same as the subgroup K_1 in [5]) and define $f^K : I_0 \to K$ to be the projection of f^U to K, so $f^K = \bigoplus_n f_n^H \oplus \bigoplus_n f_n^T$. Finding a formula for f^K is therefore the same as finding formulae for all f_n^H, f_n^T. The following formula for $f^K : I_0 \to K \subseteq \mathbb{G}_U$ will be proved in Section 5:

$$f^K(i) = \sum_{m\in\mathbb{Z}} \chi(U_m)\Bigg(\sum_{-\frac{1}{2}<k<\lfloor\frac{m}{2}\rfloor+\frac{1}{2}} h_{m-2k}\Bigg) + \sum_{m\in\mathbb{Z}} \frac{1}{2} N_m \Bigg(t_m - \sum_{-\frac{1}{2}<k<m-\frac{1}{2}} h_k\Bigg)$$

where for $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the greatest integer $\leqslant a$, and for $a, b \in \mathbb{R}$ the sum $\sum_{a<k<b}$ means the following: If $a < b$ then it is the sum over all integers $a < k < b$, if $a = b$ then the sum is 0, and if $a > b$ then $\sum_{a<k<b} = -\sum_{b<k<a}$.

For f_n^H, f_n^T we thus get:

Formula 4.1.

$$f_n^H(i) = \sum_{m\in\mathbb{Z}} \chi(U_m)\Bigg(\sum_{-\frac{1}{2}<k<\lfloor\frac{m}{2}\rfloor+\frac{1}{2}} \delta_{n,m-2k}\Bigg) - \sum_{m\in\mathbb{Z}} \frac{1}{2}N_m\Bigg(\sum_{-\frac{1}{2}<k<m-\frac{1}{2}} \delta_{n,k}\Bigg),$$

$$f_n^T(i) = \frac{1}{2}N_n$$

where $\delta_{n,r}$ is the Kronecker delta.

For $i, j \in I_0$ let $M(i, j) = M(j) - M(i)$. The following two formulae for M will be proved in Section 7:

Formula 4.2. For any diffeomorphism $h : F \to F$ such that i and $i \circ h$ are regularly homotopic,

$$M(i, i \circ h) = \big(\mathrm{rank}(h_* - \mathrm{Id})\big) \bmod 2$$

where h_* is the map induced by h on $H_1(F, \mathbb{Z}/2)$.

If $e : F \to \mathbb{R}^3$ is an embedding then $e(F)$ splits $\mathbb{R}^3$ into two pieces, one compact and one non-compact, which we denote $D^0(e)$ and $D^1(e)$, respectively. By restriction of range, e induces maps $e^k : F \to D^k(e)$, $k = 0, 1$. Let $e^k_* : H_1(F, \mathbb{Z}/2) \to H_1(D^k(e), \mathbb{Z}/2)$ be the map induced by e^k.

Formula 4.3. For two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$, $M(e, e')$ is computed as follows:

(1) Find a basis $a_1, \dots, a_n, b_1, \dots, b_n$ for $H_1(F, \mathbb{Z}/2)$ such that $e^0_*(a_i) = 0$, $e^1_*(b_i) = 0$ and $a_i \cdot b_j = \delta_{ij}$ (where $a \cdot b$ denotes the intersection form in $H_1(F, \mathbb{Z}/2)$).
(2) Find a similar basis $a'_1, \dots, a'_n, b'_1, \dots, b'_n$ using e' in place of e.
(3) Let m be the dimension of the subspace of $H_1(F, \mathbb{Z}/2)$ spanned by:

$$a'_1 - a_1, \dots, a'_n - a_n, b'_1 - b_1, \dots, b'_n - b_n.$$

Then $M(e, e') = m \bmod 2 \in \mathbb{Z}/2$.

