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Quadruple points of regular homotopies of surfaces in 3-manifolds

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Abstract

Let GI denote the space of all generic immersions of a surface F into a 3-manifold M. Let $q(H_t)$ denote the number mod 2 of quadruple points of a generic regular homotopy $H_t: F \to M$. We are interested in defining an invariant $Q: GI \to \mathbb{Z}/2$ such that $q(H_t) = Q(H_0) - Q(H_1)$ for any generic regular homotopy $H_t: F \to M$. Such an invariant exists iff $q = 0$ for any *closed* generic regular homotopy (abbreviated CGRH).

We prove that indeed $q(H_t) = 0$ for any CGRH $H_t: F \to \mathbb{R}^3$ where F is any system of surfaces. We prove the same for general 3-manifolds in place of $\mathbb{R}^3$ under certain assumptions. We demonstrate the need for these assumptions with various counter-examples. We give an explicit formula for the invariant Q for embeddings of a system of tori in $\mathbb{R}^3$. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

We begin with two remarks on terminology, followed by our two basic definitions:

1. A *surface* will always be compact and connected. A finite union of (compact connected) surfaces will be called a *system of surfaces*.
2. *Generic* immersions and *generic* regular homotopies are defined in a natural way. For a brief discussion see [5]. We will further always assume, that the initial and final immersions of a generic regular homotopy, are generic immersions.

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Definition 1.1. Let F be a system of closed surfaces and M a 3-manifold. A regular homotopy $H_t: F \to M$, $t \in [0,1]$ will be called *closed* if $H_0 = H_1$. We will denote a *closed generic regular homotopy* by CGRH.

Definition 1.2. Let F be a system of closed surfaces, M a 3-manifold and $H_t: F \to M$ a generic regular homotopy. The number mod 2 of quadruple points of H_t will be denoted by $q(H_t)$ ($\in \mathbb{Z}/2$).

Max and Banchoff in [5] proved that any generic regular homotopy of S^2 in $\mathbb{R}^3$ which "turns S^2 inside out," has an odd number of quadruple points. The main point was showing that any CGRH of S^2 in $\mathbb{R}^3$ has an *even* number of quadruple points. Goryunov in [1] expresses this from the Vassiliev Invariants point of view, as follows: Let $Imm(S^2, \mathbb{R}^3)$ be the space of all immersions of S^2 in $\mathbb{R}^3$, and let $\Delta \subseteq Imm(S^2, \mathbb{R}^3)$ be the subspace of all non-generic immersions. Choose some generic immersion $f_0: S^2 \to \mathbb{R}^3$ as a base immersion. For any generic immersion $f: S^2 \to \mathbb{R}^3$ let $Q(f) \in \mathbb{Z}/2$ be defined as $q(H_t)$ where H_t is some generic regular homotopy between f_0 and f. There exists such an H_t since $Imm(S^2,\mathbb{R}^3)$ is connected (by [6]) and this is well defined since any CGRH has $q = 0$. Furthermore, since generic immersions do not have quadruple points, Q will be constant on each connected component of $Imm(S^2, \mathbb{R}^3) - \Delta$. Goryunov then raises the question whether such a Q may be defined for any surface in $\mathbb{R}^3$, that is, whether for any CGRH of any surface F in $\mathbb{R}^3$ the number of quadruple points is $0 \bmod 2$. Q will then be specified by choosing one base immersion in each connected component of $Imm(F,\mathbb{R}^3)$.

We begin this work with a short alternative to the pictorial part of [5].

We then answer Goryunov's question to the affirmative in:

Theorem 3.9. *Let F be a system of closed surfaces, and let $H_t: F \to \mathbb{R}^3$ be any CGRH. Then $q(H_t) = 0$.*

This phenomenon is not true in general for any 3-manifold in place of $\mathbb{R}^3$, as we demonstrate in various examples in Section 4. However, we prove the following positive result:

Theorem 3.15. *Let M be an orientable irreducible 3-manifold with $\pi_3(M) = 0$. Let F be a system of closed orientable surfaces. If $H_t: F \to M$ is any CGRH in the regular homotopy class of an embedding, then $q(H_t) = 0$.*

Theorems 3.9 and 3.15 will both be proved by reduction to the following more fundamental result, which will be proved first:

Theorem 3.4. *Let M be any 3-manifold with $\pi_2(M) = \pi_3(M) = 0$. Let F be a system of closed surfaces and let $D \subseteq F$ be a system of discs, one disc in each component of F. If $H_t: F \to M$ is a CGRH that fixes D then $q(H_t) = 0$.*

In Section 5 we give an explicit formula for the above mentioned invariant Q for embeddings of a system of tori in $\mathbb{R}^3$. That is, if F is a union of tori, then for every embedding $f: F \to \mathbb{R}^3$ we assign $Q(f) \in \mathbb{Z}/2$ such that whenever H_t is a generic regular homotopy between two embeddings f and g, then $q(H_t) = Q(f) - Q(g)$.

2. S^2 in $\mathbb{R}^3$

Let $D \subseteq S^2$ be a disc, and let $Imm_D(S^2,\mathbb{R}^3)$ be the space of all immersions $f: S^2 \to \mathbb{R}^3$ such that $f|_D$ is some chosen embedding. $\pi_1(Imm_D(S^2,\mathbb{R}^3))$ is known to be $\mathbb{Z}$ (as will be explained below.) Max and Banchoff [5] presented a specific CGRH which has 0 mod 2 quadruple points, and proved that it is the generator of $\pi_1(Imm_D(S^2,\mathbb{R}^3))$. The proof used a sequence of 19 drawings of intermediate stages of the homotopy. Our following proposition is more general (and uses no pictures.)

Let $A:\mathbb{R}^3 \to \mathbb{R}^3$ be the rotation $(x,y,z) \mapsto (-x,-y,z)$, then $-A$ is the reflection $(x,y,z) \mapsto (x,y,-z)$. Let $U = S^2 - intD$ be parametrized as the unit disc in the xy-plane, and so the rotation $(x,y) \mapsto (-x,-y)$ acts on U.

Proposition 2.1. *Let D, U and A be as above. Let $f: S^2 \to \mathbb{R}^3$ be an immersion such that $f|_D$ is an embedding into the xy-plane, and such that $f|_U(-x,-y) = A \circ f|_U(x,y)$. Let $H_t: S^2 \to \mathbb{R}^3$ be a regular homotopy with $H_0 = f$, $H_1 = -A \circ f$ and which fixes D.*

Define $H'_t: F \to \mathbb{R}^3$ by $H'_t(x,y) = -H_t(-x,-y)$ on U, and H'_t is fixed on D. Finally define $G_t = H_t * H'_t$ (where $*$ denotes concatenation from left to right.)

Then G_t represents some odd power of the generator of $\pi_1(Imm_D(S^2,\mathbb{R}^3), f)$.

