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Intersection of surfaces in 3-manifolds

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Abstract

Given a pair of incompressible surfaces F and S in an irreducible 3-manifold M, we define a directed graph T which expresses the way F may be isotoped with respect to S. We study the properties of T. We use our results about T to study properties of the intersection between F and S. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Given two incompressible orientable surfaces F and S in an irreducible orientable 3-manifold M, we investigate how F may be isotoped relative to S.

We define I_0 to be the set of all embeddings of F in M, that are isotopic to the inclusion, and are transversal with respect to S. Since we are dealing with embeddings of F into the pair (M, S), the following is the natural equivalence relation on I_0: Two embeddings in I_0 are equivalent if there is an isotopy between them of the form $K_t \circ i \circ h_t$ where $i: F \to M$ is an embedding and $h_t: F \to F$, $K_t: (M, S) \to (M, S)$ are isotopies. This is equivalent to the following: Two embeddings in I_0 are equivalent if we can move from one to the other through embeddings that are all transversal with respect to S, i.e., within I_0 itself. The set of equivalence classes will be called I.

We orient M and F. This determines a preferred side of $i(F)$ in M for any embedding $i: F \to M$. An isotopy h_t will be called a directed isotopy, if at any time t, all points of F are moving into the preferred side of $h_t(F)$. We give I the structure of a directed

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graph as follows: For any $A, B \in I$ there will be a directed edge $A \to B$ if there is a directed isotopy h_t with $h_0 \in A$, $h_1 \in B$.

Our I includes many (equivalence classes of) embeddings for which the intersection between F and S says nothing about the relation between F and S in M. For example, one can always add a circle of intersection by simply changing the embedding of a small disc of F, as to intersect S. We would like to exclude this and more: We define $T_0 \subseteq I_0$ (respectively $T \subseteq I$) to be the set of all embeddings (respectively equivalence classes of embeddings) in which there are no product regions between F and S. (It is known that least area surfaces satisfy this property.) T inherits the structure of a directed graph from I. The directed graph T will be our subject of interest.

Our main results about T are:

(1) Though the directed graph structure of T was induced upon it from I, T is selfcontained in the following sense: If $i, j \in T_0$ and there is a directed isotopy h_t from i to j, then there is a directed isotopy g_t from i to j satisfying $g_t \in T_0$ for all t where the intersection of $g_t(F)$ and S is transverse (Theorem 6.1).

(2) T is a connected graph (Theorem 6.3). Furthermore, T is isometrically embedded into I with respect to the standard metric on connected graphs (Theorem 6.5). The geometric meaning of this is as follows: Given an isotopy between two embeddings $i, j \in T_0$, we may approximate it by an isotopy h_t which is a concatenation $h_t^1 * h_t^2 * \cdots * h_t^k$ of isotopies h_t^i which are alternately directed and anti-directed. (Anti-directed meaning that F is moving into it's nonpreferred side.) Theorems 6.1, 6.3, and 6.5 combined together will say, that we may replace h_t by an isotopy $g_t = g_t^1 * g_t^2 * \cdots * g_t^l$ such that $g_t \in T_0$ for all t where the intersection of $g_t(F)$ and S is transverse, and $l \leqslant k$.

(3) If M is not "circular" (Definition 7.1) then T is a "graded graph" (Theorem 7.13). This means geometrically, that for any g_t as above (which is sufficiently generic), if we count the number of times g_t passes from one equivalence class in T to another, each such passage being counted as 1 or -1 according to whether we are at a directed or anti-directed portion of g_t, then this number depends only on the pair of embeddings i, j and not on the choice of g_t. If M *is* circular, then we show T is a complete graph (Theorem 7.16).

We then use T to study properties of the intersection between F and S:

In [1, Theorems 6.6 and 6.7] it is shown that if either F or S is a torus then for any Riemannian metric on M, any pair of least area surfaces homotopic to F and S must intersect transversely, and the number of intersection curves between them will be the minimal possible for the homotopy classes of F and S. We translate these two properties of a pair F, S into our setting, and ask what is the most general assumption on the pair F, S (rather than assuming that one of them is a torus), that will guarantee each one of these two special properties. We show that the two properties are both equivalent to the property that F and S may be isotoped to satisfy the one line property. We show this by proving that each one of these three properties is equivalent to T having at most one element (Theorem 8.1). The distinction between one and zero elements is given by Theorem 2.12 stating that $T = \emptyset$ iff F is isotopic to S. We will also show that indeed if either F or S is a torus, then T has at most one element (Theorem 6.7).

In [1] some further minimality properties are shown for the intersection of two least area surfaces (Theorems 6.3 and 6.2). We show that the corresponding results in our setting, are immediate consequences of the connectivity of T (Theorem 8.11 and Corollary 8.12).

The structure of the paper is as follows:

In the remainder of this section, we give the basic definitions and assumptions of the paper. In Section 2 we study the basic properties of the surface $L = H^{-1}(S)$ where $H : F \times [0,1] \to M$ is a directed isotopy. What we will need for our purposes is that $L \subseteq F \times [0,1]$ will have no extremum points (Definition 2.24). And so in Sections 3–5 we describe ways of changing a directed isotopy H so as to avoid extremum points. In Section 6 we show the selfcontainedness and connectedness of T. In Section 7 we define a "graded graph" and show that T is a graded graph, except for when M is circular, in which case T is a complete graph. In Section 8 we prove all the geometric applications mentioned above. In Section 9 we investigate the connection between T and the graph obtained by reversing the roles of F and S.

In any result titled "Theorem" we will always give reference to the definitions of the terms that appear, except for the definitions of this section.

Assumptions and notation 1.1.

(i) M will always denote an orientable irreducible closed 3-manifold.

(ii) F and S will be two orientable incompressible closed surfaces in M.

(iii) If $H(x,t)$ is a map then for a fixed t the map H_t will be defined by $H_t(x) = H(x,t)$.

(iv) If $H : F \times [0,1] \to M$ is an isotopy, we will always assume that there are only finitely many t's where the intersection of $H_t(F)$ with S is nontransversal and in those t's we always assume the nontransversality is of generic type (i.e., there is *only one* point where the intersection is not transversal and in a neighborhood of that point the intersection is like between $z = x^2 - y^2$ and $z = 0$, or between $z = x^2 + y^2$ and $z = 0$ in $\mathbb{R}^3$). The points in $F \times [0,1]$ and the times where this occurs, will be called the singular points and singular times of H.

(v) When needed we will also assume generic relation between the sets $H_{t_i}(F) \cap S$ of the different singular times t_i. (In particular, for a singular point a at time t_i we will have $H(a) \notin H_{t_j}(F) \cap S$ for the singular times $t_j \neq t_i$.)
Assumptions (iv) and (v) constitute no restriction, since arbitrarily close to any isotopy there is an isotopy satisfying these assumptions.

(vi) If G is a surface (G will be F or S), then we will denote the two projections of $G \times [0,1]$ by: $\pi_G : G \times [0,1] \to G$ and $q : G \times [0,1] \to [0,1]$.

Definitions 1.2.

(i) I_0 will denote the set of all embeddings $i : F \to M$ that are isotopic to the inclusion map of F, and are transversal with respect to S.

(ii) We define an equivalence relation $\sim_S$ on I_0 as follows: $i_1 \sim_S i_2$ if there is an isotopy from i_1 to i_2 via embeddings that are in I_0. $[i]$ will denote the equivalence class of i and I will denote the set of equivalence classes.

(iii) For $i \in I_0$ we will say there is a product region between $i(F)$ and S if there is a surface K (with or without boundary) and an embedding $h: K \times [0,1] \to M$ such that $h(K \times \{0\}) \subseteq i(F)$ and $h(K \times \{1\} \cup \partial K \times [0,1]) \subseteq S$.

(iv) $T_0 \subseteq I_0$ (respectively $T \subseteq I$) will be the set of all i (respectively $[i]$) such that there is no product region between $i(F)$ and S.

(v) We define a directed isotopy as follows: We choose once and for all, an orientation for F and for M and this induces a choice of a preferred side of $i(F)$ for every embedding $i: F \to M$. A directed isotopy will be an isotopy $H: F \times [0,1] \to M$ such that $\partial H/\partial t(x,t) \neq 0$ and points to the preferred side of $H_t(F)$ for all $x \in F$, $t \in [0,1]$.

(vi) We give I the structure of a directed graph as follows: Let $A, B \in I$, there will be a directed edge from A to B if there are $i_1 \in A$, $i_2 \in B$ and a directed isotopy from i_1 to i_2. T then inherits the structure of a directed graph from I. Note that for any $A \in I$ there is a directed edge from A to itself (represented by a directed isotopy moving F only slightly into the preferred side).

2. Foundations

Lemma 2.1. *Let $H: F \times [0,1] \to M$ be an isotopy. Then H is directed iff H is an orientation preserving local diffeomorphism.*

Proof. $\partial H/\partial t(x,t) \neq 0$ and pointing to the preferred side of $H_t(F)$ is equivalent to $dH(x,t)$ being nonsingular and orientation preserving. □

Lemma 2.2. *Let $H: F \times [0,1] \to M$ be a directed isotopy. If $H_0(F)$ separates M then H is an embedding.*

Proof. Assume not. Let $t_0 = \inf\{t \in [0,1]\colon H|_{F\times[0,t]}$ is not 1–1$\}$. Let (x_n, t_n), (x'_n, t'_n) be two sequences such that $(x_n, t_n) \neq (x'_n, t'_n)$, $H(x_n, t_n) = H(x'_n, t'_n)$ and $t_n, t'_n \leqslant t_0 + 1/n$. By compactness we may assume the two sequences converge to (x,t) and (x',t'), respectively, and we have $H(x,t) = H(x',t')$. H is a local diffeomorphism so we cannot have $(x,t) = (x',t')$. Since H is an isotopy we can also not have $t = t'$. So at least one is $< t_0$. Say $t < t_0$. Now $t' < t_0$ will contradict the definition of t_0. So $t' = t_0$. If $0 < t < t_0$ then a small neighborhood of (x,t) will be mapped onto a neighborhood of $H(x,t)$, but there will also be points (x'', t'') close to (x',t') with $t'' < t'$, mapped into this neighborhood of $H(x,t)$ contradicting again the definition of t_0. So the only case left is $t = 0$ and the images of half ball neighborhoods in $F \times [0,t_0]$ of (x,t), (x',t') intersect only in points coming from $\partial(F \times [0,t_0])$, among these, the point $H(x,0) = H(x',t_0)$. This will give us a loop in M intersecting $H_0(F)$ transversally in one point, contradicting our assumption that $H_0(F)$ separates M. □

Corollary 2.3. *Let $H : F \times [0,1] \to M$ be a directed isotopy. If M' is a covering of M such that H lifts to a map $H' : F \times [0,1] \to M'$ with $H'_0(F)$ separating M', then H' is an embedding.*

Definition 2.4. We define the homomorphism $\psi : \pi_1(M) \to \mathbb{Z}$ by the intersection number with F. We denote by M^F the covering space of M related to $\ker\psi$. (If F separates M then $M^F = M$. If F does not separate M then M^F is the covering obtained by taking a $\mathbb{Z}$ indexed collection of copies of M, cutting each one along F and gluing one side of F in the ith copy to the other side of F in the $(i+1)$st copy.) We denote the covering map by $p : M^F \to M$.

Note that F lifts to M^F and is separating there, and so Corollary 2.3 applies to M^F.

Given a directed isotopy $H : F \times [0,1] \to M$, we can define $H'' : F \times [0,1] \to M \times [0,1]$ by $H''(x,t) = (H(x,t),t)$. H'' is an embedding. We also have $S \times [0,1] \subseteq M \times [0,1]$. We now define $L = L(H) = H''(F \times [0,1]) \cap S \times [0,1]$. We will think of L as contained in $F \times [0,1]$ (via H''), in $S \times [0,1]$, and due to Corollary 2.3, also in M^F. When necessary we will give L a subscript to say where we consider it to be contained (e.g., $L_{F\times[0,1]}$, L_{M^F}).

Remark 2.5.

(a) When identifying $L_{F\times[0,1]}$ and $L_{S\times[0,1]}$, the maps $H|_{L_{F\times[0,1]}} : L_{F\times[0,1]} \to S \subseteq M$ and $\pi_S|_{L_{S\times[0,1]}} : L_{S\times[0,1]} \to S$ become the same map. So, anything we prove about one of these maps, will be true for the other.

(b) The restriction to $L_{F\times[0,1]}$ of the projection $F \times [0,1] \to [0,1]$ coincides with the restriction to $L_{S\times[0,1]}$ of the projection $S \times [0,1] \to [0,1]$. And so we have a well defined function $q : L \to [0,1]$.

Lemma 2.6.

(a) $H|_{L_{F\times[0,1]}} : L_{F\times[0,1]} \to S \subseteq M$ *is a local diffeomorphism.*

(b) $\pi_S|_{L_{S\times[0,1]}} : L_{S\times[0,1]} \to S$ *is a local diffeomorphism.*

Proof. (a) and (b) are equivalent by Remark 2.5.

(a) is true since H is a local diffeomorphism, and $L_{F\times[0,1]}$ is simply $H^{-1}(S)$. □

Corollary 2.7. *L is an orientable surface.*

Our definition above of product region (Definition 1.2(iii)), is very inclusive. For example, if c is a circle of ∂K, then it is allowed that the product region $h(K \times [0,1])$ will contain a neighborhood in $i(F)$ and in S of $h(c \times \{0\})$. (The situation here is that $h(K \times [0,1])$ contains three quarters of a neighborhood in M of $h(c \times \{0\})$, instead of just one quarter.) See Fig. 1.

We will now show that whenever there is any product region, then there is also a convenient one:

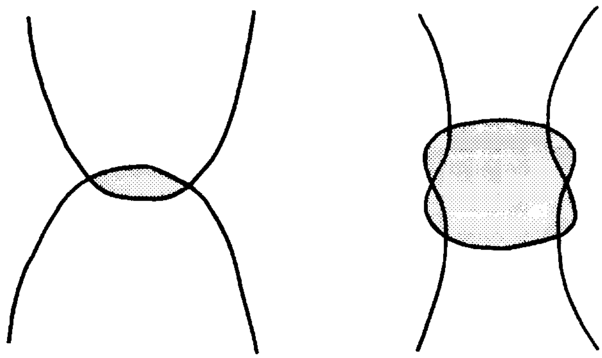

Fig. 1.

Lemma 2.8. *Assume there is a product region between $i(F)$ and S. Then there is a product region $h: K \times [0,1] \to M$ between $i(F)$ and S satisfying one of the following:*

(1) $h(K \times [0,1]) \cap (i(F) \cup S) = \partial h(K \times [0,1])$.

(2) *K is a disc and* $h(K \times [0,1]) \cap i(F) = h(K \times \{0\})$.

(In particular, the situation described before the lemma does not occur.)

Proof. Assume there is a null-homotopic circle of intersection between $i(F)$ and S. S is incompressible, so this circle bounds a disc in S. Take a minimal such disc D. ∂D bounds a disc D' in $i(F)$ since $i(F)$ is incompressible too. Since D was minimal, $D \cup D'$ is an embedded sphere. It bounds a ball B in M. We parameterize B as $K \times [0,1]$ with K a disc, $K \times \{0\} = D'$ and $K \times \{1\} \cup \partial K \times [0,1] = D$. We have then that $K \times [0,1] \cap i(F) \supseteq K \times \{0\}$. If the inclusion was strict, then since $\operatorname{int} D \cap i(F) = \emptyset$, we would have $i(F) \subseteq B$. So $K \times [0,1] \cap i(F) = K \times \{0\}$, and (2) is satisfied.

So now assume there are no null-homotopic circles of intersection. It follows that any component A of $S \cap h(K \times [0,1])$ is incompressible in $h(K \times [0,1])$. Since $\partial A \subseteq h(K \times \{0\})$, then by Proposition 3.1 of [4], A is parallel to $h(K \times \{0\})$. Take an A such that the region U bounded by A and $h(K \times \{0\})$ contains no other component of $S \cap h(K \times [0,1])$. We replace $h(K \times [0,1])$ by U. We now do the same thing with $i(F) \cap U$, to get (1). □

Remark 2.9. In the proof above, in the case there is a null-homotopic circle of intersection, we did not use the prior assumption that there is a product region. And so we have shown that whenever there is a null-homotopic circle of intersection, there is a product region.

Lemma 2.10. *$L_{F\times[0,1]}$ is incompressible in $F \times [0,1]$ iff $L_{S\times[0,1]}$ is incompressible in $S \times [0,1]$.*

Proof. Since H is π_1 injective, and S is incompressible in M: $L_{F\times[0,1]}$ is incompressible in $F \times [0,1]$ iff $H|_{L_{F\times[0,1]}}: L_{F\times[0,1]} \to S$ ($\subseteq M$) is π_1 injective on each component.

That is iff $\pi_S|_{L_{S\times[0,1]}} : L_{S\times[0,1]} \to S$ is π_1 injective on each component, iff $L_{S\times[0,1]}$ is incompressible in $S \times [0,1]$. □

Lemma 2.11. *If A is a closed component of $L_{F\times[0,1]}$ then A is incompressible, and thus boundary parallel, in $F \times [0,1]$. Furthermore $H|_A : A \to S \subseteq M$ is a homeomorphism.*

Proof. Look at A in $S \times [0,1]$. $\pi_S|_A$ is a local homeomorphism. In the proof of Proposition 3.1 of [4] we see that a surface $A \subseteq S \times [0,1]$ for which $\pi_S|_A$ is a local homeomorphism, and such that $A \cap S \times \{0\} = \emptyset$, is actually embedded by π_S. So $\pi_S|_A$ is a homeomorphism, and A is incompressible in $S \times [0,1]$. Back to $F \times [0,1]$, this means that $H|_A$ is a homeomorphism onto S, and A is incompressible in $F \times [0,1]$. □

Our first calculation of T will be in the following:

Theorem 2.12. *$T = \emptyset$ iff F is isotopic to S iff there exists a directed isotopy H (in the isotopy class of the inclusion), such that $L(H)$ has a closed component.*

Proof. If $i(F)$ is isotopic to S then by Proposition 5.4 of [4], there must be a product region between $i(F)$ and S, and so if F is isotopic to S, $T = \emptyset$.

If $T = \emptyset$ then there is a product region between F and S, and so there is a product region of a type described in Lemma 2.8. Such a region can be canceled by an isotopy. But we must then still have product regions. We continue until F and S are disjoint. A product region now means F and S are parallel, and so isotopic.

Let H be a directed isotopy in the isotopy class of the inclusion, such that $L(H)$ has a closed component. By Lemma 2.11, H_0 is homotopic to a homeomorphism onto S. And so by Corollary 5.5 of [4] F is isotopic to S.