5. Proof of the formula for f^K

We define the group $\mathbb{O}$ to be the free Abelian group with generators $\{x_n\}_{n\in\mathbb{Z}} \cup \{y_n\}_{n\in\mathbb{Z}}$. For $i \in I_0$ we define $k(i) \in \mathbb{O}$ as follows (the terms are defined in Section 4 and the sums are always finite):

$$k(i) = \sum_{m\in\mathbb{Z}} \chi(U_m)x_m + \sum_{m\in\mathbb{Z}} \frac{1}{2}N_m y_m.$$

Indeed this is an element of $\mathbb{O}$ since as we shall see below, N_m is always even. In the mean time say k attains values in the $\mathbb{Q}$ vector space with same basis.

Proposition 5.1. *The invariant k is an order* 1 *invariant, with $u(k)$ given by*:

- $u(k)(E^a_m) = u(k)(H^a_m) = x_{m+a-2} - x_{m-a}$,
- $u(k)(T^a_m) = x_{m+a-3} + x_{m-a} + y_m$,
- $u(k)(Q^a_m) = x_{m+a-4} - x_{m-a} + (a-2)y_m + (2-a)y_{m-1}$.

Proof. We use the explicit description of the CE types, as appearing in [4], where more details may be found. A model in 3-space for the different sheets involved in the self intersection near the CE, is given. The CE is obtained at the origin when setting $\lambda = 0$. We will show that for any $i \in I_1$, if $i_+ \in I_0$ is the immersion on the positive side of i with respect to the permanent co-orientation for the CE of i (if such exists, otherwise an arbitrary side is chosen) and $i_- \in I_0$ is the immersion on the other side, then indeed $k(i_+) - k(i_-)$ depends on $C(i)$ as in the statement of this proposition. By showing in particular, that this change depends *only* on $C(i)$, we show that k is indeed an invariant of order 1.

Model for E^a_m: $z = 0$, $z = x^2 + y^2 + \lambda$. The positive side is that where $\lambda < 0$, where there is a new 2-sphere in the image of the immersion, which is made of two 2-cells, and bounds a 3-cell in $\mathbb{R}^3$. The superscript a is then the number of 2-cells (0, 1 or 2) whose preferred side determined by the orientation of the surface, is facing away from the 3-cell (and m is the degree at the CE at time $\lambda = 0$). The degree of points in the new 3-cell is seen to be $m + a - 2$, and its χ is 1, and so the term x_{m+a-2}. The second change occurring, is that the region just above the plane $z = 0$, has a 2-handle removed from it, so its χ is reduced by 1, and the degree in this region is seen to be $m - a$, and so the term $-x_{m-a}$.

Model for H^a_m: $z = 0$, $z = x^2 - y^2 + \lambda$. The positive side for H^2 is that where both sheets have their preferred side facing toward the region that is between them near the origin. For H^1 a positive side is chosen arbitrarily. By rotating the configuration if necessary, say the positive side is where $\lambda < 0$. The superscript a then denotes the number of sheets (1 or 2) whose preferred side is facing toward the region that is between the two sheets near the origin, when $\lambda < 0$. The changes occurring in the neighboring regions when passing from $\lambda > 0$ to $\lambda < 0$ are that a 1-handle is removed from the region X just above the x axis, and a 1-handle is added to the region Y just below the y axis. The degree of X is seen to be $m + a - 2$ and since a 1-handle is removed, $\chi(X)$ increases by 1 and thus the term x_{m+a-2}. The degree of Y is seen to be $m - a$, and since a 1-handle is added, $\chi(Y)$ decreases by 1 and thus the term $-x_{m-a}$.

Model for T^a_m: $z = 0$, $y = 0$, $z = y + x^2 + \lambda$. The positive side for this configuration is when $\lambda < 0$, where there is a new 2-sphere in the image of the immersion, which is made of three 2-cells, and bounds a 3-cell in $\mathbb{R}^3$. The superscript a is the number of 2-cells (0, 1, 2 or 3) whose preferred side is facing away from the 3-cell. The degree in the new 3-cell is $m + a - 3$ and its χ is 1 and so the term x_{m+a-3}. The second change occurring is that a 1-handle is removed from the region near the x axis having negative y values and positive z values. The degree of this region is $m - a$ and since a 1-handle is removed, χ is increased by 1 and so the term x_{m-a}. The last change that effects the value of k is that two triple points are added, each of degree m. This increases $\frac{1}{2}N_m$ by 1 and so the term y_m.