Proof. Given independent $v_1,v_2 \in \mathbb{R}^3$ let $K(v_1,v_2) \in SO_3$ be the unique matrix in SO_3 whose first two columns are obtained from v_1, v_2 by the Gram–Schmidt process.

Given a representative J_t of an element of $\pi_1(Imm_D(S^2,\mathbb{R}^3))$, we define the following map $\bar{J}: S^3 \to SO_3$, which will be regarded as an element of $\pi_3(SO_3)$. Parametrize S^3 as the quotient space of $U \times [0,1]$ with identifications $((x,y),0) \sim ((x,y),1)$ for any $(x,y) \in U$, and $((x,y),t) \sim ((x,y),t')$ for any $(x,y) \in \partial U$, $t,t' \in [0,1]$. And so S^3 contains one copy of ∂U which we will still call ∂U. Now let $\bar{J}: S^3 \to SO_3$ be defined by $\bar{J}((x,y),t) = K((\partial/\partial x)J_t(x,y), (\partial/\partial y)J_t(x,y))$. This is well defined since J_t fixes D and so the 2-frame $((\partial/\partial x)J_t(x,y), (\partial/\partial y)J_t(x,y))$ is fixed with respect to t on $\partial U = \partial D$.

Now, the map $J_t \mapsto \bar{J}$ induces an isomorphism $\pi_1(Imm_D(S^2,\mathbb{R}^3)) \to \pi_3(SO_3) = \mathbb{Z}$. (This follows from Smale's Theorem [6], and was also done in [5]. Following the more general Smale–Hirsch Theorem, we will perform the corresponding computation for surfaces and 3-manifolds which may be non-orientable, in the proof of Theorem 3.4 below. We will then have no global 2-frame on $F - D$ and no global 3-frame on M to work with.)

Still using our parametrization of S^3, let $c: S^3 \to S^3$ be defined by:

$$((x,y),t) \mapsto \left((-x,-y), \quad \left(t + \frac{1}{2}\right) \bmod 1\right).$$

For the right identification of S^3 with our parameter space, c is the antipodal map of S^3.

Now $H'_t(x,y) = -H_t(-x,-y)$, so also $H_t(x,y) = -H'_t(-x,-y)$. Together this means that $G_t(x,y) = -G_{(t+1/2) \bmod 1}(-x,-y)$. Let dJ_t denote the pair $((\partial/\partial x)J_t, (\partial/\partial y)J_t)$. Applying this d to both sides of the last identity gives $dG_t(x,y) = dG_{(t+1/2) \bmod 1}(-x,-y)$. Composing with K gives: $\bar{G} = \bar{G} \circ c$.

And so $\bar{G}$ induces a map g from S^3/c $(=\mathbb{R}P^3)$ to SO_3 $(=\mathbb{R}P^3)$. We need to show that $\bar{G}$ represents an odd power of the generator of $\pi_3(SO^3)$. This is equivalent to g inducing an isomorphism $H^3(\mathbb{R}P^3,\mathbb{Z}/2) \to H^3(\mathbb{R}P^3,\mathbb{Z}/2)$. If $a \in H^1(\mathbb{R}P^3,\mathbb{Z}/2)$ is the generator, then a^3 is the generator of $H^3(\mathbb{R}P^3, \mathbb{Z}/2)$ and so it is enough to show that g induces an isomorphism on $H^1(\mathbb{R}P^3, \mathbb{Z}/2)$, which is equivalent to it inducing an isomorphism on $\pi_1(\mathbb{R}P^3)$. So this is what we will now show.

Let X be the subgroup of SO_3 of matrices with third column $(0,0,1)$, then X is a circle. D remains fixed inside the xy plane, and so the tangent planes of S^2 at points of $\partial U = \partial D$ are horizontal, so it follows that $\bar{G}(\partial U) \subseteq X$ or X', where X' is the coset of X of all elements of SO_3 with third column $(0,0,-1)$. By change of coordinates $(x,y) \mapsto (x, -y)$ on U if needed (this preserves the conditions of the theorem) we may assume $\bar{G}(\partial U) \subseteq X$.

We now claim that $\bar{G}|_{\partial U}: \partial U \to X$ is a map of degree ± 2. (We are not orienting ∂U and X and so the sign is meaningless.) We look at say $G_0 = f$ for computing $\bar{G}$ on ∂U. A loop going once around X describes one full rotation of $\mathbb{R}^3$ around the z-axis. And so we need to check how many rotations does the horizontal 2-frame $((\partial/\partial x)f, (\partial/\partial y)f)$ perform with respect to the 2-frame $((1,0,0), (0,1,0))$, when traveling once around ∂U. But $((1,0,0), (0,1,0))$ is globally defined on $f(D)$ and $((\partial/\partial x)f, (\partial/\partial y)f)$ is globally defined on $f(U)$ (in the sense of immersions). And so, since the Euler number of TS^2 $(=\chi(S^2))$ is 2, we must have 2 relative rotations.

Since c preserves ∂U and $\bar{G} \circ c = \bar{G}$, it follows that half of ∂U, (from some $p \in \partial U$ to $c(p)$) is mapped by g with degree 1 onto X. But X as a loop, is the generator of $\pi_1(SO_3)$. And so the generator of $\pi_1(S^3/c)$ is mapped by g to the generator of $\pi_1(SO_3)$, which is what we needed to show.

The essential part of the Max–Banchoff theorem now follows:

Theorem 2.2. $q(J_t) = 0$ *for any CGRH* $J_t: S^2 \to \mathbb{R}^3$.

Proof. There exists a generic f satisfying the conditions of Proposition 2.1 (e.g. any embedded sphere of revolution which is parametrized as such). By [6] there exists a regular homotopy, and thus also a generic regular homotopy H_t from f to $-A \circ f$ which fixes D. The G_t which is constructed from H_t in Proposition 2.1, has exactly twice as many quadruple points as H_t, and so $q(G_t) = 0$. By Lemma 1 of [5] (see Lemma 3.2 below for a more general formulation) q is well defined on $\pi_1(Imm_D(S^2, \mathbb{R}^3), f)$ (i.e. two CGRHs in the same class have the same q). Since G_t represents an odd power of the generator, and $q(G_t) = 0$, we must also have $q = 0$ for the generator, and thus $q = 0$ for any element of $\pi_1(Imm_D(S^2, \mathbb{R}^3), f)$.