Now assume F is isotopic to S. There is an embedding i of F isotopic to the inclusion such that S is disjoint from, and parallel to $i(F)$ from the preferred side. Now take a directed isotopy H that moves F across S to the other side. $L(H)$ will be a closed surface in $F \times [0,1]$. □

Definition 2.13. Let $A \subseteq F \times [0,1]$ be a surface. A will be called "lower" (respectively "upper") if $\emptyset \neq \partial A \subseteq F \times \{0\}$ (respectively $F \times \{1\}$).

Definition 2.14. For an upper surface $A \subseteq F \times [0,1]$ we define $m(A) = \min q(A)$.

Definition 2.15. Let $A \subseteq G \times [0,1]$ be a lower or upper surface. (G will be F or S.) We will say that A is *very well embedded* in $G \times [0,1]$ if the projection

$$\pi_G : G \times [0,1] \to G,$$

embeds A into G.

We will say that A is just *well embedded*, if after a level preserving homeomorphism of $G \times [0,1]$, A will become very well embedded. (A level preserving homeomorphism of $G \times [0,1]$ is a homeomorphism where each $G \times \{t\}$ is mapped into itself.)

Lemma 2.16.

(a) *Any upper component A of $L_{S\times[0,1]}$ is very well embedded in $S \times [0,1]$. (Upper component meaning a connected component which is an upper surface.)*

(b) *Any upper component A of $L_{F\times[0,1]}$ is embedded by H.*

Proof. (a) and (b) are equivalent by Remark 2.5 so we show (a). By Lemma 2.6, $\pi_S|_A$ is a local diffeomorphism. Again by the proof of Proposition 3.1 of [4] we know that an upper surface for which the projection is a local diffeomorphism is actually embedded by it. □

Definition 2.17. Let $A \subseteq G \times [0,1]$ be an upper surface. (G will be F or S.) We denote by $U_G(A)$ or $U_{G\times[0,1]}(A)$ or $U_{G\times\{1\}}(A)$, the region bounded by A and $G \times \{1\}$. (This region exists by homological considerations.) And similarly for a lower component. It will be called "A's region".

Remark 2.18. Let $A \subseteq G \times [0,1]$ be a very well embedded upper surface, and let $A' = \pi_G(A)$. Then A may be viewed as the graph of a function $f : A' \to (0,1]$ with $f(\partial A') = 1$.

$U_G(A)$ is then $\{(x,t) \in G \times [0,1]\colon x \in A' \text{ and } f(x) \leqslant t \leqslant 1\}$.

Lemma 2.19. *If $A \neq B$ are two upper components of $L_{F\times[0,1]}$ then $H(A)$ and $H(B)$ are either disjoint or one is contained in the interior of the other.*

Proof. Let $a \in A$, $b \in B$ and $H(a) = H(b)$. Now look at A and B in $S \times [0,1]$. Then we have $\pi_S(a) = \pi_S(b)$ and so we must have $q(a) \neq q(b)$ and assume $q(b) > q(a)$. Then $b \in U_S(A)$ and so $B \subseteq U_S(A)$. It follows that $\pi_S(B) \subseteq \operatorname{int} \pi_S(A)$ and so (back to $F \times [0,1]$) $H(B) \subseteq \operatorname{int} H(A)$. □

The above considerations make the following lemma clear:

Lemma 2.20. *If $A \neq B$ are two upper components of L such that $H(A) \cap H(B) \neq \emptyset$ then the following are equivalent:*

(1) *$B \subseteq U_S(A)$. (This is in $S \times [0,1]$.)*

(2) *$H(B) \subseteq H(A)$. (This and the following are in $F \times [0,1]$.)*

(3) *$H(B) \subseteq H(\operatorname{int} A)$.*

(4) *$m(B) > m(A)$.*

We now characterize the case of an A having no such B.

Lemma 2.21. *For an upper component A of L, the following are equivalent:*

(1) *$\operatorname{int} U_S(A) \cap L = \emptyset$. (This is in $S \times [0,1]$.)*

(2) *There is no upper component $B \neq A$ with $H(B) \subseteq H(A)$. (This and the following are in $F \times [0,1]$.)*

(3) *$H(\operatorname{int} A) \cap H(F \times \{1\}) = \emptyset$.*

(4) $H(\text{int}(A \cap F \times [0,t])) \cap H(F \times \{t\}) = \emptyset$ *for all* $t \in [0,1]$.
(5) $F \times \{1\} \cup U_F(A)$ *is embedded by* H.

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 2.20. Now if in $S \times [0,1]$, there is a component B of L in $\text{int}\, U_S(A)$, then B is an upper component (by Lemma 2.11 and Remark 2.5 or directly from the proof of Lemma 2.11). So if there is a $B \subseteq U_S(A)$, then $\emptyset \neq \partial B \subseteq \text{int}(\pi_S(A) \times \{1\})$, and so $\text{int}\, U_S(A) \cap L \neq \emptyset$ iff $\text{int}(\pi_S(A) \times \{1\}) \cap L \neq \emptyset$. But the latter means that at time 1, there is intersection (in M) between F and $\text{int}\, \pi_S(A) \subseteq S \subseteq M$. This is equivalent to $H(\text{int}\, A) \cap H(F \times \{1\}) \neq \emptyset$. So we have (1) $\Leftrightarrow$ (3).

But now using (1) $\Leftrightarrow$ (3) for $F \times [0,t]$, the claim (3) $\Leftrightarrow$ (4) becomes equivalent to: $\text{int}\, U_S(A) \cap L = \emptyset$ iff $(\text{int}\, U_S(A) \cap S \times [0,t]) \cap (L \cap S \times [0,t]) = \emptyset$ for all $t \in [0,1]$, which is obvious.

Now since (5) $\Rightarrow$ (3) is trivial, it remains to show (1)–(4) $\Rightarrow$ (5):

Let $U = U_F(A)$ and $A' = U \cap F \times \{1\}$, then $\partial U = A \cup A'$. We first show U is embedded by H: Let $a \in A$ and assume $b \in U$, $b \neq a$ and $H(a) = H(b)$. Then $b \notin A$ (Lemma 2.16(b)). Let B be the component of b in L, then $B \subseteq U$. It follows that B is an upper component with $H(B) \subseteq H(A)$, since $H(A) \cap H(B) \neq \emptyset$ and $m(B) > m(A)$. (B cannot be closed by Lemma 2.11.)

On the other hand, Let $a' \in A'$, and assume $b \in U$, $b \neq a'$ and $H(a') = H(b)$. Think of $F \times [0,1]$ as contained in M^F via a lifting of H. Then we must have a covering translation τ of M^F bringing a' to b. $\tau(F \times \{1\}) \cap A' = \emptyset$ so $\tau(F \times \{1\})$ must intersect $\text{int}\, A$, and so $H(F \times \{1\}) \cap H(\text{int}\, A) \neq \emptyset$. (We cannot have $\tau(F \times \{1\}) \subseteq \text{int}\, U$ since U is compact with connected boundary, and $\tau(F \times \{1\})$ separates M^F into two noncompact pieces.)

We have shown that for every $a \in \partial U = A \cup A'$ there is no $b \neq a$ in U with $H(a) = H(b)$. Since H is a local homeomorphism, it follows that U is embedded by H. (Take $\{a \in U\colon$ there is $b \in U,\ b \neq a,\ H(a) = H(b)\}$. It is closed and open and thus empty.)

Now $H(F \times \{1\} - A') \cap H(\partial U) = \emptyset$, so $H(F \times \{1\} - A') \cap H(U) = \emptyset$, and so $F \times \{1\} \cup U$ is embedded by H. $\square$

Remark 2.22. Lemmas 2.16–2.21 become trivial when F separates M, by Lemma 2.2.

Lemma 2.23. *Let* $H(x,t)$ *be a directed isotopy with* $H_0, H_1 \in T_0$. *Let* $L = L(H)$. *Then:*

(a) L *is incompressible in* $F \times [0,1]$.
(b) *All components of* L *intersect both* $F \times \{0\}$ *and* $F \times \{1\}$.

Proof. (a) Let $H' : F \times [0,1] \to M^F$ be a lifting of H. Then L_{M^F} is just $H'(F \times [0,1]) \cap p^{-1}(S)$. Since $p^{-1}(S)$ is incompressible in M^F, it is enough to show that there is no circle in $\partial H'(F \times [0,1]) \cap p^{-1}(S)$ that bounds a disc in $p^{-1}(S)$. But that would give a null-homotopic circle of intersection between either $H_0(F)$ or $H_1(F)$ and S, contradicting $H_0, H_1 \in T_0$, by Remark 2.9.

(b) By Theorem 2.12, there are no closed components. So assume there are upper or lower components. Take such a component A, say upper, with maximal $m(A)$. And so by Lemma 2.20, there is no upper component $B \neq A$ with $H(B) \subseteq H(A)$, and so by Lemma 2.21, $U = U_F(A)$ is embedded by H. This region U is a product region by Proposition 3.1 of [4] since A is incompressible and upper in $F \times [0,1]$. And so $H(U)$ is a product region between $H_1(F)$ and S. □

Definition 2.24. Let $H : F \times [0,1] \to M$ be an isotopy. A birth point, is a point in $F \times [0,1]$ where a new circle of intersection between F and S is created. Similarly, a death point is a point where a circle of intersection shrinks to a point and disappears. Birth and death points will be called extremum points.

Definition 2.25. We will say an isotopy H "moves in T" if $H_t \in T_0$ for all nonsingular t.

Lemma 2.26. *Let $H(x,t)$ be a directed isotopy with $H_0, H_1 \in T_0$. Then H moves in T iff H has no extremum points.*

Proof. A bit after (respectively before) a birth (respectively death) point there is a ball region between $H_t(F)$ and S.

Assume now that for some nonsingular $0 < t < 1$ there is a product region U between $H_t(F)$ and S, with $\partial U = A \cup B$, $A \subseteq H_t(F)$, $B \subseteq S$. Let $H' : F \times [0,1] \to M^F$ be a lifting. This induces a lifting of A to M^F. Lift the whole of U accordingly. Call the lifted sets U', A', B'. We claim $B' \subseteq H'(F \times [0,1])$.

If not, then say $H'(F \times \{1\}) \cap B' \neq \emptyset$. $H'(F \times \{1\}) \cap A' = \emptyset$ and $H'(F \times \{1\}) \cap B'$ has no null-homotopic circles (in particular, U' is not a ball). And so $F \times \{1\} \cap U'$ is incompressible in U' and so parallel to B'. The projection back to M gives a product region between $H_1(F)$ and S, contradicting $H_1 \in T_0$.

So $B' \subseteq H'(F \times [0,1])$. Pull it into $F \times [0,1]$ and call it B''. Then $B'' \subseteq L$ and $\partial B'' \subseteq F \times \{t\}$ and so the projection $q : F \times [0,1] \to [0,1]$ must have a local minimum or maximum in $\operatorname{int} B''$, i.e., an extremum point. □

3. Changing L, part 1

In view of Lemma 2.26, and of our goal, which is Theorem 6.1, we describe ways of changing L so as to avoid extremum points.

Lemma 3.1. *Let $H : F \times [0,1] \to M$ be a directed isotopy, and let $L = L(H)$. Let $0 < t_0 < 1$, and let A be a well embedded upper component of $L \cap F \times [0,t_0]$. Assume further that $H(\operatorname{int} A) \cap H(F \times \{t_0\}) = \emptyset$. Let $U = U_{F \times [0,t_0]}(A)$. Then:*

(a) *We can change H in a small neighborhood of U, such that the effect on L will be that A, and any parts of L inside U, will be pushed up into the other side of $F \times \{t_0\}$, without changing the structure of singular points of L (in particular, the number of extremum points will be unchanged).*

(b) *If furthermore* t_0 *is a singular time, and the singular point is a point on the boundary of* A *that connects* A *to another component* B *of* $L \cap F \times [0, t_0]$, *with* $B \not\subseteq U$, *then we may actually reduce the number of extremum points.*

Proof. (a) By a level preserving change of coordinates in $F \times [0, t_0]$ we may assume that A is very well embedded, and by Lemma 2.21, $F \times \{t_0\} \cup U$ is embedded by H.

Let V_1 be a product neighborhood of A in $F \times [0, t_0]$ such that $F \times \{t_0\} \cup U \cup V_1$ is still embedded by H. Let $U_1 = U \cup V_1$. By Lemma 2.21, for every $t \in [0, t_0]$, $H(F \times \{t\} - U_1) \cap H(A \cap F \times [0, t]) = \emptyset$. By compactness there is a product neighborhood $V_2 \subseteq V_1$ of A in $F \times [0, t_0]$ such that $H(F \times \{t\} - U_1) \cap H(V_2 \cap F \times [0, t]) = \emptyset$, for every $t \in [0, t_0]$. We choose V_2 such that also $A' = \mathrm{Cl}(\partial V_2 - (F \times \{t_0\} \cup U))$ is very well embedded. Let $U_2 = U \cup V_2$.

Since A' is very well embedded, there is a function $f : F \to (0, t_0]$ such that $A' = \mathrm{Cl}\{(x, t) \in F \times [0, t_0]:\ t < t_0,\ t = f(x)\}$. (The value of f outside $\pi_F(A')$ is t_0.) And so $U_2 = \mathrm{Cl}\{(x, t) \in F \times [0, t_0]:\ f(x) \leqslant t < t_0\}$.

We define $h : F \times [0, t_0] \to F \times [0, t_0]$ by $h(x, t) = (x, \min(t, f(x)))$, and on $F \times [0, t_0]$ we define $G = H \circ h$. We now show G is an isotopy. Let $t \in [0, t_0]$, we must show $G|_{F \times \{t\}}$ is an embedding. $K_1 = F \times \{t\} \cap U_1$ is embedded by G since h maps it into U_1. $K_2 = F \times \{t\} - U_2$ is embedded by G since there $G = H$. And so it remains to show that $G(K_1 - K_2) \cap G(K_2 - K_1) = \emptyset$. But this follows from the definition of V_2.

Strictly speaking, G is not a directed isotopy. Every $x \in F$ moves into the preferred side while it is moving, but any $x \in \pi_F(A')$ stops moving at time $f(x)$. We can fix this by letting them continue moving into the preferred side, inside a very thin neighborhood of $G_{t_0}(F)$. We denote this altered isotopy by G again.

We now define $h' : F \times [0, t_0] \to F \times [0, t_0]$ by $h'(x, t) = (x, \max(t, f(x)))$. $G' = H \circ h'$ is an isotopy since h' embeds each $F \times \{t\}$ in $F \times \{t_0\} \cup U_2$ and H embeds $F \times \{t_0\} \cup U_2$. (Again a slight modification of G' will make it directed in the strict sense.)

We have $G_{t_0} = G'_0$ and $G'_{t_0} = H_{t_0}$ and so $G * G' * H|_{F \times [t_0, 1]}$ is well defined and is the required isotopy since $L(G) = L(H|_{F \times [0, t_0]}) - U$ and $L(G') = (L(H|_{F \times [0, t_0]}) \cap U) \cup (\bigcup_i A_i)$ where $\{A_i\}$ are vertical annuli.

(b) We use the isotopy G as above. Just before time t_0 an intersection circle coming from B, is moving towards the boundary of $H(A)$. We let it reach $H(A)$ and touch it at a point $a \in \partial H(A)$. Let $A'' = \pi_F(A')$, i.e., that piece of F that at this stage is situated at $H(A')$. We start moving A'' vertically across $H(A)$, but we want to dictate the timing of passage of every point of A'' across $H(A)$, such that there will be no birth or death of intersection circles. This amounts to defining a Morse function on A (or A' or A'') that has no local minimum or maximum in the interior of A, and no local minimum on the boundary of A except for in a. Such a Morse function can be defined, for example, by the height function on an embedding of A in $\mathbb{R}^3$ as in Fig. 2. After A'' has passed $H(A)$, we proceed as before. The only difference is that in the preceding case, in this stage, A'' was close and parallel to $H(A)$, from the outside of $H(U)$, and now it is close and parallel to $H(A)$ from the inside of $H(U)$.

Fig. 2.

Since the original A had at least one birth point, and now it has no birth or death points, we have reduced the number of extremum points of L. □

Let $H(x,t)$ be a directed isotopy such that 0 and 1 are nonsingular times, let $L = L(H)$ and let $a = (x_0, t_0) \in L$ be a birth point.

For $t \in [0,1]$ let A^a_t be the connected component of a in $L \cap F \times [0,t]$. Let $t(a)$ be the supremum of the t's for which:

(a) A^a_t is well embedded.

(b) $A^a_t \cap F \times [0,s]$ is connected for any $s \leqslant t$ (i.e., the component of a does not merge with any other component until time t).

If $t(a) = 1$ then the connected component of a in L is a well embedded upper component with a it's unique birth point. Such a birth point will be called "of type 0".

Assume then that $t(a) < 1$, and so $t(a)$ is the first time where either A^a_t is not well embedded, or it merges with another component. (And so $t(a)$ must be a singular time.)

Let $A^a \subseteq A^a_{t(a)}$ be defined as follows: If $A^a_{t(a)} = A \cup B$ where A, B are two surfaces touching each other in a point, with $a \in A$, then $A^a = A$. Otherwise simply $A^a = A^a_{t(a)}$.

We now analyze the various possibilities for the nature of the singular point $b = (x_1, t(a))$ at time $t(a)$.

b is not a death point. If it was, look at A^a_t for close $t < t(a)$ ("close" meaning that the times $t \leqslant s < t(a)$ are nonsingular). Assume (by level preserving change of coordinates in $F \times [0,1]$), that A^a_t is very well embedded in $F \times [0,t]$. Let D be the disc in $L \cap F \times [t,1]$ where b lies. Assume D is very well embedded in $F \times [t,1]$. There are two possibilities: Either $\pi_F(A^a_t) \subseteq \pi_F(D)$, or $\pi_F(A^a_t) \cap \pi_F(D) = \partial\pi_F(D)$. In the first case, since there are no singular times in the time interval $[t, t(a))$, A^a_t must also be a disc, and together we get a sphere, which is impossible by Lemma 2.11. So we have $\pi_F(A^a_t) \cap \pi_F(D) = \partial\pi_F(D)$, and so $A^a_{t(a)}$ is still well embedded. There was also no merging with another component, and this will still be true for t's a bit larger than $t(a)$. This contradicts the definition of $t(a)$.

b is also not a birth point since that would not have affected A^a_t.

b is thus a "saddle point". There are two distinctions now to be made. The first is whether in the saddle point b, A^a touches another component B, or touches itself. The second is whether the points of $F \times [0, 1]$ directly under the saddle are contained in $U = U_{F\times[0,t(a)]}(A^a)$ or not.

We analyze each of the four combinations:

(1) Two components, U not under saddle: This means that B is not contained in U, so this is the situation described in Lemma 3.1(b).

(2) Two components, U under saddle: Here B is contained in U.

(3) One component, U not under saddle: Here A^a touches itself from the outside, and so A^a_t for close $t > t(a)$ is still well embedded, and so this case is impossible.