Model for Q^a_m: $z=0$, $y=0$, $x=0$, $z=x+y+\lambda$. On both the positive and negative side there is a simplex created near the origin, and the positive side is that where the majority of the four sheets are facing away from the simplex (and for Q^2 a positive side is chosen arbitrarily). The superscript a denotes the number of sheets (2, 3 or 4) facing away from the simplex created on the positive side, its degree thus seen to be $m+a-4$. The simplex on the negative side has $4-a$ sheets facing away from it and so its degree is $m-a$. So when moving from the negative to the positive side, a 3-cell ($\chi=1$) of degree $m-a$ is removed and a 3-cell of degree $m+a-4$ is added, and so the terms $x_{m+a-4}-x_{m-a}$. In addition to that, the degree of the four triple points of the simplex changes. On the positive side there are a triple points with degree m (namely, the triple points which are opposite the faces which are facing away from the simplex), and $4-a$ triple points with degree $m-1$. On the negative side the situation is reversed, i.e., there are $4-a$ triple points with degree m and a triple points with degree $m-1$. So the total change in N_m is $a-(4-a)=2a-4$ and the total change in N_{m-1} is $(4-a)-a=4-2a$ and so the terms $(a-2)y_m+(2-a)y_{m-1}$. □

We can now verify that indeed the values of k are in $\mathbb{O}$, i.e., no half integer coefficients appear (which means N_m is always even). From Proposition 5.1 we see that the change in the value of k is in $\mathbb{O}$ along any regular homotopy, and so it is enough to show that the value is in $\mathbb{O}$ for one immersion in any given regular homotopy class. Indeed, we show the following:

Lemma 5.2. *Let g be the genus of F. Any immersion $i: F \to \mathbb{R}^3$ is regularly homotopic to an immersion j with $k(j)=(2-g)x_0+(1-g)x_{-1}$.*

Proof. By [6], any immersion $i: F \to \mathbb{R}^3$ is regularly homotopic to an immersion whose image is of one of two standard forms: either a standard embedding, or an immersion obtained from a standard embedding by adding a ring to it. (For definition of "ring" see [3, p. 434] and for "standard embedding" see [3, Definition 8.3].) For an embedding e, $k(e)$ is either $(2-g)x_0+(1-g)x_{-1}$ or $(2-g)x_0+(1-g)x_1$, depending on whether the preferred side of $e(F)$, determined by the orientation of F, is facing the compact or the non-compact side of $e(F)$ in $\mathbb{R}^3$, respectively. Now take an orientation reversing diffeomorphism $h: F \to F$ such that $e \circ h$ is regularly homotopic to e, to see that both values are attained. (Such h exists by [6], take, e.g., an h that induces the identity on $H_1(F, \mathbb{Z}/2)$.) Now, a ring added to such embedding bounds a solid torus, whose χ is 0, and the topological type and degree of the other two components remains the same, and so by the same argument as for an embedding, the two values are attained in this case too. □

We define a homomorphism $\varphi: \mathbb{G}_U \to \mathbb{O}$ on generators as follows:

- $\varphi(h_m)=x_m-x_{m-2}$,
- $\varphi(t_m)=x_{m-1}+x_{m-2}+y_m$,
- $\varphi(h)=\varphi(q)=0$.

By Proposition 5.1, $u(k) = u(\varphi \circ f^U)$ and so $k = \varphi \circ f^U + c$ where $c \in \mathbb{O}$ is a constant. We now define a homomorphism $F : \mathbb{O} \to K$ as follows (the notation involved is defined in Section 4):

$$F(x_m) = \sum_{-\frac{1}{2} < k < \lfloor \frac{m}{2} \rfloor + \frac{1}{2}} h_{m-2k}, \qquad F(y_m) = t_m - \sum_{-\frac{1}{2} < k < m - \frac{1}{2}} h_k.$$

One checks directly that $F \circ \varphi$ maps each generator of K to itself, so $F \circ \varphi$ is the projection of $\mathbb{G}_U$ onto K, and so $F \circ k = F \circ \varphi \circ f^U + F(c) = f^K + F(c)$. Since f^U is defined up to an additive constant of our choice, we may take $f^U + F(c)$ to be our new f^U, then $f^K + F(c)$ will become our new f^K and so we will have $F \circ k = f^K$.