Now let $J_t: S^2 \to \mathbb{R}^3$ be any CGRH. By composing with a global isotopy of $\mathbb{R}^3$ we may assume $J_t|_D = f|_D$ for all t. Let $J'_t: S^2 \to \mathbb{R}^3$ be a generic regular homotopy from f to J_0 which fixes D. Then $J'_t * J_t * J'_{-t}$ represents an element of $\pi_1(Imm_D(S^2, \mathbb{R}^3), f)$ and so $q(J_t) = q(J'_t * J_t * J'_{-t}) = 0$. □

To complete the picture, we follow [5] from this point: There are two isotopy classes of embeddings of S^2 in $\mathbb{R}^3$. A regular homotopy from one of these isotopy classes to the other, is called an *eversion of the sphere*. By Theorem 2.2 any two generic eversions will have the same number mod 2 of quadruple points, so we need only to count this for one eversion. The Froissart–Morin eversion (see e.g. [4]) has exactly one quadruple point and so we get the Max–Banchoff Theorem:

Theorem 2.3. *Every generic eversion of the sphere has an odd number of quadruple points.*

3. Surfaces in 3-manifolds: positive results

Definition 3.1. Let F be a system of closed surfaces, and M a 3-manifold. Two regular homotopies $H_t, G_t : F \to M$ will be called *equivalent* if as paths in the space $Imm(F, M)$, they are homotopic relative to their endpoints. (This is stronger than just having the maps $H, G: F \times [0,1] \to M$ defined by H_t and G_t be homotopic relative $\partial(F \times [0,1])$.)

Lemma 3.2. *Let F be a system of closed surfaces, and M a 3-manifold. If $H_t, G_t : F \to M$ are equivalent generic regular homotopies then $q(H_t) = q(G_t)$.*

Proof. The proof follows exactly as in Lemma 1 of [5]. We only need to emphasize, that for a generic regular homotopy H_t, by definition, H_0 and H_1 are generic immersions, and so H_0 and H_1 do not have quadruple points. □

The following lemma is obvious:

Lemma 3.3. *Let F be a system of closed surfaces and M a 3-manifold. If $F^1, \dots, F^n$ are the connected components of F then*:

1. $Imm(F, M) = Imm(F^1, M) \times \cdots \times Imm(F^n, M)$.
2. *Any regular homotopy $H_t : F \to M$ is equivalent to a concatenation $H_t^1 * \cdots * H_t^n$ where each H_t^i fixes all components of F except F^i.*

As for the special case of S^2 in $\mathbb{R}^3$, we begin by assuming that H_t fixes discs:

Theorem 3.4. *Let M be a 3-manifold with $\pi_2(M) = \pi_3(M) = 0$. Let F be a system of closed surfaces and let $D \subseteq F$ be a system of discs, one disc in each component of F. If $H_t : F \to M$ is a CGRH that fixes D then $q(H_t) = 0$.*

Proof. We assume first that F is connected. The general case will follow easily.

The Smale–Hirsch Theorem [3] states that $d: Imm(F, M) \to Mon(TF, TM)$ is a weak homotopy equivalence, where $Mon(TF, TM)$ is the space of bundle monomorphisms $TF \to TM$ (i.e. bundle maps which are a monomorphism on each fiber) and d is the differential. From this it is easy to deduce, by means of [6] or [3], the following relative version: Let $Imm_D(F, M)$ be the space of all immersions $f: F \to M$ with $f|_D = H_0|_D$ and let $Mon_D(TF, TM)$ be the space of all bundle monomorphisms $f: TF \to TM$ with $f|_{TD} = dH_0|_{TD}$, then $d: Imm_D(F, M) \to Mon_D(TF, TM)$ is a weak homotopy equivalence. In particular: $d_*: \pi_1(Imm_D(F, M), H_0) \to \pi_1(Mon_D(TF, TM), dH_0)$ is an isomorphism.

Our H_t may be viewed as a representative of an element of $\pi_1(Imm_D(F, M), H_0)$. By Lemma 3.2, q is well defined for elements of $\pi_1(Imm_D(F, M), H_0)$. We will show that $\pi_1(Imm_D(F, M), H_0)$ $(= \pi_1(Mon_D(TF, TM), dH_0))$ is a cyclic group. We will then construct a CGRH G_t with $q(G_t) = 0$ and which is a representative of the generator of $\pi_1(Imm_D(F, M), H_0)$. It will follow that $q = 0$ for any generic representative of any element of $\pi_1(Imm_D(F, M), H_0)$, in particular for H_t.

Some definitions and notation: Given a bundle map $f: TF \to TM$ let $\hat{f}: F \to M$ denote the map that it covers. Denote by $Mon_{D,H_0}(TF,TM)$ the subspace of $Mon_D(TF,TM)$ of all $f \in Mon_D(TF,TM)$ with $\hat{f} = H_0$. Denote by $Map_D(F,M)$ the space of all maps $f: F \to M$ with $f|_D = H_0|_D$. For a space X denote by $\Omega_x X$ the loop space of X based at $x \in X$. If X is a space of maps $A \to B$ (as any of the above spaces), and $a \in \Omega_f X$ then for $t \in [0,1]$ the map $a(t)$ from A to B will be denoted by $a_t: A \to B$.

It is shown in [2] that any 3-manifold admits a connection, with parallel transport along any loop assigning either Id or $-Id$. Choose such a connection on M. Given $a \in \Omega_{H_0} Map_D(F,M)$ it defines isomorphisms $A_{xt}: T_{a_0(x)}M \to T_{a_t(x)}M$ given by parallel transport along the path $s \mapsto a_s(x), s \in [0,t]$. We will say A_{xt} is associated to a_t. We note that for any $x \in F$, A_{x1} is Id and not $-Id$, since the loop $t \mapsto a_t(x)$, $t \in [0,1]$ is null-homotopic. This is so since it is homotopic to $t \mapsto a_t(y)$, $t \in [0,1]$ with $y \in D$.

We now define a map

$$\Phi: \Omega_{dH_0} Mon_D(TF,TM) \to \Omega_{H_0} Map_D(F,M) \times \Omega_{dH_0} Mon_{D,H_0}(TF,TM)$$

as follows: Given a loop $u \in \Omega_{dH_0} Mon_D(TF,TM)$, let $A_{xt}: T_{\widehat{u_0}(x)}M \to T_{\widehat{u_t}(x)}M$ be the continuous family of isomorphisms associated to $\widehat{u_t}$. Let $\widehat{\widehat{u_t}}: TF \to TM$ be defined by $\widehat{\widehat{u_t}}|_{T_xF} = A_{xt}^{-1} \circ u_t|_{T_xF}$. Now let $\Phi(u_t) = (\widehat{u_t}, \widehat{\widehat{u_t}})$. Φ has the following inverse: given $(a,b) \in \Omega_{H_0} Map_D(F,M) \times \Omega_{dH_0} Mon_{D,H_0}(TF,TM)$, take the A_{xt} associated to a_t and define $\Phi^{-1}(a,b)$ by $(\Phi^{-1}(a,b))_t|_{T_xF} = A_{xt} \circ b_t|_{T_xF}$. And so Φ is a homeomorphism.