(4) One component, U under saddle: Here A^a touches itself from the inside, and so it is not well embedded. In U we take a small disc D modeled by $x = 0$, $-\varepsilon \leqslant z \leqslant -y^2$ in $\mathbb{R}^3$ where $z = x^2 - y^2$ models L in a neighborhood of b. Say $z = -\varepsilon$ in $\mathbb{R}^3$ corresponds to time $t < t(a)$ in $F \times [0, 1]$. Look at A^a_t. Since it is well embedded, the arc $D \cap U_{F\times[0,t]}(A^a_t)$ can be continued by an arc in A^a_t to a circle that bounds a disc D' in $U_{F\times[0,t]}(A^a_t)$. $E = D \cup D'$ is a disc in U with $\partial E \subseteq A^a$.

We now think of $F \times [0, 1]$ as contained in M^F via a lifting of H. L is then simply $F \times [0, 1] \cap p^{-1}(S)$.

Since $p^{-1}(S)$ is incompressible in M^F, the disc E that we have found implies the existence of a disc K in $p^{-1}(S)$ bounded by ∂E. There are two circles of $L \cap F \times \{t(a)\}$ that touch ∂K at b. They intersect ∂K only at b since that is the only point of ∂K $(= \partial E)$ at level $t(a)$. So one of them γ_1 is contained in K, and the other γ_2 is not. γ_1 bounds a disc K_1 in $p^{-1}(S) - A^a$ $(K_1 \subseteq K)$. In particular, we see here that at time $t(a)$, the case is, that one intersection circle splits into two, and not that two circles merge into one.

A birth point a will be called of type 1, 2 or 4 according to the type of singularity that appears at b as described above. (Birth points of type 0 were defined earlier.) For a death point a we define A^a_t, A^a, $t(a)$, and the type of a, analogously.

We summarize the above discussion about birth points of type 4 in the following lemma:

Lemma 3.2. *If a is a birth point of type 4, then at time $t(a)$, one intersection circle splits into two, and at least one of these circles bounds a disc in $p^{-1}(S) - A^a$.*

We will find ways to reduce the number of extremum points when having extremum points of type 1 and 4, under certain conditions. (Lemmas 3.4 and 5.6.) The following lemma will help us avoid type 2.

Lemma 3.3. *Let $A \subseteq F \times [0, t_0]$ be an upper surface.*

(a) *The first death point of A (if there are any), is not of type 2.*

(b) *If the last birth point a of A is of type 2, then $t(a) \geqslant t_0$ and so $A^a \supseteq A$ and A is well embedded with a its unique birth point.*

Proof. (a) Call the first death point e. If e is of type 2 then $U_{F\times[t(e),t_0]}(A^e)$ contains a lower component B that touches A^e. So $A^e \cup B$ is connected and so must all be contained in A. Now, a death point of B must come before e.

(b) Assume $t(a) < t_0$. Again, in $U_{F\times[0,t(a)]}(A^a)$ there is a component B that touches A^a and so must be contained in A, and has a later birth point than a. □

With Lemma 3.1(b) we may reduce the number of extremum points, in the presence of a birth point of type 1 satisfying a certain strong condition. We now weaken that condition.

Lemma 3.4. *Let a be a birth point of type* 1. *Assume that for any birth point b of an upper component B of $L \cap F \times [0, t(a)]$ such that $H(B) \subseteq H(A^a)$, either* (1) $t(b) > t(a)$ *or* (2) *b is not of type* 4. *Then the number of extremum points may be reduced.*

Proof. By induction on the number of B's. If there are none then we are done by Lemma 3.1(b). Otherwise take a B that is minimal, i.e., $H(B) \subseteq H(A^a)$ but there is no upper component $C \neq B$ of $L \cap F \times [0, t(a)]$ such that $H(C) \subseteq H(B)$. If B is well embedded then by Lemmas 2.21 and 3.1(a) we can push B above level $t(a)$. In doing so we did not alter A^a since it is impossible that $A^a \subseteq U_{F\times[0,t(a)]}(B)$ since by Lemma 2.20, $m(b) > m(a)$. (It is also impossible that B is the component that is touching A^a at time $t(a)$, since then $A^a \cup B$ would be an upper connected component that is not embedded by H.) So we have reduced the number of B's and the conclusion follows by induction.

So assume B is not well embedded. Let b be the last birth point of B. $t(b) \leqslant t(a)$ since otherwise B would be well embedded. So b is not of type 4 and not of type 0. By Lemma 3.3(b) it is also not of type 2. So it is of type 1. By Lemma 2.21,

$$H(\operatorname{int} A^b) \cap H(F \times \{t(b)\}) = \emptyset.$$

And so by Lemma 3.1(b) applied to A^b, we are done. □

4. Changing L, part 2

Lemma 4.1. *Let $H : F \times [0, 1] \to M$ be a directed isotopy. If $0 \leqslant t_1 < t_2 \leqslant 1$ and $N \subseteq F \times [t_1, t_2]$ is a (bounded) 3-manifold, such that $H|_{F\times[t_1,t_2]}{}^{-1}(H(n)) = \{n\}$ for every $n \in N$, then one can change H, such that the effect on L will be a change by an arbitrary homeomorphism $G : N \to N$ which is the identity on ∂N (and no change in $L - \operatorname{int} N$).*

Proof. Let $h : F \times [0, 1] \to F \times [0, 1]$ be defined by the identity on $F \times [0, 1] - \operatorname{int} N$ and by G^{-1} on N. $H \circ h$ is the required directed isotopy. □

We will use this technique with N a ball, and so we now study embeddings of surfaces in $\mathbb{R}^3$, and the way they may be isotoped.

Lemma 4.2. *Let $D \subseteq \mathbb{R}^3$ be a disc such that:*

(1) *(Neighborhood of ∂D in D) $\cap \{z = 0\} = \partial D$.*

(2) *D is in general position with respect to $\{z = 0\}$.*

(3) *Each component of $D \cap \{z \geqslant 0\}$ or $D \cap \{z \leqslant 0\}$ is boundary parallel in $\{z \geqslant 0\}$, $\{z \leqslant 0\}$, respectively.*
Call these components E_i, $i = 1, \ldots, n$. Let U_i be the region bounded by E_i and $\{z = 0\}$ and let $P_i = U_i \cap \{z = 0\}$. We assume the numbering of the E_i's is such that if E_i is contained in a disc component of $D - \operatorname{int} E_j$, then $i > j$. Our last assumption will be:

(4) $\bigcap_i \operatorname{int} P_i \neq \emptyset$.

Then $E_1, \ldots, E_{n-1}$ are annuli, E_n is a disc and $P_i \subseteq P_{i+1}$ for all i, i.e., D is of the form of the rotation surface described in Fig. 3.

Proof. By induction on n. Look at E_1. ∂D is one of it's k boundary circles, and the other $k - 1$ circles bound discs $D_1, \ldots, D_{k-1}$, in $D - E_1$.

If $k = 1$ then $E_1 = D$ and we are done.

If $k = 2$ then E_1 is an annulus and there is only D_1. D_1 satisfies the induction hypothesis, so we must only show $P_1 \subseteq P_2$. Let e be the circle $E_1 \cap E_2$. P_1 and P_2 must lie on the same side of e in $\{z = 0\}$ otherwise we would have $\operatorname{int} P_1 \cap \operatorname{int} P_2 = \emptyset$. If $n = 2$, P_2 is a disc and we are done. If $n \geqslant 3$ then by the induction hypothesis $e \subseteq U_3$ and so $E_1 \subseteq U_3$. So $P_1 \subseteq P_3$ and is on the same side of e as P_2. It follows that $P_1 \subseteq P_2$.

If $k \geqslant 3$, let E_{i_j} be the outermost E_i of D_j (i.e., i_j is the minimal i of E_i's contained in D_j). $D_1, \ldots, D_{k-1}$ satisfy the induction hypothesis and so exactly as in the previous case we get $P_1 \subseteq P_{i_j}$ for $j = 1, \ldots, k - 1$. It follows that $\partial D_1 \subseteq P_{i_2}$ and $\partial D_2 \subseteq P_{i_1}$. But then it follows that $E_{i_1} \subseteq U_{i_2}$ and $E_{i_2} \subseteq U_{i_1}$, which is impossible. □

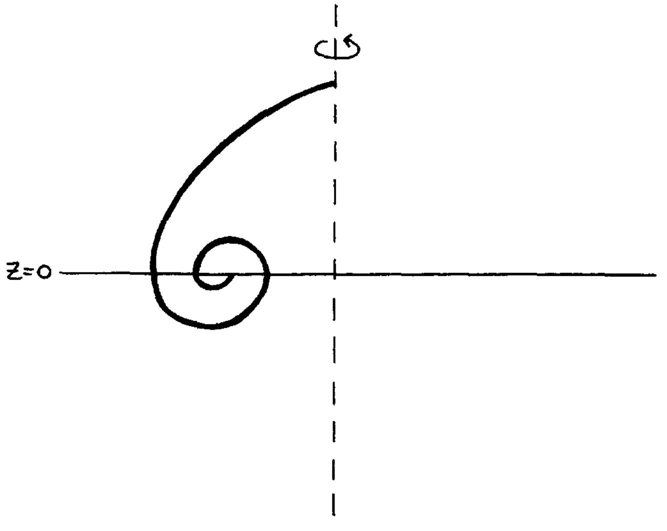

Fig. 3.

Definition 4.3. A disc satisfying the hypothesis of Lemma 4.2 will be called a "curled" disc. A regular neighborhood of a curled disc will be called a "thick curled disc" (TCD).

Lemma 4.4. *Let A be a (compact) surface. Let $f: A \to \mathbb{R}$ be a nonnegative Morse function with $f(\partial A) = 0$. Then for any point $x \in A$ there is a path $u: [a, b] \to A$, with $u(a) = x$, $(f \circ u)'(t) > 0$ for all $t \in (a, b)$, and $u(b)$ is a local maximum point for f.*

Proof. Let $y_1, \ldots, y_n$ be the singular points of f, i.e., the points where $df = 0$. For $i = 1, \ldots, n$, let U_i be a small neighborhood of y_i. In $A - (U_1 \cup \cdots \cup U_n)$ we let u be an integral curve of grad f (assuming some metric on A). And so there, surely $(f \circ u)' > 0$. If we reach one of the U_i's, it must be along an inward vector (i.e., a vector at the boundary of U_i that is pointing into U_i), and so y_i cannot be a local minimum. If y_i is a local maximum, we know how to make the last step inside U_i. If y_i is a saddle point, we also know how to move inside U_i with $(f \circ u)' > 0$ and such that we exit U_i along an outward vector. Then continue with an integral curve again. (If our initial point happened to be itself a minimum point y_i, we also know how to make the first step as to exit U_i along an outward vector.) This process must end since there are only finitely many U_i's and since $\|\operatorname{grad} f\|$ has a positive minimum in $A - (U_1 \cup \cdots \cup U_n)$. □

Lemma 4.5. *Let A be a lower surface in $G \times [0, 1]$, where G is a surface (with or without boundary), and let $x \in G \times \{0\} - \partial A$. Then there is a path $u: [a, b] \to G \times [0, 1]$ with*

(1) $u(a) = x$.
(2) $(q \circ u)'(t) > 0$ *for all* $t \in [a, b]$. ($q: G \times [0, 1] \to [0, 1]$ *is the projection.*)
(3) $u(b) \in G \times \{1\}$.
(4) *u intersects A only at local maximum points of $q|_A$. (We assume the embedding of A is generic, i.e., that $q|_A$ is a Morse function.)*

Proof. From x go straight up. Stop just before intersecting A, and then move very close to A, but not touching it, according to a path in A that is given by Lemma 4.4 with $f = q|_A$. You are now outside A but very close to a local maximum point of $q|_A$. If you happen to be above A at this stage, start everything again by just going up. If you happen to be under A, cross A right at the local maximum point, and then start going straight up. This procedure will terminate at $G \times \{1\}$. □

Lemma 4.6. *Let $B \subseteq \mathbb{R}^3$ be a TCD (thick curled disc), and assume (for convenience of presentation), that ∂B is made of horizontal and vertical discs and annuli.*

Call one of the two discs (which are necessarily horizontal) A_1. Call it's neighboring annulus A_2. Call A_2's other neighbor A_3 etc. until finally the other disc will receive the name A_{4n+1}. (See Fig. 6(a) for the case $n = 1$.)

Let $L \subseteq B$ be a surface with $\partial L \subseteq D$ where $D \subseteq \partial B$ is a small disc situated in A_{2n} or A_{2n+2}, (which are both vertical). Let U be a small ball in B with $U \cap \partial B = D$.

Then there is an isotopy of L inside B, not moving ∂L, that brings L into U, and such that in the final stage $q|_L$ has the same number of local minimum and maximum points. ($q:\mathbb{R}^3 \to \mathbb{R}$ is the projection $(x_1, x_2, x_3) \mapsto x_3$.)

Proof. We first show that there is an isotopy of B itself in $\mathbb{R}^3$, not moving D, in the end of which the whole of B is situated on one side of a vertical plane containing D, and the number of local minima and maxima of $q|_L$ is unchanged. (D is actually not flat, and so this vertical plane is not completely flat either.)

We show this by induction on n. For $n = 1$, either B is already on one side of D (Fig. 4(a)) and we are done, or it is not (Fig. 4(b)). We will perform a "flipping over" of B as to "expose" D (Fig. 5). We must do this without changing the number of minima and maxima of $q|_L$. (We will call this property, "being kind to L".)

Let the numbering be as in Fig. 6(a), and let K and K' be defined by Fig. 6(b). Let x be the midpoint of A_5. Let u be a path connecting x to A_1, with $(q \circ u)' > 0$ and with u intersecting L only in local maximum points of $q|_L$. (Lemma 4.5 for K.) By a level preserving isotopy of K, fixing $\partial K - A_1$, we can assume u is actually vertical. By another level preserving isotopy of K which pushes everything away from u, we can assume $L \cap$ (neighborhood of K') is just a union of well-embedded discs with a unique

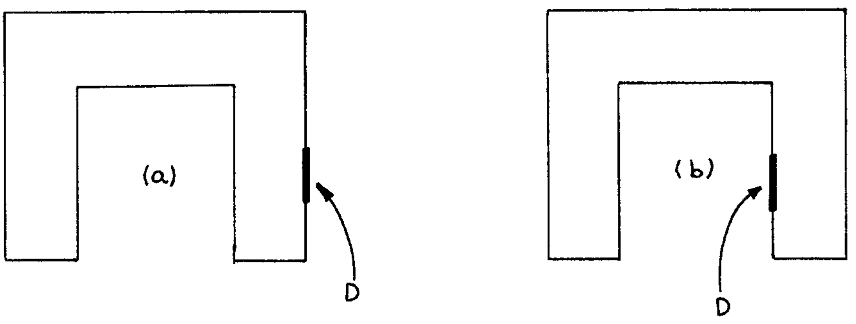

Fig. 4.

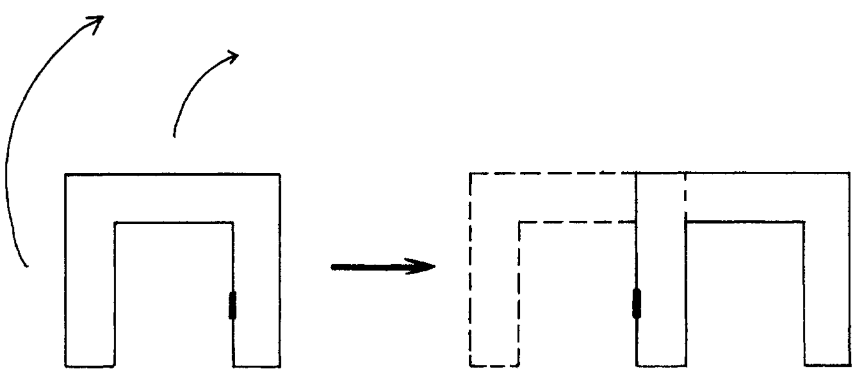

Fig. 5.

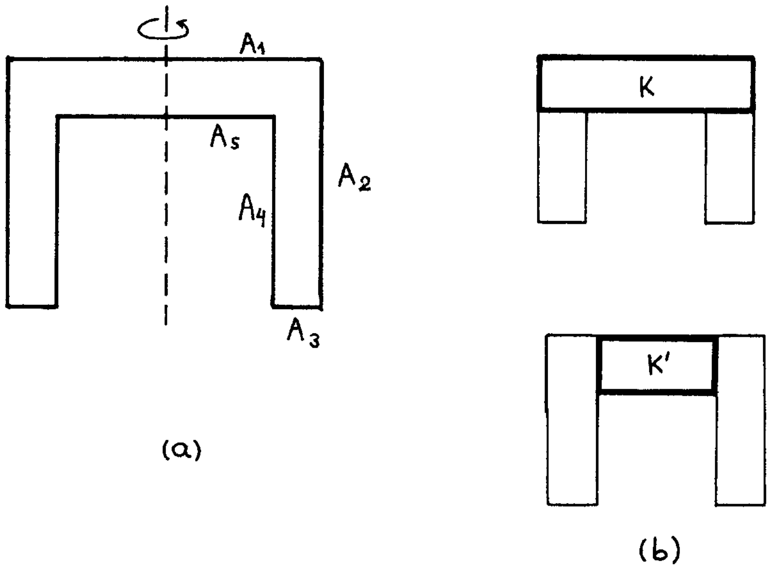

Fig. 6.

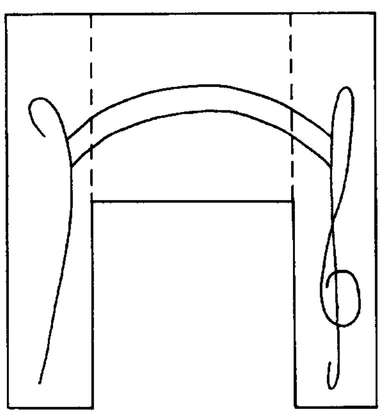

Fig. 7.

singular point, which is a maximum, and with the boundary of these discs contained in the vertical part of ∂(neighborhood of K') (Fig. 7).

It is now possible to perform the flipping over, being kind to L, as is shown in Fig. 8. We describe this flipping over transformation in detail (call it F): We divide B into 2 parts X and Y as in Fig. 9, which describes B as a rotation body around the depicted axis. We choose X such that $L \cap Y$ contains nothing but the well embedded discs obtained in the previous paragraph. (See Fig. 10(a).)

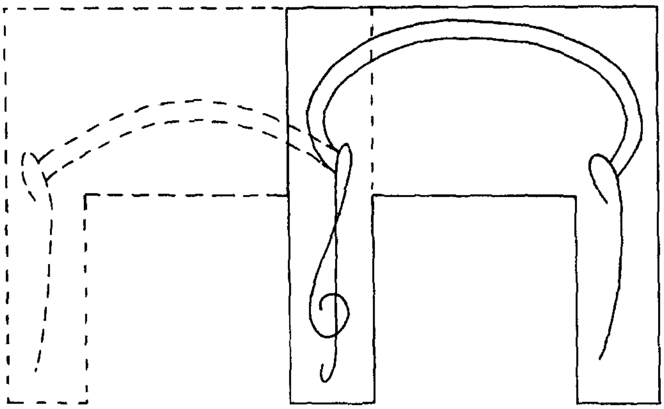

Fig. 8.