Since $\varphi : \mathbb{G}_U \to \mathbb{O}$ is not surjective, there was a certain choice for an $F : \mathbb{O} \to K$ such that $F \circ \varphi : \mathbb{G}_U \to K$ is the projection. Indeed the image of φ is the subgroup of $\mathbb{O}$ of all elements $\sum A_m x_m + \sum B_m y_m$ with $A_m, B_m \in \mathbb{Z}$ satisfying $\sum_m A_{2m} = \sum_m A_{2m+1} = \sum_m B_m$. And so any two generators x_i, x_j with i even and j odd, generate a subgroup in $\mathbb{O}$ which is a direct summand of the image of φ. Our choice for F was that $F(x_{-2}) = F(x_{-1}) = 0$. Note that by Lemma 5.2, the image of $k : I_0 \to \mathbb{O}$ is contained in a non-trivial coset of the image of φ in $\mathbb{O}$ (and so the constant c appearing above is non-zero, regardless of an additive constant for f^U). Composing the formula for F with the formula for k we obtain our formula for f^K:

$$f^K(i) = \sum_{m \in \mathbb{Z}} \chi(U_m) \Bigg(\sum_{-\frac{1}{2} < k < \lfloor \frac{m}{2} \rfloor + \frac{1}{2}} h_{m-2k} \Bigg) + \sum_{m \in \mathbb{Z}} \frac{1}{2} N_m \Bigg(t_m - \sum_{-\frac{1}{2} < k < m - \frac{1}{2}} h_k \Bigg).$$

The choice of additive constants for f^K here may be characterized by saying that in each regular homotopy class, $f^K(j) = (2 - g)h_0$ for j of Lemma 5.2.

Since $C : I_1 \to \mathcal{C}_1$ is surjective, any generator of K is attained as $f^K(j) - f^K(i)$ for some $i, j \in I_0$, and so the image of $f^K : I_0 \to K$ is not contained in any coset of any proper subgroup of K. Yet, the image of f^K is far from being the whole group K, since as we see from the formula, the coefficients of all generators t_m are always non-negative. It would be interesting to determine the precise image of $f^U : I_0 \to \mathbb{G}_U$ (up to an additive constant).

6. Applications

We present several applications, the last of which will be used in Section 7.

We will use the fact that $\varphi : \mathbb{G}_U \to \mathbb{O}$ is not surjective to obtain identities on immersions: Let $\theta_0 : \mathbb{O} \to \mathbb{Z}$ be a homomorphism defined by $\theta_0(x_{2m}) = 1$, $\theta_0(x_{2m+1}) = 0$, $\theta_0(y_m) = -1$ for all m and $\theta_1 : \mathbb{O} \to \mathbb{Z}$ defined by $\theta_1(x_{2m}) = 0$, $\theta_1(x_{2m+1}) = 1$, $\theta_1(y_m) = -1$ for all m, so $\theta_0 \circ \varphi = \theta_1 \circ \varphi = 0$. It follows that $\theta_0 \circ k$ and $\theta_1 \circ k$ are constant invariants, which are given explicitly by $\theta_0 \circ k(i) = \sum \chi(U_{2m}) - \frac{1}{2}N$ and $\theta_1 \circ k(i) = \sum \chi(U_{2m+1}) - \frac{1}{2}N$ where $N = N(i) = \sum N_m(i)$ is the total number of triple points of i. To find the values of these constants we need to evaluate them on a single immersion in every regular homotopy class.