So

$$\begin{aligned}\pi_1(Mon_D(TF,TM), dH_0) &= \pi_0(\Omega_{dH_0} Mon_D(TF,TM)) \\ &= \pi_0(\Omega_{H_0} Map_D(F,M) \times \Omega_{dH_0} Mon_{D,H_0}(TF,TM)) \\ &= \pi_0(\Omega_{H_0} Map_D(F,M)) \times \pi_0(\Omega_{dH_0} Mon_{D,H_0}(TF,TM)).\end{aligned}$$

We will first show that $\pi_0(\Omega_{H_0} Map_D(F,M)) = 0$. An element in $\Omega_{H_0} Map_D(F,M)$ is a map $h: F \times [0,1] \to M$ such that $h(x,t) = H_0(x)$ whenever $(x,t) \in F \times \{0\} \cup F \times \{1\} \cup D \times [0,1]$. Let SF be the quotient space of $F \times [0,1]$ obtained by identifying $(x,0) \sim (x,1)$ for any $x \in F$ and $(x,t) \sim (x,t')$ for any $x \in D$, $t,t' \in [0,1]$. There is a natural inclusion $F = F \times \{0\} \subseteq SF$. A map $h: F \times [0,1] \to M$ satisfying the above conditions is equivalent to a map $h: SF \to M$ with $h(x) = H_0(x)$ for $x \in F$. We claim that any two such maps are homotopic relative F, which means that $\pi_0(\Omega_{H_0} Map_D(F,M))$ has just one element. This is so, since there is a CW decomposition of SF with F being a subcomplex and $SF - F$ containing only 2-cells and 3-cells, and since $\pi_2(M) = \pi_3(M) = 0$.

We now deal with $\pi_0(\Omega_{dH_0} Mon_{D,H_0}(TF,TM))$. For $x \in M$ let $GL^+(T_xM)$ be the group of orientation preserving automorphisms of T_xM (i.e. automorphisms with positive determinant.) Let GL^+TM be the bundle over M whose fiber over x is $GL^+(T_xM)$. Let $Map_{D,H_0}(F, GL^+TM)$ be the space of all maps $f: F \to GL^+TM$ that lift H_0 and such that $f(x) = Id_{T_{H_0(x)}M}$ for every $x \in D$.

We define a map $\psi: Mon_{D,H_0}(TF,TM) \to Map_{D,H_0}(F, GL^+TM)$ as follows: Choose a Riemannian metric on M. Let Mon_x denote the space of linear monomorphisms $T_xF \to T_{H_0(x)}M$. We first define a map $S_x: Mon_x \times Mon_x \to GL^+(T_{H_0(x)}M)$ as follows: $S_x(\phi_1, \phi_2)$ is defined to be the unique element in $GL^+(T_{H_0(x)}M)$ which extends $\phi_2 \circ \phi_1^{-1}: \phi_1(T_xF) \to T_{H_0(x)}M$ and sends a unit vector normal to $\phi_1(T_xF)$ into a unit vector normal to $\phi_2(T_xF)$. Note that $S_x(\phi, \phi) = Id$ for any ϕ. Now, for $f \in Mon_{D,H_0}(TF,TM)$, define $\psi(f)(x) = S_x(dH_0|_{T_xF}, f|_{T_xF})$.

We claim that ψ is a homotopy equivalence. Its homotopy inverse $\psi' : Map_{D,H_0}(F, GL^+TM) \to Mon_{D,H_0}(TF, TM)$ is given by $\psi'(f)|_{T_xF} = f(x) \circ dH_0|_{T_xF}$. Indeed $\psi' \circ \psi = Id$. The homotopy $K : Map_{D,H_0}(F, GL^+TM) \times [0,1] \to Map_{D,H_0}(F, GL^+TM)$ showing $\psi \circ \psi' \sim Id$ is given by $K(f,t)(x) = tf(x) + (1-t)\psi \circ \psi'(f)(x)$. A priori, $tf(x) + (1-t)\psi \circ \psi'(f)(x)$ is just an endomorphism of $T_{H_0(x)}M$, but since $f(x)$ and $\psi \circ \psi'(f)(x)$ agree on the 2 dimensional subspace $dH_0(T_xF)$, and are both non-singular and orientation preserving, then any convex combination of them will also be such.

So it is enough for us to consider $Map_{D,H_0}(F, GL^+TM)$. But now we notice that GL^+TM is a trivial bundle. Indeed, choose a base point $x_0 \in M$. Our parallel transport identifies any T_xM with $T_{x_0}M$ only up to $\pm Id$, but since $-Id$ is in the center of $GL(T_{x_0}M)$, the identification $GL^+(T_xM) \to GL^+(T_{x_0}M)$ is nevertheless well defined. And so $Map_{D,H_0}(F, GL^+TM)$ is homeomorphic to the space $Map_D(F, GL^+(T_{x_0}M))$ of all maps $f : F \to GL^+(T_{x_0}M)$ with $f(x) = Id$ for every $x \in D$. Denote $GL^+ = GL^+(T_{x_0}M)$. We were interested in $\pi_0(\Omega_{dH_0} Mon_{D,H_0}(TF, TM))$. This corresponds now to $\pi_0(\Omega_{Id} Map_D(F, GL^+))$ where Id here is the constant map taking F into $Id \in GL^+$. This in turn is the same as the set of based homotopy classes $[(\Sigma F, *), (GL^+, Id)]$ where ΣF is the quotient space of $F \times [0,1]$ with $F \times \{0\} \cup F \times \{1\} \cup D \times [0,1]$ identified into one point $*$.

What we obtained so far, is an isomorphism $\pi_1(Imm_D(F, M), H_0) \to [(\Sigma F, *), (GL^+, Id)]$. But GL_n^+ is homotopy equivalent to SO_n, so we are finally interested in $[(\Sigma F, *), (SO_3, Id)]$. (The group structure on $[(\Sigma F, *), (SO_3, Id)]$ is given by concatenation along the $[0,1]$ variable of ΣF.)

Now, there is a CW decomposition of ΣF with one 0-cell, no 1-cells, some number of 2-cells and one 3-cell. Let S^3 be modelled as the quotient space of ΣF with the 2-skeleton of ΣF identified into one point $*$, and let $e : \Sigma F \to S^3$ be the quotient map. Since ΣF has no 1-cells and $\pi_2(SO_3) = 0$, any map $\Sigma F \to SO_3$ is homotopic to a map which factors through S^3. In other words, the map $e^* : [(S^3, *), (SO_3, Id)] \to [(\Sigma F, *), (SO_3, Id)]$ is an epimorphism. But $[(S^3, *), (SO_3, Id)] = \pi_3(SO_3) = \mathbb{Z}$ and so $\pi_1(Imm_D(F, M), H_0) = [(\Sigma F, *), (SO_3, Id)]$ is cyclic.