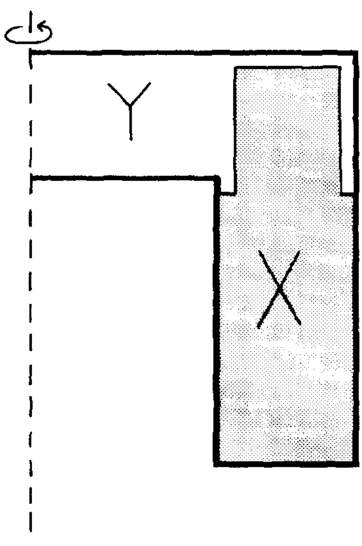

Fig. 9.

On X, F will be of the form $(x_1, x_2, x_3) \mapsto (f(x_1, x_2), x_3)$, where f is the complex function:

$$z \mapsto \left(r + \frac{1}{r}\right)i + \frac{1}{z}$$

restricted to the annulus $\{1/r \leqslant |z| \leqslant r\}$ with D located around $(1/r)i$. (With the additional requirement that D actually stands in place, and so in that area f must differ a bit from $z \mapsto (r + 1/r)i + 1/z$.)

As an intermediate stage, F is defined on Y, such that $F(B)$ is congruent as a set to B. (Fig. 10(b).) Finally we move $F(L \cap Y)$ (which by our construction is simply a union of discs), inside $F(Y)$ until each such disc has a unique singular point, that being

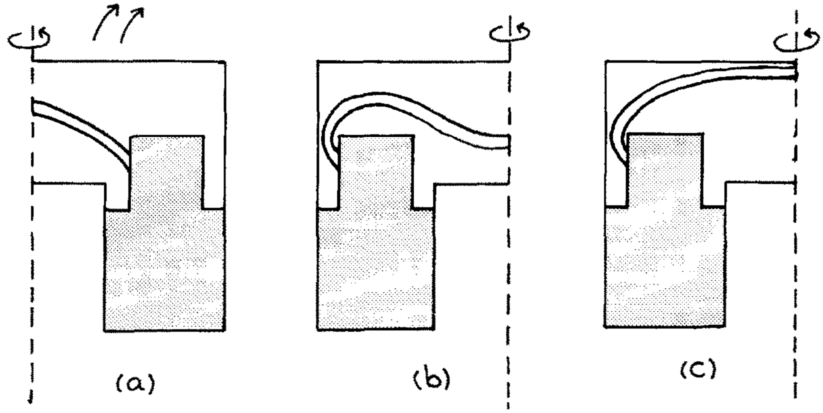

Fig. 10.

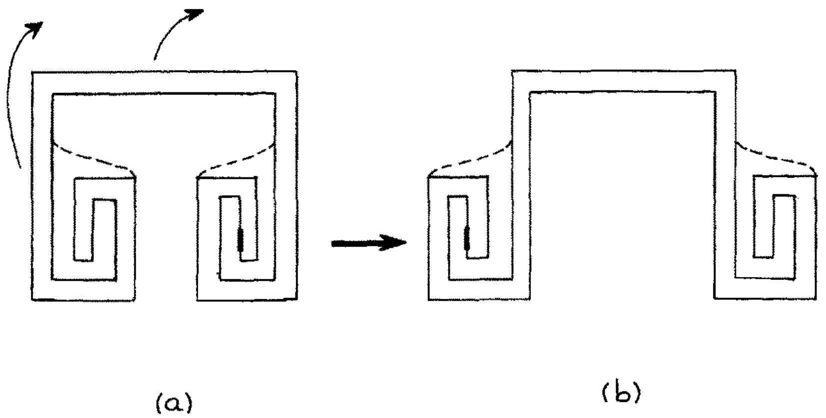

Fig. 11.

a maximum point. (Figs. 10(c) and 8.) In B each such disc had the shape $\{x_1^2+x_2^2+x_3^2 = 1,\ x_3 \geqslant 1/2\}$ (Figs. 10(a) and 7), and in $F(B)$ they have the shape $\{x_1^2+x_2^2+x_3^2 = 1,\ x_3 \geqslant -1/2\}$ (Figs. 10(c) and 8 again). Note that this fits smoothly with the definition of F on X.

Since F is level preserving on X, and $F(L \cap Y)$ has a unique singular point, which is a maximum, for each of its discs, just like in $L \cap Y$ itself, we did not change the number of local minima and maxima of $q|_L$.

Assume now $n \geqslant 2$. We enlarge B as described in Fig. 11(a), which brings us back to $n = 1$. So we perform the flipping over as in Fig. 11. Then an isotopy that is kind to L, as in Fig. 12, and another isotopy, that does not move L at all, as in Fig. 13, will let us use induction. (The next step of the recursion will of course use a reversed formulation of Lemma 4.5.)

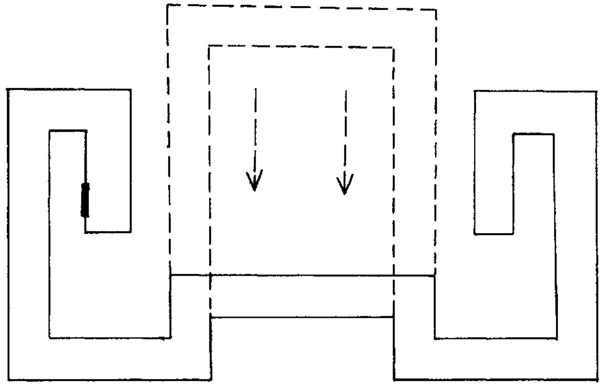

Fig. 12.

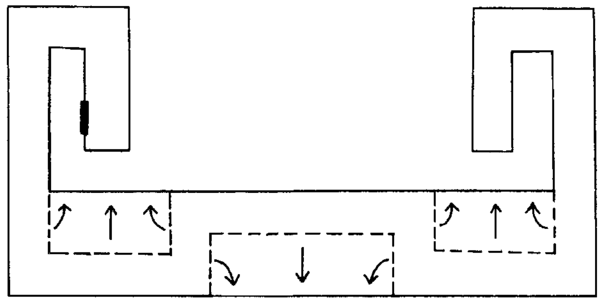

Fig. 13.

So we now have B on one side of D. It is necessarily the side in which U lies. And so we can now shrink B, by a level preserving isotopy and then by pushing it from the top and from the bottom, until it is contained in U.

We conclude, that we can deform B in $\mathbb{R}^3$, without moving D, such that at the final map, B is contained in U, and such that we are kind to L.

On the other hand, we can surely deform B inside itself, without moving D, and such that at the final map B is contained in U. The proof is complete if we notice that any two embeddings of B in U such that their restriction to D is the inclusion, are isotopic inside U with an isotopy not moving D. □

5. Changing L, part 3

Definition 5.1. For a birth point a, let A^{a*} be the union of A^a and the disc components of $p^{-1}(S) - \operatorname{int} A^a$. (Again we are thinking of $F \times [0,1]$ as embedded in M^F via a lifting of H.)

In general it is possible that A^{a*} is not contained in $F \times [0,1]$, but eventually we will be interested only in the case that it is.

Lemma 5.2. *A^{a*} is embedded by p. (And so if $A^{a*} \subseteq F \times [0,1]$, A^{a*} is embedded by H.)*

Proof. We know A^a is embedded by H. This means that after lifting $F \times [0,1]$ to M^F, A^a is embedded by p. Let D be a disc component of $p^{-1}(S) - A^a$. $\partial D \subseteq A^a$ and so it is embedded by p. It follows by Lemma 5.3 bellow that D is embedded by p. $p(A^a) \not\subseteq p(D)$ since then p would not be a local homeomorphism in a neighborhood of ∂D. So $p(A^a) \cap p(D) = p(\partial D)$, and so p embeds $A^a \cup D$. Adding the discs one by one we get A^{a*} is embedded by p. □

In the above proof we used the following easy fact (with $n = 2$):

Lemma 5.3. *Let N be a smooth n-manifold that is covered by $\mathbb{R}^n$, and let B^n be the n dimensional ball. If $f : B^n \to N$ is a local diffeomorphism such that $f|_{\partial B^n}$ is an embedding, then f is an embedding.*

If $K \subseteq S$ is a subsurface we define similarly K^* to be the union of K and the disc components of $S - \operatorname{int} K$.

Lemma 5.4. *Let $a \in L$ be a birth point. Then $p(A^{a*}) = p(A^a)^*$. (And so if $A^{a*} \subseteq F \times [0,1]$, $H(A^{a*}) = H(A^a)^*$.)*

Proof. We have seen A^{a*} is embedded by p. This implies $p(A^{a*}) \subseteq p(A^a)^*$. Now let D be a disc component of $S - p(A^a)$. Lift D to $p^{-1}(S)$ to get a disc component of $p^{-1}(S) - A^a$ that is mapped onto D. □

Lemma 5.5. *Let $H(x,t)$ be a directed isotopy, and let $L = L(H)$. Let $a \in L$ be a birth point of type* 4, *and assume $A^{a*} \subseteq F \times [0,1]$ and $\partial A^{a*} \neq \emptyset$. (The latter is automatically satisfied when $T \neq \emptyset$ by Theorem* 2.12.) *Then*:

(a) *For any birth or death point $b \in A^{a*}$, $b \neq a$, we have $A^{b*} \subsetneq A^{a*}$.*

(b) *For any birth or death point $b \in L$ with $H(A^b) \subseteq \operatorname{int} H(A^{a*})$, we have $p(A^{b*}) \subsetneq p(A^{a*}) (= H(A^{a*}))$.*

Proof. (a) If $A^b \subseteq \operatorname{int} A^{a*}$ then since $p^{-1}(S) - A^{a*}$ has no disc components we must have $A^{b*} \subseteq \operatorname{int} A^{a*}$. So assume $A^b \not\subseteq \operatorname{int} A^{a*}$ and so $A^b \cap \partial A^{a*} \neq \emptyset$. If b is a birth point (respectively death point), then look at $B = A^b \cap F \times [0, t(a)]$ (respectively $\cap F \times [t(a), 1]$). Since $\partial A^{a*} \subseteq F \times \{t(a)\}$, we have $\partial B \cap \partial A^{a*} \neq \emptyset$ (in particular, $B \neq \emptyset$). It also follows that $F \times \{t(a)\}$ cuts A^{a*} into upper components of $L \cap F \times [0, t(a)]$ and lower components of $L \cap F \times [t(a), 1]$, and B must be one of these components (since $b \in B$). Since $b \neq a$ (and a is the only birth point in A^a), B is contained in one of the discs of $A^{a*} - A^a$. If it is attached to a boundary circle of A^a then that circle is not in the boundary of A^{a*}.

So the only way we can have $\partial B \cap \partial A^{a*} \neq \emptyset$ is that it is attached to one of the two touching circles of the boundary of A^a while the other one does not bound a disc in $p^{-1}(S) - A^a$. (We see in particular that b was a death point.) So B merges at time $t(a)$ with another component of $L \cap F \times [t(a), 1]$, and so $A^b = B$. Since it is contained in a disc component of $A^{a*} - A^a$, A^{b^*} is contained in that same disc, and so $A^{b^*} \subsetneq A^{a*}$.

(b) $p(A^b) \subseteq \operatorname{int} p(A^{a*})$. But $p(A^{a*}) = p(A^a)^*$ so $S - p(A^{a*})$ has no disc components, so $p(A^{b^*}) = p(A^b)^* \subseteq \operatorname{int} p(A^{a*})$. □

In Lemma 3.4 we simplified L given a singular point of type 1. Now we do the same for type 4:

Lemma 5.6. *Let H be a directed isotopy, and assume there is a birth point a of type 4 such that $p(A^{a*}) \cap H_i(F)$ contains no null-homotopic circle for $i = 0, 1$ (in particular, $A^{a*} \subseteq F \times [0, 1]$), and $\partial A^{a*} \neq \emptyset$. Then the number of extremum points of H may be reduced.*

Proof. Let a be a birth or death point of type 4 that satisfies the hypothesis of the lemma, and such that in addition $p(A^{a*})$ is minimal with respect to set inclusion, among the birth and death points of type 4. (This exists since if A^{a*} satisfies the hypothesis, and $p(A^{b^*}) \subseteq p(A^{a*})$ then also A^{b^*} satisfies it.)

Assume a is a birth point. If there is some birth or death point a' in A^{a*} of type 1, we show Lemma 3.4 applies to a'. Say a' is a birth point. Let B be an upper component of $F \times [0, t(a')]$ such that $H(B) \subseteq H(A^{a'})$, and let $b \in B$ be a birth point with $t(b) \leqslant t(a')$. We will show $p(A^{b^*}) \subsetneq p(A^{a*})$ and so by the minimality condition on a, b is not of type 4. $A^b \subseteq B$ since $t(b) \leqslant t(a')$, and so $H(A^b) \subseteq H(B) \subseteq \operatorname{int} H(A^{a'})$. By Lemma 5.5(a), $A^{a'} \subseteq A^{a*}$, so $H(A^b) \subseteq \operatorname{int} H(A^{a'}) \subseteq \operatorname{int} H(A^{a*})$. So by Lemma 5.5(b), $p(A^{b^*}) \subsetneq p(A^{a*})$.

So assume there are no birth or death points of type 1 in A^{a*}. And so (by Lemma 5.5(a) and the minimality of A^{a*}) all birth and death points of A^{a*} are of type 2, except for a itself which is of type 4. By Lemma 3.2, there is at least one disc in $A^{a*} - A^a$. Since the boundary of these discs lie in $F \times \{t(a)\}$, $F \times \{t(a)\}$ cuts them into upper components of $F \times [0, t(a)]$ and lower components of $F \times [t(a), 1]$. By Lemma 3.3, all these components are well embedded, and all of them including A^a, have a unique extremum point. We will refer to them as "the pieces of $A^{a*} - A^a$". (Or, if we want to include A^a itself we will say the pieces of A^{a*}.)

Let c and c' be the two circles of ∂A^a meeting at the singular point $a' \in F \times \{t(a)\}$. If they both bound discs in $p^{-1}(S) - A^a$ (and so in $A^{a*} - A^a$), then if E and E' are the lower components of $F \times [t(a), 1]$ having c and c' in their boundary, then at least one of them is of type 1. (Or rather it's death point is of type 1.) So at most one of c and c' bounds a disc in $p^{-1}(S) - A^a$, and we know at least one of them does, say it is c'. If $c \not\subseteq U_{F \times [t(a),1]}(E')$, then again, E' is of type 1. So $c \subseteq U_{F \times [t(a),1]}(E')$. It follows that c is null-homotopic (since it may be isotoped into E'), and so it bounds a disc in $p^{-1}(S)$. Since c does not bound a disc in $p^{-1}(S) - A^a$, this disc must contain A^a, and so A^{a*} is simply a disc, with $c = \partial A^{a*}$.

We now show all the pieces of A^{a*} have c contained in their region. We know this for A^a and for the E' mentioned above. All other pieces are disjoint from c and so it is enough to show c intersects their region, and it will follow that the whole of c is there. Assume there are pieces that are say, upper components in $F \times [0, t(a)]$ that do not have points of c in their region. Let E be a minimal one, i.e., one that does not contain other pieces of A^{a*} in its region. (Such exists since if E does not have points of c in its region, then any piece inside its region cannot have points of c in its region.) Now E's unique birth point e, is of type 2. Let B be the component in $U_{F\times[0,t(e)]}(A^e)$ that touches A^e. $A^e \subseteq A^{a*}$ (Lemma 5.5) and so $B \cap A^{a*} \neq \emptyset$. On the other hand if $B \subseteq A^{a*}$, then $B \cap F \times [0, t(a)]$ would contradict the minimality of E. (It is not empty since all of A^{a*}'s birth points are under $F \times \{t(a)\}$.) So we must have $B \cap \partial A^{a*} \neq \emptyset$. So there are points of c in $U_{F\times[0,t(e)]}(A^e)$, and so there are points of c in $U_{F\times[0,t(a)]}(E) = U_{F\times[0,t(e)]}(A^e) \cap F \times [0, t(a)]$ since $c \subseteq F \times \{t(a)\}$. As we said, it follows that the whole of c is there.

Now let E be a piece of $A^{a*} - A^a$, and let $P = U_{F\times\{t(a)\}}(E) \cap F \times \{t(a)\}$. Then $P \subseteq F \times \{t(a)\}$ is a planar surface (since it is homeomorphic to E), with null-homotopic boundary circles (since they are contained in E). And so it is contained in a disc $P' \subseteq F \times \{t(a)\}$. By a level preserving isotopy, we may have $E \subseteq \pi_F(P') \times [0, 1]$. We may do this with all E's together, and also with A^a, by thinking of a regular neighborhood of $U_{F\times[0,t(a)]}(A^a)$ instead of just $U_{F\times[0,t(a)]}(A^a)$ itself. It is clear now from the connectivity of A^{a*}, that there is one maximal P' such that $A^{a*} \subseteq \pi_F(P') \times [0, 1]$. We fix an open disc $K \subseteq F$ with $A^{a*} \subseteq K \times [0, 1]$.

So by Lemma 4.2 each of the discs $D_1, \ldots, D_m$ of $A^{a*} - A^a$ is a curled disc. (Our pieces satisfy more than is assumed in Lemma 4.2, they are well embedded with unique extremum point. And so Fig. 3 is an almost accurate model for our D_i. Assumption 4 is satisfied since we have shown c or $c - \{pt\}$ is in that intersection.)

We first show $m \leqslant 2$. Assume D_1 is the disc who's boundary touches c. Look at $A^{a*} - D_1$. This is a disc with two of its boundary points touching each other. Separate these two points only a very small distance inside $K \times \{t(a)\}$, as to get a nonsingular disc which we will call D'. D' is made of all the pieces of $D_2, \ldots, D_m$, and a modified A^a, a modification which brings A^a back to the form of A^a_t, for close $t < t(a)$. Now c is in the region of all the pieces of $D_2, \ldots, D_m$ but does not touch the pieces themselves. (c only touches the outermost piece of D_1.) And so points close to c are also inside all these regions. So take a point that is close to c, and also inside the region of the modified A^a. This point will be in the regions of all pieces of D', and so we conclude D' is a curled disc. And so the modified A^a, which is the outermost piece of D' is either a disc or an annulus, and so has either 1 or 2 boundary components. It follows that the real A^a has 2 or 3 boundary circles (two of which touch each other), and so $m = 1$ or 2.

If $m = 1$ there is only one possibility for A^{a*}: A^a is a disc touching itself from the inside, and D_1 is cut by $K \times \{t(a)\}$ into two pieces, a lower annulus in $K \times [t(a), 1]$, and an upper disc in $K \times [0, t(a)]$. (See Fig. 14.)