For the immersion j of Lemma 5.2, $\theta_0 \circ k(j) = 2 - g$ and $\theta_1 \circ k(j) = 1 - g$, so we get the following two identities:

Proposition 6.1. *Let g be the genus of F. For any $i \in I_0$,*

$$\sum_m \chi(U_{2m}) - \frac{1}{2}N = 2 - g \quad \textit{and} \quad \sum_m \chi(U_{2m+1}) - \frac{1}{2}N = 1 - g.$$

For our second application, let $U : I_0 \to \mathbb{Z}$ be the order one invariant defined by $u(U)(H^2_m) = 1$, $u(U)(T^2_m) = 0$ for all m and $u(U)(H^1_0) = u(U)(Q^2_0) = 0$. (In other words $U = \sum_{n\in\mathbb{Z}} f^H_n$.) By the 7-step procedure we have

$$u(U)\big(H^2_m\big) = u(U)\big(E^2_m\big) = 1 \quad \text{for all } m, \qquad u(U)\big(E^0_m\big) = -1 \quad \text{for all } m,$$

and $u(U)$ is 0 on all other symbols in $\mathcal{C}_1$. That is, for any $i, j \in I_0$, $U(j) - U(i) \in \mathbb{Z}$ is the signed number of *un*-matching tangencies occurring in any regular homotopy from i to j (thus the name U for this invariant) where each such tangency is counted as ± 1 according to its permanent co-orientation and the prescription

$$u(U)\big(H^2_m\big) = u(U)\big(E^2_m\big) = 1, \qquad u(U)\big(E^0_m\big) = -1.$$

Following the definition of U we define $\eta : K \to \mathbb{Z}$ on generators as follows: $\eta(h_m) = 1$ and $\eta(t_m) = 0$ for all m. Then $u(U) = u(\eta \circ f^K)$ and so (up to choice of constants) $U = \eta \circ f^K$. So from our formula for f^K we get an explicit formula for U:

Proposition 6.2.

$$U(i) = \sum_{m\in\mathbb{Z}} \chi(U_m) \left\lfloor \frac{m+2}{2} \right\rfloor - \sum_{m\in\mathbb{Z}} \frac{1}{2} m N_m.$$

Again we may characterize the choice of constants by saying that $U(j) = 2 - g$ for j of Lemma 5.2.

We denote $U(i, j) = U(j) - U(i)$. For two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$ we would like to compute $U(e, e')$. For $e : F \to \mathbb{R}^3$ an embedding let $c(e) \in \mathbb{Z}$ be the degree of the points in the compact side of $i(F)$ in $\mathbb{R}^3$, so $c(e) = \pm 1$. By Propositions 6.1 and 6.2 we have $U(e) = (2 - g) + (1 - g)\lfloor (c(e) + 2)/2 \rfloor$ and so

$$U(e, e') = U(e') - U(e) = (1 - g)\left(\left\lfloor \frac{c(e') + 2}{2} \right\rfloor - \left\lfloor \frac{c(e) + 2}{2} \right\rfloor\right) = (1 - g)\epsilon(e, e')$$

where $\epsilon(e, e')$ is 0 if $c(e) = c(e')$, is 1 if $c(e) = -1, c(e') = 1$ and is -1 if $c(e) = 1$, $c(e') = -1$.

Now for $i \in I_0$ and $h : F \to F$ a diffeomorphism such that i and $i \circ h$ are regularly homotopic, we would like to compute $U(i, i \circ h)$. If h is orientation preserving then from the formula for $U(i)$ it is clear that $U(i) = U(i \circ h)$ and so $U(i, i \circ h) = 0$. Now let $h : F \to F$

be orientation reversing. If $p \in \mathbb{R}^3 - i(F)$ then $d_p(i \circ h) = -d_p(i)$ and if $p \in \mathbb{R}^3$ is a triple point of i then $d_p(i \circ h) = 3 - d_p(i)$ and so we get:

$$U(i \circ h) - U(i) = \sum_m \chi\big(U_m(i)\big)\left(\left\lfloor \frac{-m+2}{2} \right\rfloor - \left\lfloor \frac{m+2}{2} \right\rfloor\right) - \sum_m \frac{1}{2}(3 - m - m)N_m(i).$$

Using Propositions 6.1 and 6.2 and the fact that $\lfloor(-m+2)/2\rfloor - \lfloor(m+2)/2\rfloor = -2\lfloor(m+2)/2\rfloor + k(m)$ where $k(m)$ is 2 for m even and 1 for m odd, we get:

$$U(i, i \circ h) = (1-g) + 2\big(2 - g - U(i)\big).$$

Note that the $U(i)$ appearing here on the right, stands for our specific formula for the invariant U, and not for the abstract invariant which is defined only up to a constant. This equality for h orientation reversing can be interpreted as $U(i, i \circ h) = U(j, j \circ h) + 2U(i, j)$ for j of Lemma 5.2, hinting on another way for proving the equality.