We construct the following CGRH $G_t : F \to M$ which fixes D and has $G_0 = H_0$. Let $B \subseteq M$ be a small ball such that $B \cap H_0(F)$ is an embedded disc, disjoint from $H_0(D)$. Parametrize B such that $H_0(F) \cap B$ looks in B like $(s - d) \cup t$ where s is a sphere in B, t is a very thin tube connecting s to ∂B, and $d \subseteq s$ is the tiny disc deleted from s in order to glue t. Let $i : s \to B$ denote the inclusion map. Let G'_t be a generating CGRH of $\pi_1(Imm_d(s, B), i)$. By changing G'_t slightly, we may also assume that no quadruple point occurs in d, and no triple point passes t. Now, identify $u = s - d$ with its preimage in F, and define $G_t : F \to M$ as G'_t on u and as fixed on $F - u$. The conditions on G'_t guarantee that the number of quadruple points of $G'_t : s \to B$ and $G_t : F \to M$ is the same, and so $q(G_t) = q(G'_t) = 0$ by Theorem 2.2. Our proof (for the case F connected) will be complete if we show G_t is a generator of $\pi_1(Imm_D(F, M), H_0)$.

The above mentioned CW decomposition of ΣF comes from the product structure on $F \times [0,1]$ and so there is no problem choosing B such that $int\ u \times [0,1] \subseteq \Sigma F$ will be contained in the open 3-cell of ΣF, and so our model of S^3 will contain a copy of $int\ u \times [0,1]$. As done for F, Σs will denote the quotient space of $s \times [0,1]$ with $s \times \{0\} \cup s \times \{1\} \cup d \times [0,1]$ identified into one point $*$. Σs also contains a copy of $int\ u \times [0,1]$. Define $f : S^3 \to \Sigma s$ ($\cong S^3$) by $f(x) = x$ for all $x \in int\ u \times [0,1]$, and $f(S^3 - int\ u \times [0,1]) = *$. f is clearly of degree 1, and so $f^* : [(\Sigma s, *), (SO_3, Id)] \to [(S^3, *), (SO_3, Id)]$ is an isomorphism. And so $(f \circ e)^* : [(\Sigma s, *), (SO_3, Id)] \to [(\Sigma F, *), (SO_3, Id)]$ is an epimorphism. Denote by $\bar{J}$ the map $\Sigma F \to SO_3$ which we have attached to a CGRH J_t. For $\overline{G'}$ to be defined we must choose a connection and metric on B. If we use the restrictions to B of the connection and metric chosen for M, then we get $(f \circ e)^*(\bar{G}') = \bar{G}$. And so G'_t generating $\pi_1(Imm_d(s,B),i)$ implies $\bar{G}'$ generates $[(\Sigma s,*),(SO_3,Id)]$ implies $\bar{G}$ generates $[(\Sigma F,*),(SO_3,Id)]$ implies finally that G_t generates $\pi_1(Imm_D(F,M),H_0)$.

We now deal with the general case where F may be non-connected. Let $F^1,\dots,F^n$ be the components of F and $D^i \subseteq F^i$ the components of D. As in Lemma 3.3, $Imm_D(F,M) = Imm_{D^1}(F^1,M) \times \cdots \times Imm_{D^n}(F^n,M)$ and so

$$\pi_1(Imm_D(F,M),H_0) = \pi_1(Imm_{D^1}(F^1,M),H_0|_{F^1}) \times \cdots \times \pi_1(Imm_{D^n}(F^n,M),H_0|_{F^n}).$$

So $\pi_1(Imm_D(F,M),H_0)$ is generated by the n generators $G_t^i \in \pi_1(Imm_{D^i}(F^i,M),H_0|_{F^i})$. But by our proof, we may choose the generator G_t^i to move F^i only in a small ball B^i which does not intersect any other component. And so the number of quadruple points of G_t^i when thought of as an element of $\pi_1(Imm_{D^i}(F^i,M),H_0|_{F^i})$ or $\pi_1(Imm_D(F,M),H_0)$ is the same. Since $q(G_t^i) = 0$, and G_t^i are generators, the theorem follows. □

Our aim from now on will be, to be able to reduce a general CGRH, to a CGRH that fixes such a system of discs.

Lemma 3.5. *Let M be any 3-manifold. Let $\delta \subseteq M \times M$ be the diagonal, i.e. $\delta = \{(x,x)\colon x \in M\}$. There exists an open neighborhood $\delta \subseteq U \subseteq M \times M$ and a continuous map $\Phi : U \to Diff(M)$ where $Diff(M)$ is the space of self diffeomorphisms of M, such that*:

1. *$\Phi(x,x) = Id_M$ for any $x \in M$.*
2. *$\Phi(x,y)(x) = y$ for any $(x,y) \in U$.*

Proof. Let $B_r \subseteq \mathbb{R}^3$ denote the ball of radius r (about the origin). Choose an isotopy $f_r : \mathbb{R}^3 \to \mathbb{R}^3$, $r \in [0,\infty)$ with the following properties: (1) f_r is the identity outside B_{2r} (2) $f_r(0,0,0) = (0,0,r)$ (3) For any $A \in O_3$ which fixes the z-axis, $f_r \circ A = A \circ f_r$.

Choose a Riemannian metric on M. Let $d(x,y)$ denote the distance in M and let $B_r(x)$ denote the ball of radius r about $x \in M$. Let $\delta \subseteq U \subseteq M \times M$ be a thin neighborhood such that for any $(x,y) \in U$, the ball in T_xM of radius $2d(x,y)$ is embedded by the exponential map.

For $(x,y) \in U$ let $r = d(x,y)$, and define $\Phi(x,y)$ as follows: $\Phi(x,y)$ will be the identity outside $B_{2r}(x)$. In $B_{2r}(x)$ take normal coordinates (i.e. an orthonormal set of coordinates on T_xM which is induced onto $B_{2r}(x)$ via the exponential map), with the additional requirement that the z-axis runs from x to y, i.e. that in these coordinates $x = (0,0,0)$ and $y = (0,0,r)$. We let $\Phi(x,y)$ act on $B_{2r}(x)$ as f_r with respect to these coordinates. Due to the symmetry of f_r, this definition does not depend on the freedom we still have in the choice of the normal coordinates. It is easy to verify that $\Phi : U \to Diff(M)$ is indeed continuous. □

Lemma 3.6. *Let M be any 3-manifold and F a (connected) closed surface. Let $H_t : F \to M$ be a regular homotopy. Let $p \in F$ and let $h : [0,1] \to M$ be defined by $h(t) = H_t(p)$.*

If $g : [0,1] \to M$ is homotopic to h relative end points, then there exists a regular homotopy $G_t : F \to M$ which is equivalent to H_t and with $G_t(p) = g(t)$.

In particular, if H_t is closed, and $h(t)$ is null-homotopic, then H_t is equivalent to a CGRH which fixes p.

Proof. Let $k:[0,1]\times[0,1]\to M$ satisfy $k(t,0)=h(t)$, $k(t,1)=g(t)$, $k(0,s)=h(0)$, $k(1,s)=h(1)$. Let U and $\Phi(x,y)$ be as defined in Lemma 3.5. There is a partition $0=s_0<s_1<\cdots<s_n=1$ such that $(k(t,s_{i-1}),k(t,u))\in U$ for any i, t and $s_{i-1}\leqslant u\leqslant s_i$.