If $m = 2$, then A^a is an annulus, with one of it's boundary circles touching itself from the inside. There are two cases:

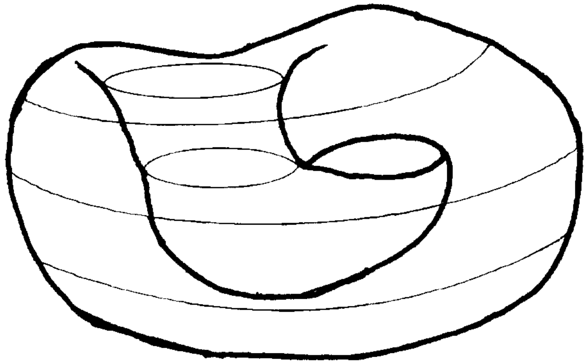

Fig. 14.

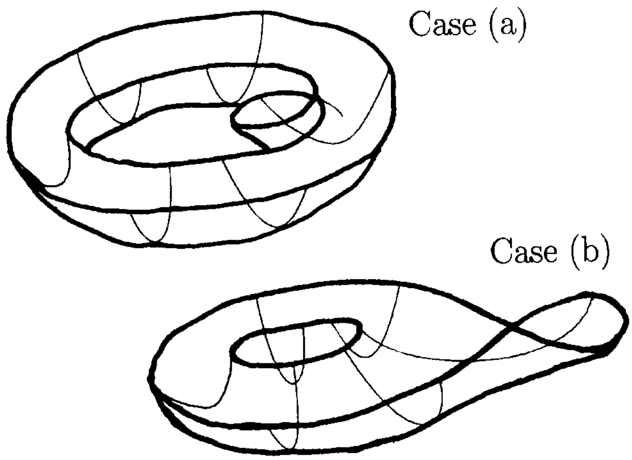

Fig. 15.

Case (a): The inner circle of the annulus is touching itself. (Inner in $K \times \{t(a)\}$, i.e., the circle such that the disc bounded by the other circle in $K \times \{t(a)\}$ contains it.)

Case (b): The outer circle of the annulus is touching itself. (See Fig. 15.)

Given $n_2 =$ the number of pieces in D_2, then $D_2 \cup A^a$ is determined completely, since the modified $D_2 \cup A^a$ (which we called D'), is a curled disc. It remains to determine D_1. First we must find out which of the two touching circles of ∂A^a is ∂D_1, and which is c. Call the circle closer to ∂D_2: e_1, and the one further: e_2, i.e., e_1 separates in $K \times \{t(a)\}$ between ∂D_2 and e_2. (See Fig. 16.)

We show $\partial D_1 = e_1$ and $c = e_2$. Assume on the contrary, that $\partial D_1 = e_2$ and $c = e_1$. Call the outermost piece of D_2: E, and the outermost and second outermost pieces of D_1: E' and E''.

For case (a): $c \subsetneq U_{F\times[t(a),1]}(E)$ and so $e_1 \cup e_2 \subseteq U_{F\times[t(a),1]}(E)$ and so $E' \subseteq U_{F\times[t(a),1]}(E)$. Also $c = e_1 \subseteq U_{F\times[t(a),1]}(E')$. So $\partial E' = e_2 \cup e_3$ where e_3 is essential

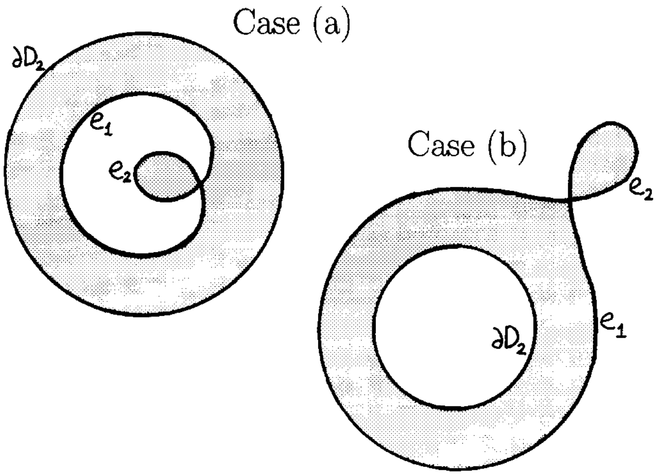

Fig. 16.

in $U_{F\times[0,t(a)]}(A^a)$. But then $E'' \subseteq U_{F\times[0,t(a)]}(A^a)$, E'' is not a disc, and $\partial E'' = e_3 \cup e_4$ where e_4 is enclosed in e_2 and so null-homotopic in A^a, contradiction.

For case (b): E must be an annulus with $e_1 \cup e_2 \subseteq U_{F\times[t(a),1]}(E)$. So $E' \subseteq U_{F\times[t(a),1]}(E)$, and $\partial E' = e_2 \cup e_3$ where e_2 is null-homotopic in $U_{F\times[t(a),1]}(E)$, and e_3 is not, since it must enclose $c = e_1$, contradiction.

Now that we know which circle is ∂D_1, we can determine D_1 and thus the whole of A^{a*}: D_2 is determined by n_2 which must be odd for case (a) and even for case (b). Given D_2, there are two possibilities for D_1, one with $n_1 = n_2$, and one with $n_1 = n_2 + 2$ (where $n_1 =$ the number of pieces in D_1). And so the pair (n_1, n_2) (where $n_1 = n_2$ or $n_2 + 2$), determines A^{a*} completely. (This includes the pair (2,0) for the case $m = 1$.) See Fig. 17 for the four possible combinations. In these figures, there is one part that is drawn as it is, and this is the little bump of A^a, the rest is a rotation surface around the depicted axis. Compare to the actual drawing of A^a in Fig. 15. The shaded area is the ball B' bounded by the sphere $A^{a*} \cup C$ where C is the disc in $K \times \{t(a)\}$ bounded by c.

Denote by B the apple shaped ball which is the union of the regions of all the pieces of A^{a*} (Fig. 18). Then $B' \subseteq B$. The boundary of B is made of three parts: Two pieces of A^{a*} which are a disc and an annulus, and a disc which is contained in $K \times \{t(a)\}$. Call them E', E'', and N, respectively. (E' and E'' are the innermost and second innermost pieces of D_1 or D_2, and N is the disc in $K \times \{t(a)\}$ bounded by $\partial(E' \cup E'')$.) For the case $(1,1)$, A^a is in the boundary of B and so B is actually a ball with a pinch at the singular point a' of A^a. E'' is A^a itself, and so it is a singular annulus, and N is a singular disc. This will cause us no disturbance.

We continue the proof of the lemma by induction on the number ℓ of circles in $H(A^{a*}) \cap H(F \times \{t(a)\} - A^{a*})$.

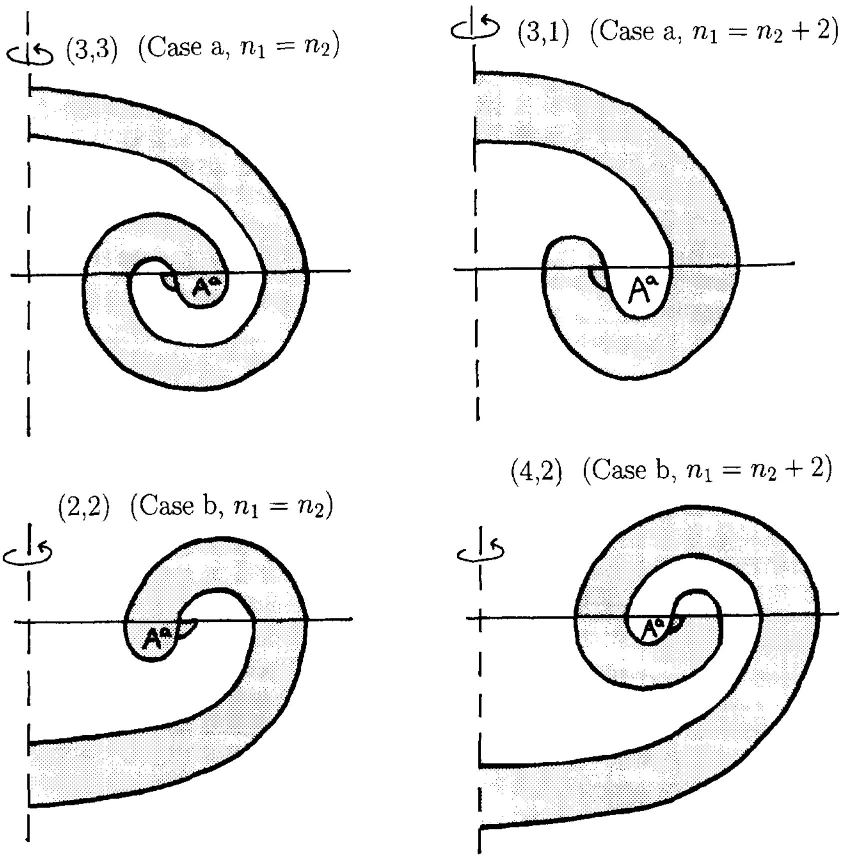

Fig. 17.

Assume $\ell = 0$. We show H embeds B. $N \subseteq F \times \{t(a)\}$ and so is embedded by H. $E' \cup E'' \subseteq A^{a*}$ and so is also embedded by H (Lemma 5.2). $\ell = 0$ implies $H(E' \cup E'') \cap H(\operatorname{int} N) = \emptyset$, and so ∂B is embedded by H, and so finally, by Lemma 5.3, B is embedded by H. It follows also that $B \cup F \times \{t(a)\}$ is embedded by H since $H(F \times \{t(a)\} - B) \cap H(\partial B) = \emptyset$, by the assumption $\ell = 0$. And so also some neighborhood of $B \cup F \times \{t(a)\}$ is embedded. By using Lemma 3.1 twice, once with E' and once with E'' we can turn this neighborhood into a neighborhood of the form $F \times [t_1, t_2]$ without changing the structure of the singular points of L. So now we have $B \subseteq F \times [t_1, t_2]$, with $F \times [t_1, t_2]$ embedded by H. So we can use Lemma 4.1 to change $L \cap B'$ inside B' in any way we please. (N of Lemma 4.1 may be taken here as the whole of $F \times [t_1, t_2]$, or just a neighborhood of B'.)

So we will now look into the structure of B'. (Fig. 17.) For the case $n_1 = n_2$, the shape of B' is of a TCD with an additional flat headed bump. (See Fig. 20 and the left side of Fig. 17.)

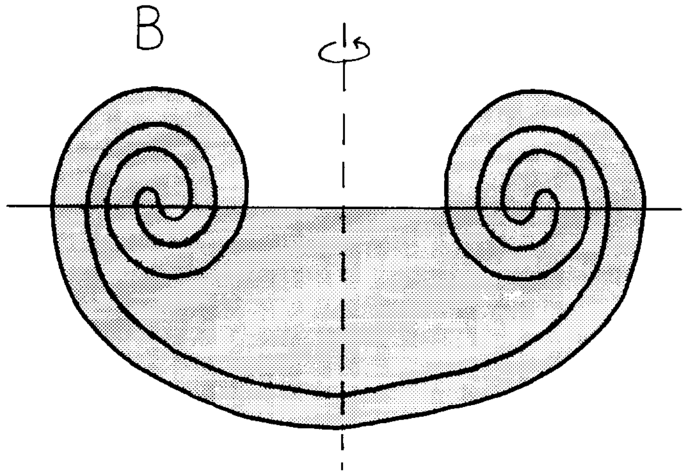

Fig. 18.

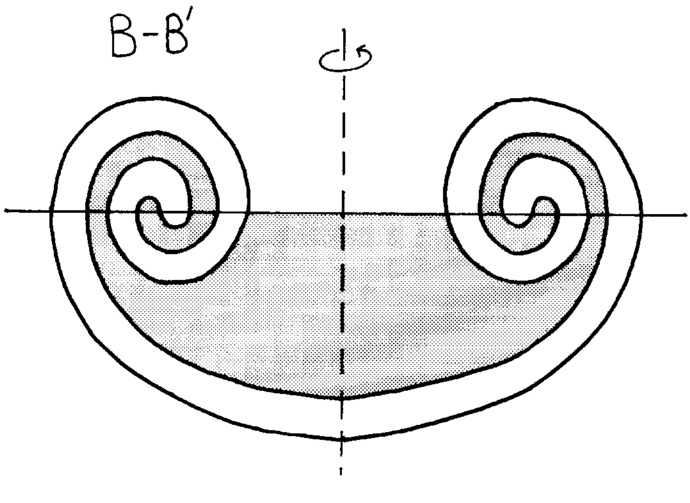

Fig. 19.

For $n_1 = n_2 + 2$, this bump is drilled out of the TCD. (See right side of Fig. 17.) We cut B' above c (Fig. 21), and think of the ball that is cut off B' as the bump, and all the rest, as the TCD.

So in any case, B' is a TCD with a bump. Call the TCD B'', and call the intersection of B'' and the bump, D''. By a level preserving change of coordinates we can assume $\partial B''$ is made of horizontal and vertical pieces, as in Lemma 4.6, or almost so (Fig. 22). We notice that D'' is situated in $\partial B''$ exactly in the place as assumed in Lemma 4.6.

What we now want to do, is to shrink A^{a*} inside B' until it has just one birth point and one death point. (In case B' reaches C from beneath, which happens when $n_1 = n_2$, we can actually get just one birth point. But if B' reaches C from above, which happens

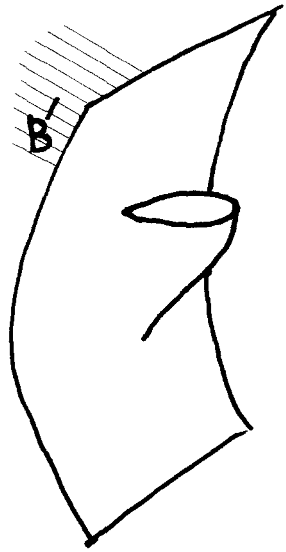

Fig. 20.

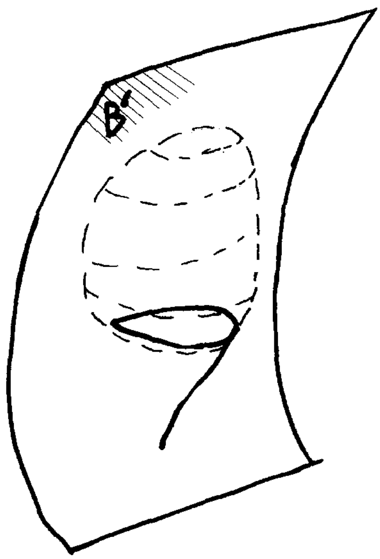

Fig. 21.

when $n_1 = n_2 + 2$, the best we can get is one birth point and one death point. This is because A^{a*} reaches ∂C from beneath.) But A^{a*} has at least three extremum points in all cases. So if we do this without changing the number of extremum points in the rest of L, we will reduce the number of extremum points of L.

We can perform such a shrinking of A^{a*} by moving only $A^{a*} \cap B''$ inside B''. Let U'' be the ball bounded by the shrunken $A^{a*} \cap B''$, and D'', though we did not perform

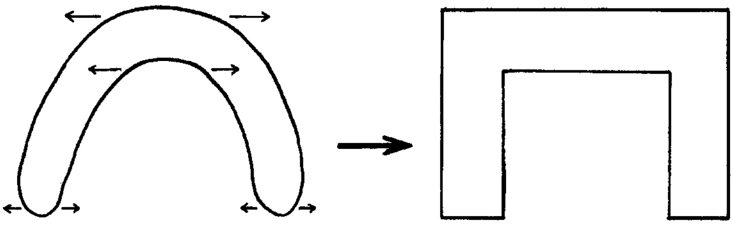

Fig. 22.

the shrinking yet. Let $L'' = L \cap \operatorname{int} B''$. Now use Lemma 4.6, with B'', L'', D'', U'' to deform L'' inside B'', until it is contained in U'', and such that the number of extremum points is unchanged. Finally we shrink $A^{a*} \cap B''$ inside B'' to the position we planned for it, thus reducing the number of extremum points.

So now assume $\ell > 0$.

By Lemmas 2.21 and 2.20 there is either an upper component A in $F \times [0, t(a)]$ or a lower component A in $F \times [t(a), 1]$ with $A \not\subseteq A^{a*}$ and $H(A) \subseteq \operatorname{int} H(A^{a*})$. If there is such an A which is not well embedded, say it is upper in $F \times [0, t(a)]$, we look at its last birth point e. By Lemmas 3.3(b), 5.5(b), and the minimality property of A^{a*}, we can use Lemma 3.4 to reduce the number of extremum points.

So assume from now on that they are all well embedded. If there is such an A, such that there is no point of A^{a*} in its region, and such that $H(\operatorname{int} A) \cap H(F \times \{t(a)\}) = \emptyset$, we can use Lemma 3.1(a) to push it to the other side of $F \times \{t(a)\}$ and thus reduce ℓ. We do not move A^{a*} by doing so since we assumed there are no points of A^{a*} in A's region.

If when doing this, A^{a*} loses its minimality property, i.e., if after this process, there is an extremum point b with $H(A^{b*}) \subsetneq H(A^{a*})$ then take an A^{c*} with $H(A^{c*}) \subseteq H(A^{b*})$, such that $H(A^{c*})$ is minimal, and start all over again. The induction here will be on the number of singular points in $\pi_S^{-1}(\pi_S(A^{a*}))$ (where A^{a*} is now viewed in $S \times [0, 1]$). This number is reduced, since A^{a*} has the singular point a' in it's boundary. (Remember our Assumption 1.1(v), this property can also be preserved whenever we perform a change in H_t. Note also that the procedure of Lemma 3.1(a) does not change the location in S of the singular occurrences, only their time, and also does not change H_0 and H_1.)

We now show that in fact there is always such an A, with no point of A^{a*} in its region, and such that $H(\operatorname{int} A) \cap H(F \times \{t(a)\}) = \emptyset$. We will work with case (b), as in Fig. 18. (Case (a) will follow exactly the same.) Denote $U = B \cap F \times [0, t(a)]$, $V = \operatorname{Cl}(B \cap F \times (t(a), 1])$. Then U is the region of a well embedded disc, and V is the region of a well embedded annulus.