Let $\widehat{U} : I_0 \to \mathbb{Z}/2$ be the mod 2 reduction of U. The reduction mod 2 of the above results reads as follows:

Proposition 6.3.

(1) *For embeddings $e, e' : F \to \mathbb{R}^3$, $\widehat{U}(e, e') = (1-g)\hat{\epsilon}(e, e')$ where $\hat{\epsilon}(e, e') \in \mathbb{Z}/2$ is* 0 *if $c(e) = c(e')$ and is* 1 *if $c(e) \neq c(e')$.*
(2) *For $i \in I_0$ and $h : F \to F$ a diffeomorphism such that i and $i \circ h$ are regularly homotopic, $\widehat{U}(i, i \circ h) = (1-g)\epsilon(h)$ where $\epsilon(h) \in \mathbb{Z}/2$ is* 0 *if h is orientation preserving and is* 1 *if h is orientation reversing.*

7. Proof of the formula for M

For $i \in I_0$ and $h : F \to F$ a diffeomorphism such that i and $i \circ h$ are regularly homotopic, let $M'(i, i \circ h)$ denote our proposed formula for $M(i, i \circ h)$ (Formula 4.2). So we must show that indeed $M(i, i \circ h) = M'(i, i \circ h)$. Similarly, for regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$, let $M'(e, e')$ denote the proposed value for $M(e, e')$ (Formula 4.3), so we must show $M(e, e') = M'(e, e')$.

In [3] it is shown that $Q(i, i \circ h) = M'(i, i \circ h) + (1-g)\epsilon(h)$ and $Q(e, e') = M'(e, e') + (1-g)\hat{\epsilon}(e, e')$. In view of Proposition 6.3, this means $Q(i, i \circ h) = M'(i, i \circ h) + \widehat{U}(i, i \circ h)$ and $Q(e, e') = M'(e, e') + \widehat{U}(e, e')$. So showing $M = M'$ in these two settings is equivalent to showing $Q = M + \widehat{U}$ in these settings, which means that the number mod 2 of quadruple points occurring in any regular homotopy between such two immersions or embeddings, is equal to the number mod 2 of all tangencies occurring (matching and un-matching). So, it remains to prove the following:

Proposition 7.1. *Let $i, j \in I_0$ such that either there is a diffeomorphism $h : F \to F$ such that $j = i \circ h$ or i, j are both embeddings. Then in any regular homotopy between i and j, the number* mod 2 *of quadruple points occurring, is equal to the number* mod 2 *of tangencies occurring.*

Proof. For a closed 3-manifold N and a stable immersion $f: N \to \mathbb{R}^4$, there is defined a closed surface S_f and an immersion $g: S_f \to \mathbb{R}^4$ such that the image $g(S_f) \subseteq \mathbb{R}^4$ is precisely the multiple set of f. It is shown in [1] that the number mod 2 of quadruple points of f is equal to $\chi(S_f)$ mod 2.

Now let $i, j: F \to \mathbb{R}^3$ be as in the assumption of this proposition and let $H_t: F \to \mathbb{R}^3$, $0 \leqslant t \leqslant 1$, be a regular homotopy with $H_0 = i$, $H_1 = j$. We define an immersion $f: F \times [0,1] \to \mathbb{R}^3 \times [0,1]$ by $f(x,t) = (H_t(x), t)$.