Let $I^i:[0,1]\times[s_{i-1},s_i]\to Diff(M)$ be defined as $I^i(t,u)=\Phi(k(t,s_{i-1}),k(t,u))$. Extend I^i to $[0,1]\times[0,1]$ by letting $I^i(t,u)=Id_M$ for $0\leqslant u\leqslant s_{i-1}$ and $I^i(t,u)=I^i(t,s_i)$ for $s_i\leqslant u\leqslant 1$. Now let $K(t,u)=I^n(t,u)\circ\cdots\circ I^1(t,u)\circ H_t$, then $K(t,0)=H_t$ and $G_t=K(t,1)$ is the required equivalent regular homotopy. □

Lemma 3.7. *Let M be any 3-manifold, F a (connected) closed surface and $p\in F$. If H_t is a CGRH that fixes p, and such that p is not in the multiplicity set of H_0, then there exists a generic regular homotopy G_t and discs D,D' with $p\in D\subseteq int\,D'\subseteq F$ such that*:

1. $G_0=H_0$.
2. *G_t fixes D.*
3. *Either $G_1=H_1$ $(=H_0)$ or $G_1=H_1\circ d$ where $d:F\to F$ is a Dehn twist performed on the annulus $A=D'-D$.*
4. $q(G_t)=q(H_t)$.

Proof. Let $p'=H_t(p)$, $t\in[0,1]$. By slightly changing H_t in a small neighborhood of p we may assume that there are coordinates on a neighborhood $p'\in U\subseteq M$ (with the origin corresponding to p'), and a small disc $p\in D'\subseteq F$ such that $H_t(D')\subseteq U$, $t\in[0,1]$ and such that with respect to these coordinates on U the following holds: (1) $H_0(D')$ is an actual flat round disc. (2) $H_t|_{D'}=I_t\circ H_0|_{D'}$ where $I_t\in SO_3$, $I_0=I_1=Id$.

We continue to work in the chosen coordinates, and use the natural norm $\|x\|$ of the coordinates. We may assume the radius of $H_0(D')$ is 2, and let $D\subseteq D'$ be the disc with $H_0(D)$ having radius 1. Let $B_r\subseteq U$ denote the ball of radius r, then by our assumption on p we may also assume $B_2\cap H_0(F)=H_0(D')$. Now let $J_t:M\to M$ be the following isotopy: On B_1, $J_t=I_t^{-1}$. For $1\leqslant\|x\|\leqslant 2$, $J_t(x)=(I_{(2-\|x\|)t})^{-1}(x)$. For $x\in M-B_2$, $J_t(x)=x$ for all t. Let $H'_t=J_t\circ H_t$. Then H'_t satisfies 1,2 and 4 of the lemma, and $H'_1=H_1$ $(=H_0)$ on $F-A$. For $x\in A$, $H'_1(x)=(I_{2-\|H_1(x)\|})^{-1}(H_1(x))$. The map $k:[1,2]\to SO_3$ defined by $k(s)=(I_{2-s})^{-1}$ has $k(1)=k(2)=Id$, and so there is a homotopy $K:[1,2]\times[0,1]\to SO_3$ with $K(s,0)=k(s)$, $K(1,u)=K(2,u)=Id$ and $K(s,1)$ is either the constant loop on Id, or the loop that describes one full rotation about the axis perpendicular to $H_1(D')$. (The loops here have domain $[1,2]$.) We now define $H''_t:F\to M$ as follows: For $x\in F-A$, $H''_t(x)=H_1(x)$. For $x\in A$, $H''_t(x)=K(\|H_1(x)\|,t)(H_1(x))$. Finally, our desired G_t is $H'_t*H''_t$. □

We combine Lemmas 3.2, 3.3, 3.6 and the proof of 3.7 to get:

Proposition 3.8. *Let M be any 3-manifold and let $F=F^1\cup\cdots\cup F^n$ be a system of closed surfaces. Let $H_t:F\to M$ be a CGRH and let $p^i\in F^i$.*

If for each i, the loop $t\mapsto H_t(p^i)$ is null-homotopic, then there is a generic regular homotopy $G_t:F\to M$ and discs $D^i\subseteq int\,D'^i\subseteq F^i$ such that:

1. $G_0=H_0$.

2. *G_t fixes $\bigcup_i D^i$.*
3. *For each i, either $G_1|_{F^i} = H_1|_{F^i}$ $(= H_0|_{F^i})$ or $G_1|_{F^i} = H_1|_{F^i} \circ d$ where $d: F^i \to F^i$ is a Dehn twist performed on the annulus $A^i = D'^i - D^i$.*
4. *$q(G_t) = q(H_t)$.*

Proof. If $t \mapsto H_t(p^i)$ is null-homotopic, then the same is true for any other $p \in F^i$, and so we may assume p^i is not in the multiplicity set of H_0. By Lemmas 3.2, 3.3 and 3.6 we may assume that $H_t = H_t^1 * \cdots * H_t^n$ with each H_t^i fixing p^i and fixing all $F^j, j \neq i$. Now replace each H_t^i with G_t^i which fixes all $F^j, j \neq i$, and such that $G_t^i|_{F^i}$ is the regular homotopy constructed from $H_t^i|_{F^i}$ as in the proof of Lemma 3.7, and making sure that each B_2^i of that proof satisfies $B_2^i \cap H_0(F) = H_0(D'^i)$ (not just the intersection with $H_0(F^i)$.)

When it is the turn of F^i to move, then F^j for $j < i$ have already performed their movement, but notice that whether a Dehn twist appeared or not, $G_1^j(F^j) = H_1^j(F^j)$. And so since $H_t'^i$ differs from H_t^i only in B_2^i, and $H_t''^i$ moves only in B_2^i, we have $q(G_t^i) = q(H_t^i)$. And so finally $q(G_t) = q(H_t)$. □

We are now ready to prove:

Theorem 3.9. *Let F be a system of closed surfaces, and let $H_t: F \to \mathbb{R}^3$ be any CGRH. Then $q(H_t) = 0$.*