Look at A^{a*} in $S \times [0, 1]$, and look at $\pi^{-1}(\pi(A^{a*}))$ ($\pi = \pi_S$). It includes A^{a*} itself, and some sheets above and under A^{a*}. They are all embedded by π (as in the proof of Proposition 3.1 of [4], since A^{a*} separates each one of them from either $S \times \{0\}$ or $S \times \{1\}$). Take such a sheet E and assume it is above A^{a*}. (E is a disc since otherwise there was a disc bounding circle of intersection between L and $\pi(A^{a*}) \times \{1\}$, and so a null-homotopic circle of intersection between $H(A^{a*})$ and $H_1(F)$ contradicting our assumption on A^{a*}.) E lies above A^{a*} in $S \times [0,1]$, and so ∂E is above $S \times \{t(a)\}$. (If ∂E has parts in $S \times \{1\}$ then they are surely above $S \times \{t(a)\}$. All other parts lie exactly above ∂A^{a*}, which is in $S \times \{t(a)\}$, and so are also above $S \times \{t(a)\}$.) Furthermore, the fact that E is above A^{a*} implies that the maximal point of E is higher than the maximal point of A^{a*}, and the minimal point of E is higher than the minimal point of A^{a*}. All the above applies of course also to the sheets E that are under A^{a*}, only now of course ∂E is under $S \times \{t(a)\}$, and the minimal and maximal points of E are lower than the minimal and maximal points of A^{a*}, respectively.

In this setting ℓ is simply the number of intersection circles between all the sheets of $\pi^{-1}(\pi(A^{a*}))$ except A^{a*} itself, and $S \times \{t(a)\}$.

Assume first that there are sheets which are under A^{a*}, that intersect $S \times \{t(a)\}$. Take a lowest one, call it E. $S \times \{t(a)\}$ cuts E into pieces. Since ∂E is under level $t(a)$, any piece A above $S \times \{t(a)\}$ is a lower surface in $S \times [t(a), 1]$, and we have $\pi(A) \subseteq \operatorname{int} \pi(A^{a*})$. Looking at A in $F \times [0,1]$, this means that A is a lower surface in $F \times [t(a), 1]$ and $H(A) \subseteq \operatorname{int} H(A^{a*})$, and so by our assumption above, A is well embedded in $F \times [t(a), 1]$. In $S \times [0,1]$ we have $\operatorname{int} U_{S\times[t(a),1]}(A) \cap L = \emptyset$ since E was chosen lowest, and so by Lemma 2.21, $H(\operatorname{int} A) \cap H(F \times \{t(a)\}) = \emptyset$.

So we must only show that such an A (when viewed in $F \times [0,1]$), does not have points of A^{a*} in its region. Such an A cannot have V in its region since then E would have a point above the highest point of A^{a*}. In particular, there is no such A that intersects both $F \times \{t(a)\} - B$ and N. Without such an A, it is impossible to get (in E) from points of $F \times [0, t(a)] - B$ into B. Since E has points lower than the lowest point of A^{a*}, it follows that E has no points in B. We conclude that all the pieces of E in $F \times [t(a), 1]$ do not have V in their region, and are not contained in V. And so their region is disjoint from V. In particular, they do not have points of A^{a*} in their region.

So we may assume that there is no sheet of $\pi^{-1}(\pi(A^{a*}))$ below A^{a*} that intersects $S \times \{t(a)\}$. So now A^{a*} itself is the lowest sheet intersecting $S \times \{t(a)\}$, and so as above it follows that all pieces A of A^{a*} which are above $S \times \{t(a)\}$ satisfy $\operatorname{int} U_{S\times[t(a),1]}(A) \cap L = \emptyset$. In particular, that piece A that when viewed in $F \times [0,1]$, has V as it's region. By Lemma 2.21, $V \cup F \times \{t(a)\}$ is embedded by H.

Since $\ell > 0$, there must be sheets of $\pi^{-1}(\pi(A^{a*}))$ above A^{a*} that intersect $S \times \{t(a)\}$. Take a highest one E. If there is a piece of E under $F \times \{t(a)\}$ (i.e., which is an upper component in $F \times [0, t(a)]$) that does not have points of A^{a*} in it's region then we are done as before. So assume there are none such. In particular, there are none with their region disjoint from U. There may also be none that contain U in their region since then E would have points below the lowest point of A^{a*}. We conclude that all such pieces are inside U. In particular, E does not intersect $F \times \{t(a)\} - B$. ∂E is above time $t(a)$. In $S \times [0,1]$ its points are either in $S \times \{1\}$ or in $\pi^{-1}(\pi(\partial A^{a*}))$. And so in $F \times [0,1]$ its points are either in $F \times \{1\}$, or mapped by H into $H(\partial A^{a*})$. Since V is embedded by H, and $\partial A^{a*} \subseteq V$, ∂E cannot intersect V. So ∂E is above $F \times \{t(a)\}$ and outside B.

We show E does not intersect B' (or else we may finish). It is enough if we show E does not intersect C (the disc in $F \times \{t(a)\}$ bounded by c). Since ∂E is outside B, the piece of E that contains ∂E may intersect $F \times \{t(a)\}$ only in N, and so does not intersect C. And so if E intersects C, then there is a piece of E which is an upper component in $F \times [0, t(a)]$ or a lower component in $F \times [t(a), 1]$ that intersect C. These are assumed to be well embedded.

A well embedded upper component A in $F \times [0, t(a)]$ that intersects C must be contained in $U_{F \times [0,t(a)]}(A^a)$. But then $U_{F \times [0,t(a)]}(A)$ can intersect $F \times \{t(a)\}$ only in C, and so has no points of A^{a*} in its region, and we are done. (Actually we assumed this does not happen.)

On the other hand, a well embedded lower component A in $F \times [t(a), 1]$ that intersects C, must be contained in the region of the piece of A^{a*} that touches c. Again A's region can intersect $F \times \{t(a)\}$ only in C, and we have no points of A^{a*} in A's region. We are done again, since now $A \subseteq V$, and $V \cup F \times \{t(a)\}$ is embedded by H, and so $H(\operatorname{int} A) \cap H(F \times \{t(a)\}) = \emptyset$.

So we now have E disjoint from B'. On the other hand in order for it to intersect $F \times \{t(a)\}$, it must intersect B. (Since it does not intersect $F \times \{t(a)\} - B$.) And so it intersects $B - B'$. $B - B'$ is more or less a TCD with a bump (or missing a bump), only that the innermost part of it, when cut by $F \times \{t(a)\}$ (the part containing N), is mushroom shaped (upside-down mushroom), rather than being thick-disc shaped. (See Fig. 19.)

Call this mushroom part K_1, call its neighboring part of $B - B'$, K_2, etc. Let i be the maximal such that E intersects K_i. Let A be a piece of E there. Since i is maximal, $\partial A \subseteq K_{i-1}$, and so A contains no part of B' in its region, and so no points of A^{a*}. A is either under $F \times \{t(a)\}$ or in V. In both cases we are done as before. □

6. Selfcontainedness and connectedness

Theorem 6.1. *If $i_1, i_2 \in T_0$ and there is a directed isotopy $H(x,t)$ from i_1 to i_2, then there is a directed isotopy from i_1 to i_2 that moves in T (Definition 2.25). Thus having no extremum points (by Lemma 2.26).*

In this sense, T is selfcontained. Alternatively, we may say that this theorem justifies our defining the directed graph structure of T by simply inducing it from I.

Proof. By Lemma 2.26 it is enough to show that there is a directed isotopy from i_1 to i_2 with no extremum points.

If there is a birth or death point of type 4, then by Remark 2.9 and Theorem 2.12, we can use Lemma 5.6 to reduce the number of extremum points.

So assume there are no extremum points of type 4. Take the last birth point a in L. It is not of type 4 since we assumed there are none such. It cannot be of type 2 since then there must be a later birth point. It can also not be of type 0 by Lemma 2.23(b). So it is of type 1. There are no upper components B of $L \times [0, t(a)]$ such that $H(B) \subseteq H(A^a)$ since that would again imply (by Lemma 2.20), that there are later birth points. So by Lemma 2.21 we may use Lemma 3.1(b) to reduce the number of extremum points. □

Proposition 6.2. *T and I are transitive, i.e., if there are directed edges $A \to B \to C$ then there is a directed edge $A \to C$.*

Proof. Let $H(x,t)$, $G(x,t)$ be directed isotopies such that $H_1 \sim_S G_0$. Then there is an isotopy $K : F \times [0,1] \to M$ from H_1 to G_0 with no singular points. Since K has no singular points, there is an isotopy $k_t : F \to F$ (with $k_0 = \mathrm{Id}_F$), such that $K'_t = K_t \circ k_t$ has $K_t'^{-1}(S)$ a fixed set, not changing with t. And so there will be an isotopy $K'' : M \times [0,1] \to M$ with $K''_0 = \mathrm{Id}_M$, $K''_t \circ H_1 = K'_t$ and $K''_t(S) = S$ for all t. Let $G'_t = G_t \circ k_1$. $(K''_1 \circ H) * G'$ is then a directed isotopy from $K''_1 \circ H_0$ to G'_1. But $K''_1 \circ H_0 \sim_S H_0$ and $G'_1 \sim_S G_1$. □

I is a connected graph, i.e., between any two elements of I there is a nondirected path. This is true since in the PL category, any isotopy is a sequence of basic isotopies, which occur inside a single simplex, and these basic isotopies are directed. (Actually they are not. But like in the proof of Lemma 3.1 we can boost them with a very slow movement of the rest of F in the right direction.)

We will now prove:

Theorem 6.3. *T is a connected graph.*

Proof. Let $A, B \in T$. Since I is connected, there is a nondirected path in I from A to B, i.e., there are $A = A_1, A_2, \ldots, A_n = B$, with edges say $A_1 \to A_2$, $A_2 \leftarrow A_3$, $A_3 \to A_4, \ldots$. (We used transitivity here.) To each such path we associate a pair of natural numbers (k,l), where k is the total number of intersection circles of F and S in $A_2, \ldots, A_{n-1}$ and l is the total number of extremum points in some choice of isotopies representing the edges $A_1 \to A_2$, $A_2 \leftarrow A_3$ etc. The proof will be by induction on the (k,l)'s which will be well ordered by lexicographic ordering. Let $H_t^{1\to 2}$, $H_t^{3\to 2}$, $H_t^{3\to 4}, \ldots$ be the chosen representing directed isotopies. By the proof of Proposition 6.2 we can assume $H_1^{1\to 2} = H_1^{3\to 2}$, $H_0^{3\to 2} = H_0^{3\to 4}$ etc.

If $A_2, \ldots, A_{n-1}$ are all in T, we are done since our path is actually in T. So assume that is not the case. Assume first that there exist null-homotopic circles of intersection between F and S for some of the A_i's.

Let c be such a circle, such that the disc D it bounds in S is minimal related to set inclusion among all such circles, and assume c appears in A_k. Say the two edges in our sequence are pointed to A_k (rather than being both pointed from A_k), i.e., we have $H_t^{(k-1)\to k}$ and $H_t^{(k+1)\to k}$. Call them H_t and H'_t, respectively.

There is a disc $E \subseteq H_1(F)$ $(= H'_1(F))$ such that $\partial E = \partial D$, and since D was minimal, $\operatorname{int} D \cap \operatorname{int} E = \emptyset$ and so $D \cup E$ is a sphere. Let B be the ball it bounds in M. If the preferred side of $H_1(F)$ is pointing from E into B, then there is a directed isotopy K_t with $K_0 = H_1$, that moves E across B (note that $H_1(F) \cap B = E$ by the minimality of D) and thus reduces the number of intersection circles of F and S. We replace H and H' by $H * K$ and $H' * K$. (k is reduced and l is increased.)

So assume now that the preferred side is pointed outward from B. The meaning of this is that if we consider $F \times [0,1]$ as contained in M^F via a lifting of H, and if we lift B to

M^F such that E is contained in $F \times \{1\}$, then a neighborhood in D of ∂D will be lifted into $F \times [0,1]$. (The minimality of D was used here again.) But since D is minimal, $\operatorname{int} D$ may not intersect $F \times \{0\}$ and $F \times \{1\}$ and so the whole of D is contained in $F \times [0,1]$. So we can consider D as an upper component of $L = L(H) \subseteq F \times [0,1]$. For the same reason $H(\operatorname{int} D) \cap H(F \times \{1\}) = \emptyset$. All this is true also for H' (with $L' = L(H')$). So if D is well embedded in $F \times [0,1]$, both as an upper component of L, and as an upper component of L', then we can use half of the proof of Lemma 3.1(a) to get rid of D, and whatever it bounds together with $F \times \{1\}$, both for H and for H'. The effect on both $H_1(F)$ and $H'_1(F)$ will be that they will be pushed across B, and so the new H and H' will satisfy $H_1 \sim_S H'_1$ as needed. (Here both k and l are reduced.)

So now assume D is not well embedded say as an upper component of L. If there is a birth or death point a of type 4 in L with $H(A^a) \subseteq H(D)$, then it will satisfy the assumptions of Lemma 5.6 by the minimality of D, and so we can reduce the number of extremum points. (k is unchanged and l is reduced.)

So assume there are no extremum points of type 4 with $H(A^a) \subseteq H(D)$. Let a be the last birth point in D. It must be of type 1, and it will satisfy the assumptions of Lemma 3.4. So again we may reduce the number of extremum points.

So we can now assume there are no null-homotopic intersection circles for any of the A_i's. Not all A_i are in T so for some of them there is a product region N between F and S. Let $F_N \subseteq F$, $S_N \subseteq S$ be the two parts of ∂N. Take N such that S_N is minimal among all N's in all A_i's. We will now repeat everything we did in the previous case, with S_N and N in place of D and B. The only difference is that in the previous case, we always ruled out intersection with D by the simple fact that a circle in a disc bounds a smaller disc, contradicting the minimality of D. In the new case, we will rule out intersections with S_N by the fact that an incompressible surface in N who's boundary is contained in S_N is boundary parallel, and so will give us a product region with smaller S_N. □

Corollary 6.4. *Let $[i], [j] \in T$. Then there is a (nondirected) isotopy H_t between i and j that moves in T and with no extremum points.*

Proof. This is clear from the proofs of Theorems 6.3 and 6.1. It also follows from Theorems 6.3 and 6.1 themselves together with the fact that we can fit isotopies together as in the proof of Proposition 6.2. □

There is a standard metric defined on connected graphs, namely, the distance between two elements is the minimal length of a path between them. With respect to this metric we have:

Theorem 6.5. *T is isometrically embedded into I.*

Proof. This is clear from the proof of Theorem 6.3, where we start with a path in I between two elements of T, and replace it by a path in T of the same length. □

Lemma 6.6. *Let $[i] \in T$ and assume T has more than one element. Then there is a directed isotopy $H_t : F \to M$ that satisfies:*

(1) *H has exactly one singular point, that being a saddle point.*

(2) *H moves in T.*

(3) *$H_0 \sim_S i$ or $H_1 \sim_S i$.*

(4) *H is an embedding.*

Proof. T is a connected graph with more than one element. So any element must have at least one edge connecting it to another element. Say there is an edge coming out of $[i]$ into another element. By Theorem 6.1 this edge may be represented by a directed isotopy G_t having no extremum points, and such that $G_t \in T_0$ for all nonsingular t. G_t does have singular points since it connects two distinct elements. Let t_0 be the first singular point. Take $G|_{F\times[t_0-\varepsilon,t_0+\varepsilon]}$. □

Theorem 6.7. *If either F or S is a torus then T has at most one element.*

Proof. Assume there is more than one element. Take H_t of Lemma 6.6 and let $L = L(H)$. L has a component which is a sphere with three holes. But by Lemmas 2.23(a) and 2.10, L is incompressible in both $F \times [0,1]$ and $S \times [0,1]$, so if either F or S is a torus, we have a contradiction. □

7. Structure of T

We now go a little deeper into the structure of T. By Theorem 6.7, if either F or S is a torus then T has at most one element, so whenever it is needed or comfortable we will assume that F and S are not tori.

Definition 7.1. M will be called circular with respect to F, if M is homeomorphic to $F \times [0,1]/(x,0) \sim (\varphi(x),1)$ where $\varphi : F \to F$ is a homeomorphism *of finite order* up to isotopy, and F corresponds to $F \times \{0\}$.

Shortly we will see that if M is circular with respect to F then it is circular with respect to any other incompressible surface in M (not a torus), and so we may just say: M is circular.

Lemma 7.2. *Let $H : F \times [0,1] \to M$ be a homotopy, with $H_0 = H_1$ an embedding in the isotopy class of the inclusion. Let $H' : F \times [0,1] \to M^F$ be a lifting, and assume $H'_0 \neq H'_1$. Then M is circular with respect to F.*

Proof. Take $c = \{*\} \times [0,1] \subseteq F \times [0,1]$. $H'_1(F)$ is a nontrivial translation of $H'_0(F)$ in M^F, and so $H'(c)$ is an open path, and so by definition of M^F, $H(c)$ has nonzero intersection number with $H_0(F)$. In particular, any power of $H(c)$ as an element of $\pi_1(M)$ does not lie in $H_{0*}(\pi_1(F))$. It follows that the map $h : F \times S^1 \to M$ induced by

H is π_1 injective, and so by Theorem 6.1 of [4], h is homotopic to a covering map h'. By the proof of that theorem, we can have $h'^{-1}(F) = F \times K$ with finite $K \subseteq S^1$, and with $h'|_{F\times\{k\}}$ a homeomorphism for each $k \in K$ (which is automatic if $F \neq$ torus). Since F is nonseparating, the conclusion follows. □

Theorem 7.3. *If M is circular with respect to F (Definition 7.1) then it is circular with respect to any other incompressible $S \neq$ torus, in M.*

Proof. By a theorem of Nielsen, we may assume φ is actually of finite order k (not just up to isotopy). And so the isotopy $K : M \times [0,1] \to M$ moving each point with uniform constant speed along its fiber, k times around M, has $K_0 = K_1 = \mathrm{Id}_M$. And so $G = K|_{S\times[0,1]}$ is an isotopy with $G_0 = G_1$. By Lemma 7.2 it is enough to show that for a lifting $G' : S \times [0,1] \to M^S$ of G, $G'_0 \neq G'_1$. By definition of M^S it is enough to show that the intersection number of $c = G(\{*\} \times [0,1])$ and S is $\neq 0$. The circle c lifts to the natural k-fold covering $\pi : F \times S^1 \to M$ as a fiber $\{*\} \times S^1$. The intersection number of c and S in M is equal to that of $\{*\} \times S^1$ and $\pi^{-1}(S)$ in $F \times S^1$. But this intersection number is nonzero since $\pi^{-1}(S)$ is not a torus and so by Theorem 5.2 of [3], $\pi^{-1}(S)$ is isotopic in $F \times S^1$ to a surface S' such that the restriction to S' of the projection $F \times S^1 \to F$, is a covering map. □

We will first consider the structure of T for M that is not circular.

Definition 7.4. For two vertices A, B in a directed graph G, we will write $A \rightarrowtail B$ if $A \neq B$ and there is a directed edge from A to B. (The distinction between $A \rightarrowtail B$ and $A \to B$ is like between $A < B$ and $A \leqslant B$ in a partially ordered set.)

Definition 7.5. A sequence $a_0 \rightarrowtail a_1 \rightarrowtail \cdots \rightarrowtail a_n$ in a directed graph G will be called a *chain* if for all $1 \leqslant i \leqslant n$, there is no x with $a_{i-1} \rightarrowtail x \rightarrowtail a_i$.