In case i, j are embeddings, let A_0, A_1 be the compact regions in $\mathbb{R}^3$ bounded by i, j, respectively. Construct a closed 3-manifold N from $F \times [0,1]$, A_0, A_1 by gluing ∂A_0, ∂A_1 to $F \times \{0\}$, $F \times \{1\}$ in the manner such that f and the inclusions $A_0, A_1 \subseteq \mathbb{R}^3$ induce a (non-smooth) map $N \to \mathbb{R}^3 \times [0,1] \subseteq \mathbb{R}^4$. By slightly pushing the interiors of A_0, A_1 into $\mathbb{R}^3 \times (-\infty, 0]$, $\mathbb{R}^3 \times [1, \infty)$, respectively, so as to smooth the induced map near the gluing, we obtain an immersion $\bar{f}: N \to \mathbb{R}^4$ with self intersection being precisely the original self intersection of $F \times [0,1]$. The projection $\mathbb{R}^3 \times [0,1] \to [0,1]$ induces a Morse function on $S_{\bar{f}}$ with singularities precisely wherever a tangency CE occurs in the regular homotopy H_t, and so by the Morse formula $\chi(S_{\bar{f}})$ is equal mod 2 to the number of tangencies. By [1] then, the number mod 2 of quadruple points of H_t which is the number mod 2 of quadruple points of $\bar{f}$ is equal to the number mod 2 of tangencies.

In case $j = i \circ h$, let N be the 3-manifold obtained from $F \times [0,1]$ by gluing its two boundary components to each other via h so that there is induced an immersion $\bar{f}: N \to \mathbb{R}^3 \times S^1$. Composing $\bar{f}$ with an embedding of $\mathbb{R}^3 \times S^1$ in $\mathbb{R}^4$, we see again that the number of quadruple points of H_t is equal mod 2 to $\chi(S_{\bar{f}})$ which is equal mod 2 to the number of tangencies of H_t. Note that we apply the Morse formula here to a Morse function with values in S^1. Indeed the Morse formula is valid for such a function: Let M be any closed manifold, and $h: M \to S^1$ a Morse function. Let $s \in S^1$ be a regular value, and let $\widehat{M}$ be the manifold with boundary obtained from M by cutting it along $h^{-1}(s)$. Then h induces a Morse function $\hat{h}: \widehat{M} \to [0,1]$. By the usual Morse formula, $\chi(\widehat{M}) = \chi(h^{-1}(s)) + \Sigma$ where Σ is the sum of indices of the singular points of $\hat{h}$. Now, gluing M back again reduces χ by $\chi(h^{-1}(s))$, giving the required result. $\square$

We remark that one can prove Formulae 4.2 and 4.3 directly, without resorting to the result of [1], by going along the lines of [3]. Proposition 7.1 would then be obtained as a corollary.

Appendix A

We conclude with some remarks relating the present work to previous work. We first note that our explicit formula for f^K serves as an independent proof of the existence of f^K, i.e., the existence of an order 1 invariant f with the prescribed $u(f)$. This is in contrast with our formulae for M, and the analogous formulae for Q appearing in [3], which rely on the existence. This fact may be used to simplify the proof of the existence of f^U. Indeed since the existence of f^K is now established by explicit construction, it remains to show the existence of M and Q. This may be done for M as has already been done for Q in [2], where the argument is somewhat simpler than that appearing in [4] for the whole f^U.

Both the simpler argument appearing in [2] and the full argument appearing in [4] use an odd power of the generator of the fundamental group of the space of immersions of S^2 in $\mathbb{R}^3$ which fix a disc $D \subseteq S^2$, this element being constructed in [2]. Though only an odd power of the generator was needed for our purposes, we take this opportunity to show that from the construction appearing in [2], an actual generator may also be obtained.