Proof. Replace H_t with the G_t of Proposition 3.8 ($t \mapsto H_t(p^i)$ are of course null-homotopic). For brevity, we will denote both F^i itself, and its immersed image, by F^i. Denote $\hat{F}^i = \bigcup_{j \neq i} F^j$. We continue G_t to get a CGRH as follows. Each F^i for which $G_1|_{F^i} = H_1|_{F^i}$ will remain fixed. If for some F^i a Dehn twist appeared, then we undo this Dehn twist by rotating $F^i - D'^i$ by a rigid rotation about the line l_i perpendicular to F^i at p^i, while keeping D^i and $\hat{F}^i$ fixed. (We may assume that l_i is generic with respect to F in the sense that it intersects it generically, and that this rigid rotation of F^i while keeping $\hat{F}^i$ fixed, is a generic regular homotopy of F.) We do this one by one, to each one of the components F^i for which a Dehn twist appeared. We claim that each such rigid rotation contributes $0 \bmod 2$ quadruple points. Indeed, by our genericity assumption, such a quadruple point may occur in one of three ways: A triple point, double line or sheet of F^i crosses, respectively, a sheet, double line or triple point of $\hat{F}^i$. We will show that each of the three types separately contributes $0 \bmod 2$ quadruple points: (1) A triple point of F^i traces a circle during this rigid rotation. Since $\mathbb{R}^3$ is contractible, the $\mathbb{Z}/2$ intersection of any $\mathbb{Z}/2$ 1-cycle (this circle) with any $\mathbb{Z}/2$ 2-cycle ($\hat{F}^i$) is 0. (2) Each double line of F^i is an immersed loop, and so it traces a (singular) torus during the rigid rotation. Again, the $\mathbb{Z}/2$ intersection of this torus, with each immersed double loop of $\hat{F}^i$ is 0. (3) This is symmetric to case 1. (One may think of $\hat{F}^i$ as rotating around l_i and F^i remaining fixed.)

And so we have managed to complete G_t to a CGRH without changing q. But this CGRH fixes $D = \bigcup_i D^i$. And so by Theorem 3.4 we are done. □

We proceed towards Theorem 3.15.

Fig. 1 (resp. 2) describes a regular homotopy of an annulus K (resp. disc D) which is properly immersed in a ball B^3. (The ball is not drawn.) The regular homotopy is assumed to fix a neighborhood of ∂K (resp. ∂D.) This regular homotopy will be called move A (resp. move B). Move A begins with an embedding, and adds one circle of intersection, and one Dehn twist.

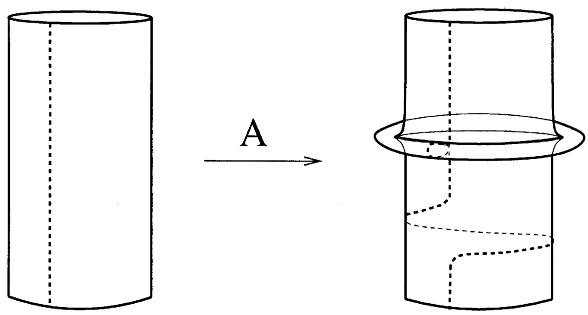

Fig. 1. Move A.

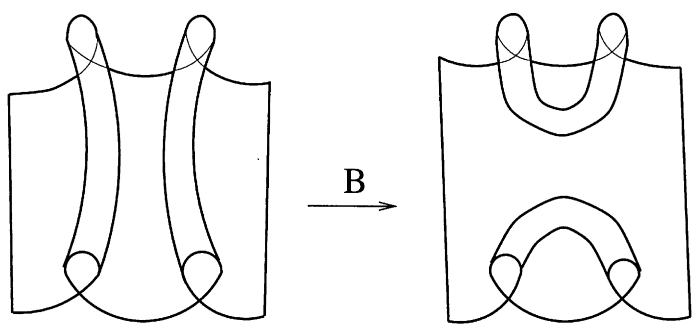

Fig. 2. Move B.

Move B begins with two arcs of intersection, and replaces them with two different arcs of intersection.

The torus that is added to the image of K by an A move, will be called "the *ring* formed by the A move." We allow this ring to face either side of K in B^3. (Fig. 1 describes one of the two possible choices.) One may also choose on what side of the ring the Dehn twist will appear, and what the orientation of the Dehn twist will be.

We do not specify the regular homotopies themselves, we only specify the initial and final immersions. It is easy to see that the desired regular homotopies exist, by the Smale–Hirsch Theorem: For move B (Fig. 2) there is nothing to check since any two immersions of a disc D into B^3 which coincide on a neighborhood of ∂D, are regularly homotopic fixing a neighborhood of ∂D. For move A one needs to check that the map $K \to SO_3$ associated to the initial immersion, is homotopic relative ∂K to the one associated to the final immersion. Since $\pi_2(SO_3) = 0$, we need to check this only for some spanning arc, say the dotted one, in Fig. 1.

Moves A and B will be applied as local moves to surfaces in 3-manifolds, i.e. performing move A or B on a surface means that this regular homotopy is performed in a small ball in M, and the rest of F remains fixed.

If c is the essential circle of the annulus K, then we will say that "move A was applied to the circle c". If $f: F \to M$ is an immersion, then by definition, move A may be applied to a circle $c \subseteq F$ iff there is an (embedded) disc $E \subseteq M$ with $E \cap f(F) = \partial E = f(c)$. The move is then performed in a thin neighborhood of E in M. In particular, move A may always be applied to a circle $c \subseteq F$ which bounds a disc D in F, with D containing no multiple points of f. In this case we may undo the Dehn twist simply by rotating D.

We will now prove that $q(A) = q(B) = 1$, or more precisely:

Lemma 3.10. *Let K, D, B^3 denote an annulus, disc and ball, respectively.*

1. *For any generic regular homotopy $A_t : K \to B^3$ that realizes an A move, $q(A_t) = 1$.*
2. *For any generic regular homotopy $B_t : D \to B^3$ that realizes the B move, $q(B_t) = 1$.*

Proof. 1. Think of B^3 as contained in $\mathbb{R}^3$ and identify K with $A_0(K)$. Cap off the two boundary components of K with two discs D_1, D_2 in $\mathbb{R}^3 - B^3$, obtaining a 2-sphere S in $\mathbb{R}^3$. Let $H_t : S \to \mathbb{R}^3$ be the regular homotopy defined as A_t on K, and fixed on D_1, D_2. We may choose D_1, D_2 so that the ring formed by H_t will face the outside of S in $\mathbb{R}^3$. (This choice will be different for the two kinds of A moves.) H_t may now be continued to complete an eversion of S in $\mathbb{R}^3$ with no additional quadruple points. And so by Theorem 2.3 we have $q(A_t) = q(H_t) = 1$.