Definition 7.6. A directed graph G will be called graded, if there exists a function $d : G \to \mathbb{Z}$ such that for any $A \rightarrowtail B \in G$:

(a) There exists a chain $A = a_0 \rightarrowtail a_1 \rightarrowtail \cdots \rightarrowtail a_n = B$ in G.

(b) Any such chain has $n = d(B) - d(A)$.

Assume F separates M. For $i \in T_0$ let $M'(i)$ be the part of M on the preferred side of $i(F)$. For $[i] \in T$ define:

$$D([i]) = \chi\bigl(S \cap M'(i)\bigr).$$

Lemma 7.7. *Assume F separates M. If $H : F \times [0,1] \to M$ is a directed isotopy with a unique singular point, that point being a saddle point, then $D(H_1) = D(H_0) + 1$.*

Proof. By Lemma 2.2, H is an embedding. So, in the process of H, the change taking place in $S \cap M'(H_0)$, is that a sphere with three holes is removed from it. The conclusion follows since the Euler characteristic of a sphere with three holes is -1. □

Corollary 7.8. *Assume F separates M. If $H : F \times [0,1] \to M$ is a directed isotopy with n singular points, all being saddle points, then $D(H_1) = D(H_0) + n$.*

Corollary 7.9. *Assume F separates M. If $A \rightarrowtail B \in T$ then $D(A) < D(B)$.*

Proof. There is a directed isotopy $H : F \times [0,1] \to M$ with $H_0 \in A$ and $H_1 \in B$. By Theorem 6.1 we can take an H that has only saddle points. Since $A \neq B$ there is at least one, so by Corollary 7.8, $D(A) < D(B)$. □

Lemma 7.10. *Assume F separates M. Let $A \rightarrowtail B \in T$. Then $D(B) = D(A) + 1$ iff there is no $x \in T$ with $A \rightarrowtail x \rightarrowtail B$.*

Proof. If there is an $x \in T$ with $A \rightarrowtail x \rightarrowtail B$, then $D(A) < D(x) < D(B)$ and so $D(B) \geqslant D(A) + 2$. Assume now $D(B) \geqslant D(A) + 2$: By Theorem 6.1 there is a directed isotopy H with $H_0 \in A$ and $H_1 \in B$ such that H moves in T and has only saddle points. By Corollary 7.8 there are exactly $D(B) - D(A) \geqslant 2$ singular points. Take any nonsingular t_0 after the first singular time and before the last one and we will get: $A \to [H_{t_0}] \to B$ and $D(A) < D([H_{t_0}]) < D(B)$ and so $A \rightarrowtail [H_{t_0}] \rightarrowtail B$. □

Corollary 7.11. *Assume F separates M. Then $a_0 \rightarrowtail a_1 \rightarrowtail \cdots \rightarrowtail a_n \in T$ is a chain iff $D(a_n) - D(a_0) = n$.*

Now assume F does not separate M but M is not circular:

Choose $i_0 \in T_0$, and one lifting of i_0, $i'_0 : F \to M^F$. Now let $i \in T_0$ be arbitrary. Take an isotopy H_t with $H_0 = i_0$, $H_1 = i$. Take the lifting H'_t to M^F with $H'_0 = i'_0$. By Lemma 7.2, any two such Hs will give the same lifting H'_1 of i. So for any $i \in T_0$ we have chosen a canonical lifting to M^F which will be denoted i'.

We now define d on T_0 as follows: Let $i \in T_0$. Take two translations F_1, F_2 of $i'_0(F)$ in M^F such that the part N of M^F bounded by $F_1 \cup F_2$ contains both $i'_0(F)$ and $i'(F)$. Define D on N as above, and define $d(i) = D(i') - D(i'_0)$. It is clear d does not depend on the translations F_1 and F_2. We must show that if $i \sim_S j$ then $d(i) = d(j)$. Let H_t be an isotopy between i and j with no singular points. A lifting H'_t will also have no singular points. Again by Lemma 7.2, if we choose H' such that $H'_0 = i'$, then $H'_1 = j'$. And so $d(i) = d(j)$, and so d is defined on T. Now Lemmas/Corollaries 7.7–7.11 will all go through for F nonseparating and M not circular, with d in place of D.

So we now reformulate Lemmas/Corollaries 7.7–7.11 for the general case where M is not circular (setting $d = D$ when F does separate M):

Theorem 7.12. *If M is not circular (Definition 7.1), then:*

(1) *If $H : F \times [0,1] \to M$ is a directed isotopy with n singular points, all being saddle points, then $d(H_1) = d(H_0) + n$.*
(2) *If $A \rightarrowtail B \in T$ then $d(A) < d(B)$.*
(3) *$a_0 \rightarrowtail a_1 \rightarrowtail \cdots \rightarrowtail a_n \in T$ is a chain iff $d(a_n) - d(a_0) = n$.*

We are now ready to show T is graded:

Theorem 7.13. *If M is not circular (Definition 7.1), then T is a graded graph (Definition 7.6). If F separates M, then d is bounded on T and* $\max d - \min d \leqslant |\chi(S)|$.

Proof. (b) of the definition is true by Theorem 7.12(3).

For (a): If $A \rightarrowtail B \in T$, take a directed isotopy H with $H_0 \in A$, $H_1 \in B$ and such that H has only saddle points, and $H_t \in T_0$ for all nonsingular t (Theorem 6.1). By Theorem 7.12, if we take $0 = t_0 < t_1 < \cdots < t_n = 1$ nonsingular times such that there is exactly one singular time between t_{i-1} and t_i for all i, then $A = [H_{t_0}] \rightarrowtail [H_{t_1}] \rightarrowtail \cdots \rightarrowtail [H_{t_n}] = B$ is a chain.

For the bound on $\max d - \min d$ when F separates M: For any $i \in T_0$, both $S \cap M'(i)$ and $S - \operatorname{int} M'(i)$ have no disc components, and so they both have nonpositive Euler characteristic. So from $\chi(S \cap M'(i)) + \chi(S - \operatorname{int} M'(i)) = \chi(S)$ it follows that $\chi(S) \leqslant D(i) \leqslant 0$. □

Corollary 7.14. *If M is not circular then there are no $A \neq B \in T$ with $A \to B \to A$.*

Proof. If there were, then we would get $d(A) < d(B) < d(A)$, contradiction. □

Corollary 7.15. *If M is not circular, then if we replace the relation $\to$ by the relation $\leqslant$ we obtain a partially ordered set.*

We now deal with the case M is circular:

Theorem 7.16. *If M is circular (Definition 7.1) then T is a complete directed graph (i.e., there is an edge $A \to B$ for any pair A, B).*

Proof. We will show that if there is a directed edge $A \to B$ then there is also a directed edge $B \to A$. The conclusion will follow by the connectivity and transitivity of T.

Let $H : F \times [0,1] \to M$ be a directed isotopy with $H_0 \in A$, $H_1 \in B$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be, such that $H|_{F\times[t_i,t_{i+1}]}$ is an embedding for all i. It is enough that we show that for all i, there is a directed isotopy from $H_{t_{i+1}}$ to H_{t_i}. Since $H|_{F\times[t_i,t_{i+1}]}$ is an embedding, $M - \operatorname{int} H(F \times [t_i, t_{i+1}])$ is also of the form $F \times [0,1]$. So there is a directed isotopy from $H_{t_{i+1}}$ to an embedding j with $j(F) = H_{t_i}(F)$. By going now a finite number of times around M we can arrive at H_{t_i}. □

So if M is circular, T cannot be graded.

We would like to define a sort of grading on T after all. Let the homeomorphism $\varphi : F \to F$ that defines M be of order k and let $n = k \cdot |\chi(S)|$. We will define a "grading" $d : T \to \mathbb{Z}/n$ such that if $H : F \times [0,1] \to M$ is a directed isotopy with a unique singular point that being a saddle, with $H_0 \in A$, $H_1 \in B$ then $d(B) = d(A) + 1 \pmod n$.

The following is the complementary to Lemma 7.2:

Lemma 7.17. *Let M be circular, and k the order of the homeomorphism $\varphi: F \to F$ defining M. Let $H: F \times [0,1] \to M$ be a homotopy, with $H_0 = H_1$ an embedding in the isotopy class of the inclusion. Let $H': F \times [0,1] \to M^F$ be a lifting, then $H'_1 = (\tau^k)^m \circ H'_0$ for some integer m, where τ is the generating deck transformation of $M^F \to M$.*

Proof. $M^F = F \times \mathbb{R}$, with $\tau: F \times \mathbb{R} \to F \times \mathbb{R}$ given by $(x,t) \mapsto (\varphi(x), t+1)$. Since $H_0 = H_1$, $H'_1 = \tau^l \circ H'_0$, for some l. Let $\pi: F \times \mathbb{R} \to F$ be the projection, then $\pi \circ \tau = \varphi \circ \pi$. So we get $\pi \circ H'_1 = \varphi^l \circ \pi \circ H'_0$. Since $\pi \circ H'_i$ $(i = 0,1)$ are homotopic to Id_F, we get $\varphi^l = \mathrm{Id}_F$ and so $l = m \cdot k$. □

So we repeat the construction of the grading, with the single difference that the lifting of an $i \in T_0$ is defined only up to τ^k and so $d(i)$ is defined only $\bmod\, k \cdot |\chi(S)|$. d will be called a "circular" grading.

Note that there is a basic difference between the grading for noncircular M and the circular grading for circular M. A grading on a *connected* graph, if it exists, is unique up to an additive constant. So though in the case of noncircular M we defined the grading via the geometry, it could have been read (up to a constant) from the directed graph T alone. This is not true for the cyclic grading of a cyclic M, since nothing can be read off a complete graph. (So the cyclic grading embodies additional geometric information.)

We conclude this section with the following, which holds in all cases:

Corollary 7.18. *If H is a directed isotopy moving in T with a unique singular point (that being a saddle), then $[H_0] \neq [H_1]$.*

Proof. If either F or S is a torus then such an H does not exist, as in the proof of Theorem 6.7. If M is not circular, the conclusion is true by the grading. If M is circular, it is true by the circular grading since as we have said, S cannot be a torus, and so $n = k \cdot |\chi(S)| \geqslant 2$. □

8. Applications

If F and S are least area surfaces in a Riemannian manifold, and F and S are transverse, then there are no product regions between F and S. In case F and S are least area but not transverse then the situation is as follows: $F \cap S$ is a graph whose vertices are precisely the points of tangency between F and S, and with any slight movement that resolves the tangencies, product regions will not appear.

In [1] it is shown that if F and S are least area surfaces, and either F or S is a torus, then the intersection between them must be transverse, and the number of circles of intersection between them is the minimal possible in their homotopy classes. (In fact [1] deals with immersed surfaces and so curves of intersection rather than circles, are counted.)

In view of the above discussion, we would like to know:

(A) What is the most general topological condition on F and S that will guarantee that whenever we isotope them to be nontransverse (but with $F \cap S$ a graph as described above), then there will be arbitrarily small movements which will create product regions (and so any least area surfaces isotopic to them must be transverse).

(B) What is the most general topological condition on F and S that will guarantee that whenever we isotope them to be transverse and without product regions, then the number of circles of intersection will be the minimal possible for the isotopy classes of F and S. (And so any transverse least area surfaces isotopic to them must have the minimal possible number of intersection circles.)

The answer to both questions is given in the following:

Theorem 8.1. *The following four conditions on F and S are equivalent:*

(1) *T has at most one element.*

(2) *F and S may be isotoped to satisfy the one line property (Definition 8.2 below).*

(3) *Whenever $i(F)$ and S are not transverse (i an embedding in the isotopy class of the inclusion), but $i(F) \cap S$ is a graph whose vertices are precisely the tangency points between $i(F)$ and S, then there are embeddings of F, arbitrarily close to i, and with product regions.*

(4) *For any $i \in T_0$, the number of circles of $i(F) \cap S$ is the minimal possible in I_0.*

Note that by Theorem 6.7, if either F or S is a torus then indeed the conditions of the theorem hold. We will prove the theorem by showing for each of conditions (2)–(4) separately, that it is equivalent to condition (1). And so we break the theorem into three separate Theorems 8.6, 8.7 and 8.9.

Definition 8.2. Let $p: \widetilde{M} \to M$ be the universal covering of M. $[i] \in I$ will be said to satisfy the one line property, if the intersection of any component of $p^{-1}(i(F))$ with any component of $p^{-1}(S)$, is empty or consists of one line.

Definition 8.3. Let $p: \widetilde{M} \to M$ be the universal cover. Let $i \in I_0$ and let $D \subseteq \widetilde{M}$ be a disc. D will be called a bigon (with respect to i), if $\partial D = c_1 \cup c_2$ where

(a) c_1 and c_2 are arcs and $c_1 \cap c_2 = \partial c_1 = \partial c_2$.

(b) $c_1 \subseteq p^{-1}(i(F))$ and $c_2 \subseteq p^{-1}(S)$.

(c) D is transversal with respect to $p^{-1}(i(F))$ and $p^{-1}(S)$.

If both points of $c_1 \cap c_2$ lie in lines of $p^{-1}(i(F)) \cap p^{-1}(S)$ (rather than circles), then they may lie in the same line, or in two distinct lines. D will then be called a 1-line bigon or a 2-line bigon, respectively.

If $D \cap p^{-1}(i(F) \cup S) = \partial D$, then D will be called a minimal bigon.

The above terms will also be applied to a disc $D \subseteq M$ simply by lifting it to $\widetilde{M}$.

Lemma 8.4. *Assume $T \neq \emptyset$. If $[i] \in I$ satisfies the one line property then $[i] \in T$.*

Proof. Assume $[i]$ satisfies the one line property but there is a product region between $i(F)$ and S. If there are null-homotopic circles of intersection between $i(F)$ and S then they will lift to the universal cover, contradicting the one line property. So we have a product region as in (1) of Lemma 2.8 with K not a disc, and by Theorem 2.12, $\partial K \neq \emptyset$. Take an arc $(\gamma, \partial\gamma) \subseteq (K, \partial K)$ that may not be homotoped into ∂K. Let $D \subseteq M$ be the disc corresponding to $\pi^{-1}(\gamma)$ where $\pi : K \times [0,1] \to K$ is the projection. D is a minimal bigon in M. We lift D to the universal covering space $\widetilde{M}$. Let $\widetilde{F}$ and $\widetilde{S}$ be the components of $p^{-1}(i(F))$ and $p^{-1}(S)$, respectively, that touch D in $\widetilde{M}$. $\widetilde{F}$ and $\widetilde{S}$ intersect in a line ℓ containing $c_1 \cap c_2$ (c_1, c_2 of Definition 8.3). Let E be the disc in $\widetilde{F}$ bounded by c_1 together with the segment of ℓ bounded by $c_1 \cap c_2$. Project E to $K \times \{0\} \subseteq M$. E gives a homotopy of $\gamma \times \{0\}$ into $\partial K \times \{0\}$. Contradiction. □

Lemma 8.5. *Assume $i \in T_0$ and i does not satisfy the one line property, then there is a minimal 2-line bigon in M.*

Proof. We will first find a 2-line bigon in $\widetilde{M}$, then a minimal 2-line bigon in $\widetilde{M}$ and finally a minimal 2-line bigon in M.

2-line bigon in $\widetilde{M}$:

Take a component $\widetilde{F}$ of $p^{-1}(i(F))$ and a component $\widetilde{S}$ of $p^{-1}(S)$ that intersect each other in more than one line. $\widetilde{F}$ cuts $\widetilde{S}$ into pieces. Take one such piece A which intersects $\widetilde{F}$ at more than one line (those are boundary lines of A). In $\widetilde{F}$ let γ be an arc with $\gamma \cap A = \partial\gamma$ and γ connects two distinct lines of $A \cap \widetilde{F}$. Let $B \subseteq \widetilde{M}$ be a ball contained in the same side of $\widetilde{F}$ as A, and such that $U = B \cap \widetilde{F}$ $(= \partial B \cap \widetilde{F})$, is a small regular neighborhood of γ in $\widetilde{F}$, so that $A \cap U$ consist of just two little segments a and b (forming with γ the shape of the letter I). Assume ∂B intersects A transversally and $\partial B - \operatorname{int} U$ intersects $\widetilde{F}$ transversally. Furthermore assume that B is large enough such that it contains a path in A connecting the two endpoints of γ, meaning that there is a component A' of $A \cap B$ connecting the two endpoints of γ. The segments a and b lie on the boundary of A'. We will now show that a and b are contained in the same boundary circle of A'. Assume on the contrary that a and b are contained in distinct circles a' and b' of $\partial A'$. Let a'' be a circle in $\operatorname{int} A'$ which is close and parallel to a'. a'' bounds a disc E in $\widetilde{S}$ and $a'' \cap \widetilde{F} = \emptyset$, and so also $E \cap \widetilde{F} = \emptyset$ since otherwise we would have a circle of intersection between $\widetilde{S}$ and $\widetilde{F}$, contradicting $i \in T_0$. Since $E \cap \widetilde{F} = \emptyset$, E cannot contain a' ($a' \cap \widetilde{F} = a \neq \emptyset$), so it lies on the other side of a'' than a'. So the disc E' which is E together with the thin annulus between a'' and a' must contain A'. But then E must contain b, contradicting the fact that $E \cap \widetilde{F} = \emptyset$. So we have shown a and b lie in the same boundary circle of A'. This means that there is an arc c in $\partial A'$ connecting an endpoint of a to an endpoint of b (c lies in $\partial B - \operatorname{int} U$). c together with the appropriate arc c' of ∂U bound together a piece of ∂B which is a 2-line bigon in $\widetilde{M}$.

Minimal 2-line bigon in $\widetilde{M}$:

Take a 2-line bigon in $\widetilde{M}$. If it is minimal we are done. We proceed by induction on the number of points in $D \cap p^{-1}(i(F)) \cap p^{-1}(S)$. Assume there is a circle of intersection of D with $p^{-1}(i(F))$ which does not intersect $p^{-1}(S)$ (or vice versa). And so the whole disc it bounds in $p^{-1}(i(F))$ is disjoint from $p^{-1}(S)$. Take a minimal such disc in $p^{-1}(i(F))$ and homotope D to the other side of it, to reduce the number of such circles. So assume there are none such circles, so there must be a minimal bigon D' in D. If it is a 2-line bigon we are done. Otherwise it bounds a ball B together with a piece $f \subseteq p^{-1}(i(F))$ and a piece $s \subseteq p^{-1}(S)$. Isotope D' together with any other parts of D in B to the other side of $f \cup s$. (If $D' \cap \partial D \neq \emptyset$ and so $D' \cap \partial D$ is one of the arcs of $\partial D'$ then the isotopy is assumed to move this arc in $p^{-1}(i(F))$ or $p^{-1}(S)$.) The number of circles or arcs of intersection might go up, but the number of points of $D \cap p^{-1}(i(F)) \cap p^{-1}(S)$ is reduced.