Let $D, U \subseteq S^2$ be the unit discs in the two standard coordinate charts for S^2 as the Riemann sphere, so $D \cup U = S^2$, $D \cap U = \partial D = \partial U$, and on a neighborhood of ∂U, the change of coordinates function is $z \mapsto \frac{1}{z}$, i.e., $(x, y) \mapsto \frac{1}{x^2+y^2}(x, -y)$. Let $i_0 : S^2 \to \mathbb{R}^3$ be an immersion such that $i_0|_D$, in the coordinates of D, is the natural inclusion $(x, y) \mapsto (x, y, 0)$. Let E be the space of all immersions $i : S^2 \to \mathbb{R}^3$ such that $i|_D = i_0|_D$, then $\pi_1(E) = \mathbb{Z}$. We define an involution $\tau : E \to E$ using the coordinates of U as follows: For $i \in E$, $\tau(i)(x, y) = -i(-x, -y)$. Now let $h(t)$ be any path in E from i_0 to $\tau(i_0)$, then it is shown in [2, Proposition 2.1] that the loop $h * (\tau \circ h)$ (where $*$ denotes concatenation) is an odd power of the generator of $\pi_1(E, i_0)$. We now improve this result by showing that there exists a choice of h, for which this loop represents an actual generator. Assume $h * (\tau \circ h) = c^{2k+1}$ where $c \in \pi_1(E, i_0)$ is a generator, and define $g = h * (\tau \circ c^{-k})$, then g is a path from i_0 to $\tau(i_0)$, for which we show $g * (\tau \circ g) = c$. Indeed we have $g * (\tau \circ g) = h * (\tau \circ c^{-k}) * (\tau \circ h) * c^{-k} = (h * (\tau \circ c^{-k}) * h^{-1}) * (h * (\tau \circ h)) * c^{-k} = (h * (\tau \circ c^{-k}) * h^{-1}) * c^{k+1}$ and so it remains to show that $h * (\tau \circ c^{-k}) * h^{-1} = c^{-k}$. The automorphism φ of $\pi_1(E, i_0)$ defined by $\varphi(x) = h * (\tau \circ x) * h^{-1}$ is either Id or $x \mapsto x^{-1}$ as these are the only automorphisms of an infinite cyclic group. So to identify φ it is enough to check its value on one non-trivial element. We take $x = h * (\tau \circ h)$ $(= c^{2k+1} \neq 1)$, getting $\varphi(x) = h * ((\tau \circ h) * h) * h^{-1} = h * (\tau \circ h) = x$ and so $\varphi = \mathrm{Id}$ and so indeed $h * (\tau \circ c^{-k}) * h^{-1} = c^{-k}$.

Finally, we would like to relate to [4, Proposition 6.1]. If e denotes the inclusion $S^2 \subseteq \mathbb{R}^3$ and e' is the embedding obtained from e by a reflection, then by Formula 4.1 for every n, $f_n^H(e') - f_n^H(e) = -\delta_{n,1}$ and $f_n^T(e') - f_n^T(e) = 0$. With the notation of Section 3 we thus obtain that any regular homotopy from e to e' (i.e., any eversion of the sphere) satisfies, for every n:

$$N(H_n^2) + N(E_n^2) - N(E_n^0) + N(T_n^0) + N(T_n^3) - N(T_{n+1}^0) - N(T_{n+1}^3) + N(Q_n^4) - 2N(Q_{n+1}^4) + N(Q_{n+2}^4) = -\delta_{n,1}$$

and

$$N(T_n) + N(Q_n^3) - N(Q_{n+1}^3) + 2N(Q_n^4) - 2N(Q_{n+1}^4) = 0.$$

As one may verify directly, the above equations are equivalent to the equations appearing in [4, Proposition 6.1], which used a different choice of order 1 invariants, which in turn followed from a different choice of generators for K. But note that whereas the right-hand side of the equalities in [4] was obtained by direct inspection of a specific eversion of the sphere, in the present work it has been provided by our explicit formulae for the invariants.

References

[1] M.A. Asadi-Golmankhaneh, P.J. Eccles, Double point self-intersection surfaces of immersions, Geom. Topol. 4 (2000) 149–170.

[2] T. Nowik, Quadruple points of regular homotopies of surfaces in 3-manifolds, Topology 39 (2000) 1069–1088.

[3] T. Nowik, Automorphisms and embeddings of surfaces and quadruple points of regular homotopies, J. Differential Geom. 58 (2001) 421–455.

[4] T. Nowik, Order one invariants of immersions of surfaces into 3-space, Math. Ann. 328 (2004) 261–283.

[5] T. Nowik, Higher order invariants of immersions of surfaces into 3-space, Pacific J. Math., in press.

[6] U. Pinkall, Regular homotopy classes of immersed surfaces, Topology 24 (4) (1985) 421–434.