Minimal 2-line bigon in M:

Take a minimal 2-line bigon D in $\widetilde{M}$. D need not be embedded into M by p. But we do know that $(p|_D)^{-1}(i(F) \cup S) = \partial D$ and $(p|_D)^{-1}(i(F) \cap S)$ consists of exactly two points. By moving D a bit if necessary, we may assume that these two points are not mapped by p into the same point. Cut M along $i(F) \cup S$, and let the component where $p(D)$ lies be called N. $p|_{\partial D} : \partial D \to \partial N$ is essential, since it being null-homotopic would imply that ∂D bounds a disc in the boundary of the manifold $\widetilde{N}$ obtained by cutting up $\widetilde{M}$ along $p^{-1}(i(F) \cup S)$. But such a disc must contain an arc of intersection of $p^{-1}(i(F))$ and $p^{-1}(S)$ connecting the two points of $p^{-1}(i(F)) \cap p^{-1}(S)$ on ∂D, contradicting the fact that D was a 2-line bigon. So by the more detailed formulation of the loop theorem [2, 4.10], there is an embedded disc D' in N with $\partial D'$ essential in ∂N and $\partial D'$ crosses $i(F) \cap S$ at most twice. If there are no such crossings, then $\partial D'$ is contained in say $i(F)$, and must bound a disc there, contradicting the fact that $\partial D'$ is essential in ∂N and $i(F)$ and S do not intersect in null-homotopic circles. So $\partial D'$ crosses $i(F) \cap S$ at least once. Just once is impossible, so we have exactly two crossings. So D' is a minimal bigon in M. It is infact a minimal 2-line bigon, since if it was 1-line, then when lifting D' to $\widetilde{N}$ we would find a disc bounded by $\partial D'$ in $\partial \widetilde{N}$. Projecting that disc back to ∂N would be a null-homotopy of $\partial D'$ in ∂N. □

Theorem 8.6. *T has at most one element iff there is an $[i] \in I$ satisfying the one line property.* (*Definition* 8.2.)

Proof. If $T = \emptyset$ this is clear by Theorem 2.12, so assume $T \neq \emptyset$.

Assume there is an $[i] \in I$ satisfying the one line property, then by Lemma 8.4 $[i] \in T$. If T has more than one element, take the isotopy H_t given by Lemma 6.6. Say $H_0 \sim_S i$, and so H_0 satisfies the one line property. The saddle point means that there is a minimal bigon D in M and H_t moves F along D. We lift D to the universal covering space $\widetilde{M}$. Let $\widetilde{F}$ and $\widetilde{S}$ be the components of $p^{-1}(H_0(F))$ and $p^{-1}(S)$, respectively, that touch D in $\widetilde{M}$. $\widetilde{F}$ and $\widetilde{S}$ intersect in a line containing $c_1 \cap c_2$. (c_1, c_2 of Definition 8.3.) Look at all the translations of D in $\widetilde{M}$ that touch $\widetilde{F}$ and $\widetilde{S}$. They will all be on the same side of $\widetilde{F}$ and on the same side of $\widetilde{S}$ since these sides are well defined by the orientations of F, S and M. So they will all lie in the same quarter space defined by the pair of planes $\widetilde{F}$, $\widetilde{S}$. Call the boundary of this quarter space N. (N is a bent plane made of half of $\widetilde{F}$ and half of $\widetilde{S}$.) Any such translate of D bounds with N a ball region U. Take such a translate D' with minimal region. (Actually one can show that these regions are all disjoint and so they are all minimal.) Now, as H_t moves F along D, $\widetilde{F}$ moves along D' and creates a circle of intersection between $\widetilde{F}$ and $\widetilde{S}$, and so there is a null-homotopic circle of intersection between $H_1(F)$ and S. This contradicts $H_1 \in T_0$. (One can show that U is embedded by p, and so $p(U)$ becomes the ball region between $H_1(F)$ and S.)

We now prove the converse: Assume there is no $[i] \in I$ satisfying the one line property. Take some $i \in T_0$. i does not satisfy the one line property, so there is a minimal 2-line bigon D in M (Lemma 8.5). Take a directed isotopy with $H_0 = i$ (or $H_1 = i$), moving F along D to create one saddle move, and with H an embedding. If also $H_1 \in T_0$ then by Corollary 7.18 we are done. So assume $H_1 \notin T_0$. So there is a product region U between $H_1(F)$ and S, as in (1) of Lemma 2.8, since even if K of the lemma is a disc (which will actually turn out to be impossible), the number of null-homotopic circles of intersection at time 1 is at most one, since at time 0 there are none, and times 0 and 1 differ by a single saddle. U must lie on the other side of $H_1(F)$ than $H(F \times [0,1])$ since otherwise U would intersect $H_0(F)$ and we would get a product region between $H_0(F)$ and S. Moving back in time from H_1 to the saddle point, we examine what might happen to U. There are essentially the four possibilities described in the discussion preceding Lemma 3.2, where $A = S \cap U$ is in place of the well embedded surface in $F \times [0,1]$ and time is going backwards.

Type 4 is impossible since by the analog of Lemma 3.2, we would have a null-homotopic circle of intersection between $H_0(F)$ and S.

Types 3 and 0 are impossible since we would then still have a product region at time 0.

Type 2 is impossible since that would contradict the fact that we are in (1) of Lemma 2.8.

So we have shown that we must have type 1. In particular, $D \cap S$ separates the component of $S - H_0(F)$ in which it lies.

A is not a disc since that would contradict the fact that D is a 2-line bigon. So there is a minimal bigon $D' \subseteq U$ which does not separate U and which does not touch ∂A in the area that is about to merge (in reversed time), with another piece of S. This may be extended to a minimal bigon D'' between $H_0(F)$ and S which is disjoint from D. Now use D'' inplace of D to define an H with a single saddle. H_1 must be in T_0 since this time $D'' \cap S$ does not separate $S - H_0(F)$. So we are done by Corollary 7.18. □

Theorem 8.7. *T has at most one element iff whenever the intersection between $i(F)$ and S is nontransverse, but $i(F) \cap S$ is a graph whose vertices are the tangency points, then we can change i in arbitrarily small neighborhoods of the tangency points (and without introducing new little circles of intersection in those neighborhoods), to get an embedding which is transverse and with product regions.*

Proof. Assume T has at most one element. Move i (according to the restrictions) such that there will be a single point p where the intersection is not transverse, that point being a saddle point. There are two sides to which we can move F in a neighborhood of p to resolve this singularity. If both of them are in T_0 then we would have a directed isotopy moving in T having exactly one singular point that being a saddle. By Corollary 7.18 this isotopy connects two distinct elements of T, contradicting our assumption. So we move F (according to the restrictions) to a side which is not in T_0, which means we will have product regions.

Now assume T has at least two elements. Take H_t of Lemma 6.6. Take $i = H_{t_0}$ where t_0 is the time of the unique saddle. Then any small movement will move us into either $[H_0]$ or $[H_1]$ which are both in T. □

Definition 8.8. For $i \in I_0$, $d(i(F), S)$ will denote the number of intersection circles between $i(F)$ and S.

Theorem 8.9. *T has at most one element iff for any $[i] \in T$, $d(i(F), S)$ is the minimal possible in I_0. (Definition* 8.8.)

Proof. Assume T has at most one element. If $T = \emptyset$ then the conclusion is (vacuously) satisfied. So assume T has exactly one element. Due to Lemma 2.8, given any $j \in I_0$ we can eliminate the product regions one by one, by isotopies, each time reducing the number of intersection circles. We finally must arrive at the unique element $[i]$ of T. So $d(i(F), S)$ is the minimal possible in I_0.

Now if T has at least two elements. Take H_t of Lemma 6.6. Then $d(H_0(F), S)$ and $d(H_1(F), S)$ differ by exactly 1, so at least one of them is not minimal. □

This completes the proof of Theorem 8.1.

We now consider a second type of intersection number between F and S:

Definition 8.10. Let $p: M' \to M$ be the covering of M corresponding to $\pi_1(F)$. Let $i: F \to M$ be isotopic to the inclusion and $i': F \to M'$ a lifting to M'. $D(i(F), S)$ is defined to be the number of components of $p^{-1}(S)$ that intersect $i'(F)$. (See [1].)

Theorem 8.11. *For any $[i] \in T$, $D(i(F), S)$ is the minimal possible for any embedding isotopic to the inclusion. (Definition* 8.10.)

Proof. First we show that for any $[i], [j] \in T$, $D(i(F), S) = D(j(F), S)$. By Corollary 6.4 there is an isotopy H_t between i and j with only saddle points. Let H_t' be a lifting of H_t to M'. H_t' too has only saddle points. Such a "saddle move" is something happening within one component of $p^{-1}(S)$, this component intersects $H_t'(F)$ before and after the move, and so the number of components intersecting $H_t'(F)$ remains the same.

We now show that if $i: F \to M$ is isotopic to the inclusion and $i \notin T_0$ then $D(i(F), S) \geqslant D(j(F), S)$ for some $j \in T_0$. By a small isotopy not changing D we can assume $i \in I_0$. Due to Lemma 2.8, we can eliminate the product regions one by one, by isotopies, until there are no product regions. Every such move, when looking in M', involves only components of $p^{-1}(S)$ that were already intersecting the lifting of F at that time, and so D can only be reduced. □

The following may be viewed as a special case of Theorem 8.6 or Theorem 8.11. We will deduce it directly from Corollary 6.4.

Corollary 8.12. *Let $[i] \in T$, j isotopic to i, and $j(F) \cap S = \emptyset$. Then*
(1) *T has one element. (In particular, $[i] = [j]$ and $i(F) \cap S = \emptyset$.)*
(2) *There is an isotopy between i and j in $M - S$.*

Proof. $j(F) \cap S = \emptyset$ and so a product region between $j(F)$ and S would imply F is isotopic to S and so by Theorem 2.12, $T = \emptyset$, contradicting $[i] \in T$. So $[j] \in T$. By Corollary 6.4 there is an isotopy H_t from j to i with no extremum points. Since $j(F) \cap S = \emptyset$, we must have $H_t(F) \cap S = \emptyset$ for all t. (The first t where $H_t(F) \cap S \neq \emptyset$ must be a birth point.) In particular, $[i] = [j]$. And since any $i' \in T_0$ is isotopic to this same j, T has just one element. □

Note that in case $T = \emptyset$, we have (unless $M = S \times S^1$) two isotopic embeddings of F with images disjoint from S, but not isotopic in $M - S$.

9. Reversing the roles of F and S

In the definition of the directed graph T, F and S had different roles. Denote by $\widehat{T}$ the graph obtained when the roles of F and S are reversed. We will give a natural bijection between the set of vertices of T and $\widehat{T}$. Given T, we will define $r(T)$, which will be a new, nondirected graph structure on the set of vertices of T, closely related to the directed graph structure T. We will show that the above bijection is an isomorphism of graphs between $r(T)$ and $r(\widehat{T})$. Since this is trivial when either F or is S is a torus (Theorem 6.7), we will assume throughout this section that *F and S are not tori.*

We first prove the following, which is probably well known. (In the proofs of [4] something a bit different appears.)

Lemma 9.1. *Assume F is not a torus. Let $h: M \to M$ be a homeomorphism that is isotopic to the identity and such that $h(x) = x$ for all $x \in F$. Then there is an isotopy $K: M \times [0,1] \to M$ from the identity to h with $K(x,t) = x$ for all $x \in F$, $t \in [0,1]$.*

Proof. Assume first that M is not circular. By the proof of Theorem 7.1 of [4], it is enough to show that there is a homotopy K satisfying the conclusion. For this it is enough to show that if $H: F \times [0,1] \to M$ is a map with $H_0(x) = H_1(x) = x$ for all $x \in F$ then H is homotopic, keeping $\partial(F \times [0,1])$ fixed, to the map $F \times [0,1] \to M$ given by $(x,t) \mapsto x$. For this it is enough to show that a circle $c = H(\{*\} \times [0,1])$ is null-homotopic in M. If no power of c lies in $\pi_1(F)$ then M is circular as in Lemma 7.2. So there is such a power k. If $c^k \neq 1$ we would get a nontrivial element in the center of $\pi_1(F)$ (since $\{*\} \times [0,1]$ as a circle in $F \times S^1$, is in the center of $\pi_1(F \times S^1)$). So $c^k = 1$, and so $c = 1$.

Now assume M is circular. The conclusion now follows from the fact that *any* homeomorphism $F \times [0,1] \to F \times [0,1]$ that is the identity on $\partial(F \times [0,1])$ is isotopic to the identity with $\partial(F \times [0,1])$ kept fixed. (By similar reasoning, again using the fact that the center of $\pi_1(F)$ is trivial.) □

We can now define the map $u : T \to \widehat{T}$. Let $[i] \in T$. There is an isotopy $H : F \times [0,1] \to M$ from the inclusion of F to i. H can be extended to an isotopy $H' : M \times [0,1] \to M$ with $H'_0 = \mathrm{Id}_M$. Define $u([i]) = [H_1'^{-1}|_S]$. We must show that the definition does not depend on the choice of representative i of $[i]$, the choice of isotopy H and the choice of it's extension H'. But first we show we get an element of $\widehat{T}$: $H_1'^{-1}|_S$ is isotopic to the inclusion of S by the isotopy $H_t'^{-1}|_S$. $H_1'^{-1}|_S(S)$ and F are transversal, and there are no product regions between them since H'_1 maps them into S and $i(F)$, respectively.

Now assume G is another isotopy from the inclusion of F to i, and G' an extension to M as above. (The case of two extensions of the same isotopy will just be a special case of this.) Look at the following isotopy of M: $K = H_1'^{-1} \circ (-H' * G')$ where $-H'$ is $H'(x, 1-t)$. Then $K_0 = \mathrm{Id}_M$ and $K_1(x) = x$ for all $x \in F$. By Lemma 9.1 there is an isotopy F_t between Id_M and $K_1 = H_1'^{-1} \circ G'_1$ that is constant on F. So finally $J_t = F_t \circ G_1'^{-1}$ is an isotopy between $G_1'^{-1}$ and $H_1'^{-1}$ that is constant on $i(F)$, and maps it onto F. $J_t|_S$ is the isotopy showing $G_1'^{-1}|_S \sim_F H_1'^{-1}|_S$.

So u is a well defined map from T_0 to $\widehat{T}$. Now assume $i, j \in T_0$ and $i \sim_S j$. Let H_t be an isotopy from the inclusion of F to i. Let K_t be an isotopy between i and j with no singular point (with respect to S). Let H' be an extension of H to M with $H'_0 = \mathrm{Id}_M$. Let K' be an extension of K to M with $K'_0 = H'_1$. Use H' and $G' = H' * K'$ to define $u(i)$ and $u(j)$. $K_t'^{-1}|_S$ is an isotopy between $H_1'^{-1}|_S$ and $G_1'^{-1}|_S$ with no singular points (with respect to F), showing $u(i) = u(j)$. So u is a well defined function from T to $\widehat{T}$.

We define $v : \widehat{T} \to T$ in the same manner. We show $v \circ u = \mathrm{Id}_T$ (and in the same way $u \circ v = \mathrm{Id}_{\widehat{T}}$.) Let $[i] \in T$. Let $H' : M \times [0,1] \to M$ be an isotopy with $H'_0 = \mathrm{Id}_M$ and $H'_1|_F = i$. So $u([i]) = [H_1'^{-1}|_S] \in \widehat{T}$, i.e., $H_1'^{-1}|_S$ is a representative of $u([i])$, and so $H_t'^{-1}$ can be used to define $v(u([i]))$. So $v(u([i])) = [(H_1'^{-1})^{-1}|_F] = [H'_1|_F] = [i]$.

We now define the nondirected graph structure $r(T)$ on the set of vertices of T as follows: there is an edge between A and B in $r(T)$ if there is a directed isotopy H with a unique singular point (that point being a saddle point), and with either $H_0 \in A$, $H_1 \in B$ or $H_0 \in B$, $H_1 \in A$. This is equivalent to the following: There is an $i \in A$ and a minimal bigon $D \subseteq M$ with respect to i, such that if one moves F (i.e., isotopes i), along D, one arrives at an element of B. (Note that by Corollary 7.18, an edge in $r(T)$ is always between distinct elements.)

Since T is connected, it is clear (using Theorem 6.1) that also $r(T)$ is connected.

We now show that u is an isomorphism of graphs between $r(T)$ and $r(\widehat{T})$, i.e., we must show that there is an edge between A and B in $r(T)$ iff there is an edge between $u(A)$ and $u(B)$ in $r(\widehat{T})$. It is enough to show the "only if" since the same will be true for v. So assume there is an edge between A and B in $r(T)$. As we have said above, this means that there is an $i \in A$ and a minimal bigon $D \subseteq M$ with respect to i, such that when we move F along D, we arrive at a representative j of B. Call this isotopy between i and j, H. Let K' be an isotopy with $K'_0 = \mathrm{Id}_M$ and $K'_1|_F = i$. Let H' be an extension of H to M with $H'_0 = K'_1$. So $u(A) = [H_0'^{-1}|_S]$ and $u(B) = [H_1'^{-1}|_S]$. $H_t'^{-1}|_S$ is an isotopy between them, moving S along $H_0'^{-1}(D)$. So there is an edge between $u(A)$ and $u(B)$.

Again there is a difference between the case of M noncircular and circular. When M is noncircular, then $r(T)$ may be read off T: There is a nondirected edge between A and B in $r(T)$ iff there is a directed edge between them in T and $|d(A) - d(B)| = 1$. (And d itself, as we have mentioned, may be read off T.) Furthermore, given $r(T)$ and d, we may reconstruct T: Each edge of $r(T)$ is given a direction according to the grading, then we pass to the directed transitive closure, and add all the edges $A \to A$.

When M is circular then again, nothing may be read off T.

For M noncircular, we may define the double grading $(d_1, d_2): r(T) \to \mathbb{Z} \times \mathbb{Z}$ by $d_1 = d$, $d_2 = \hat{d} \circ u$ where d, $\hat{d}$ are the gradings of T, $\widehat{T}$, respectively. We will write $\bar{d}(A) = (d_1(A), d_2(A))$. Note that if there is an edge between A and B in $r(T)$ then $\bar{d}(A)$ differs from $\bar{d}(B)$ by $(\pm 1, \pm 1)$. $r(T)$ is connected and so it follows that the parity of $d_1 + d_2$ is constant on $r(T)$. By adding a constant to say d_1, we can have that $d_1 + d_2$ is always even. So actually $\bar{d}: r(T) \to \overline{\mathbb{Z} \times \mathbb{Z}}$ where $\overline{\mathbb{Z} \times \mathbb{Z}} = \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}\colon z_1 + z_2 \text{ is even}\}$.

$r(T)$ together with this double grading is an object where F and S have symmetric roles, and that embodies in it the directed graph structures of both T and $\widehat{T}$.

We may define such a double circular grading on $r(T)$ for M circular too. In this case it will embody more information than just T and $\widehat{T}$ (which are simply complete graphs).

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