2. We describe the above sort of argument in shorter form: Start with an embedding $S^2 \to \mathbb{R}^3$. Perform two A moves at two remote regions of S^2, with both rings facing the outside of S^2. Perform one B move to merge the two rings into one. We may continue as before with no additional quadruple points to get an eversion. And so $2q(A) + q(B) = 1$ and so $q(B) = 1$. □

Lemma 3.11. *Let F be a (connected) closed surface and M a 3-manifold. Let $i : F \to M$ be a 2-sided embedding. Let D, D' be two discs $D \subseteq \operatorname{int} D' \subseteq F$. Then, there exists a generic regular homotopy $H_t : F \to M$ which satisfies:*

1. *$H_0 = i$*
2. *$H_1 = i \circ d$ where $d : F \to F$ is a Dehn twist performed on the annulus $D' - D$. (We may choose the orientation of the Dehn twist.)*
3. *H_t fixes D.*
4. *$q(H_t) = \chi(F) \bmod 2$.*

Proof. Let $E \subseteq M$ be a disc with $\partial E = i(F) \cap E$ an essential loop in $D' - D$. Perform move A in a thin neighborhood of E disjoint from $i(D)$. Now choose a Morse function on $F - \operatorname{int} D'$, $\partial D'$ being the minimal level curve. Let the ring created by the A move, move along with the increasing level curves. Whenever there is a singularity of the Morse function, namely a local minimum, local maximum or saddle point, then we will perform move A, A^{-1} or B, respectively, in order for the rings to continue following up with the level curve. If in all A moves, we create the ring on the same side of $i(F)$ in M, then the B moves will always be possible. This we do until we pass the maximum of the Morse function, and so we remain with no rings at all. Only the Dehn twist of the initial A move remains, since all others may be resolved by rotating the relevant disc. (The relevant disc for the initial A move is D, which we are keeping fixed.) So the count of singularities will match the count of A, A^{-1}, B moves. Together with our initial A move to match the disc D', we have by Lemma 3.10, and by the relationship between singularities of Morse functions and Euler characteristic: $q(H_t) = \chi(F) \bmod 2$. □

The assumption in the above lemma that the embedding i is 2-sided, may actually be dropped. This will be sketched in Example 4.4.

Proposition 3.12. *Let M be a 3-manifold with $\pi_2(M) = \pi_3(M) = 0$. Let $F = F^1 \cup \cdots \cup F^n$ be a system of closed surfaces each having even Euler characteristic. Let $H_t : F \to M$ be a CGRH in the regular homotopy class of a two-sided embedding. Let $p^i \in F^i$. If for each i, the loop $t \mapsto H_t(p^i)$, $t \in [0,1]$ is null-homotopic, then $q(H_t) = 0$.*

Proof. Let $J_t : F \to M$ be a generic regular homotopy from an embedding to H_0. By passing to $J_t * H_t * J_{-t}$ we may assume H_0 is a (two-sided) embedding. Replace H_t by the G_t of Proposition 3.8. For each i for which there appeared a Dehn twist, we cancel the Dehn twist by concatenating with the regular homotopy of Lemma 3.11, performed in a thin neighborhood of $H_0(F^i)$. By our assumption on the Euler characteristic of the F^is, this does not change q. And so we have replaced H_t with a CGRH that fixes $\bigcup_i D^i$ and has the same q. We are done by Theorem 3.4. □

Lemma 3.13. *Let M be a 3-manifold with $\pi_2(M) = 0$. Let $F \subseteq M$ be an embedded system of closed surfaces, and let $D \subseteq M$ be a compressing disc for F i.e. $D \cap F = \partial D$. Let $F' \subseteq M$ be the system of surfaces obtained from F by compressing along D, in a thin neighborhood of D. Let $i : F \to M$, $i' : F' \to M$ denote the inclusion maps.*

If $i' : F' \to M$ has the property that any CGRH $G_t : F' \to M$ with $G_0 = i'$ has $q(G_t) = 0$, then $i : F \to M$ has the same property.

Proof. Let H_t be a CGRH with $H_0 = i$. By slightly relocating D, we may assume H_t is generic with respect to ∂D (considered here as a loop in F). Let $\hat{F}$ denote the 2-complex $F \cup D$. By means of [6] or [3], H_t may be extended to a generic regular homotopy $H'_t : \hat{F} \to M$, with H'_0 the inclusion and such that H'_1 coincides with H'_0 on $F \cup N(\partial D)$ where $N(\partial D)$ denotes a neighborhood of ∂D in D. Since $\pi_2(M) = 0$ we may further continue H'_t (as a generic regular homotopy) until $H'_0 = H'_1$ on the whole of $\hat{F}$. This regular homotopy of $\hat{F}$ induces a CGRH $G_t : F' \to M$ with $G_0 = i'$ in an obvious way. By assumption, $q(G_t) = 0$. H'_t being a *generic* regular homotopy of $\hat{F}$ means in particular that any quadruple point of H'_t occurs away from ∂D. And so there are two types of quadruple points of H'_t: (1) The quadruple points of H_t. (2) Quadruple points which involve at least one sheet coming from D. We assume the compression along D was performed in a very thin neighborhood of D and that G_t follows H'_t very closely, and so G_t will inherit the quadruple points of H'_t in the following way: Every quadruple point of type (1) will contribute one quadruple point to G_t. For type (2), any sheet of D carries with it two sheets of F', and so each sheet of D involved in a quadruple point of H'_t will double the number of corresponding quadruple points counted for G_t (e.g. if all four sheets involved in a quadruple point of H'_t come from D, then this occurrence will contribute $2^4 = 16$ quadruple points to G_t). And so there is an even number of quadruple points for G_t in addition to those inherited from H_t. And so $q(H_t) = q(G_t) = 0$. □

The proof of the following lemma is a gathering of arguments from [7]:

Lemma 3.14. *Let M be an irreducible orientable 3-manifold, let F be a (connected) closed orientable surface and $H_t: F \to M$ a homotopy such that H_0 and H_1 are two incompressible embeddings with $H_0(F) = H_1(F)$. Then there is a generic regular homotopy $G_t: F \to M$ such that the maps $H, G: F \times [0,1] \to M$ defined by H_t and G_t are homotopic relative $\partial(F \times [0,1])$, and such that the highest multiplicities of G_t are double curves.*

Proof. Denote $F' = H_0(F)$ ($= H_1(F)$). We may homotope H relative $\partial(F \times [0,1])$ so that it will be transverse with respect to F' and $H^{-1}(F')$ will be a system of incompressible surfaces in $F \times [0,1]$. By further homotoping we may assume $H^{-1}(F') = F \times \{s_0, s_1, \ldots, s_n\}$ where $0 = s_0 < s_1 < \cdots < s_n = 1$.

Let $p \in F$ be a chosen base point for F and let $p' = H_0(p)$ be chosen as basepoint for M. We will think of $\pi_1(F) = \pi_1(F,p)$ as contained in $\pi_1(M) = \pi_1(M,p')$ via H_{0*}. Let $k: \hat{M} \to M$ be the covering corresponding to $\pi_1(F)$ and let $\hat{F} = k^{-1}(F') \subseteq \hat{M}$. Let $\hat{H}: F \times [0,1] \to \hat{M}$ be the lifting of H, then $\hat{H}^{-1}(\hat{F}) = H^{-1}(F')$. Let $\hat{F}_i$ be the component of $\hat{F}$ with $\hat{H}(F \times s_i) \subseteq \hat{F}_i$.

We proceed by induction on n. Let $n = 1$ and so $s_0 = 0, s_1 = 1$. We distinguish two cases:

1. $\hat{F}_0 = \hat{F}_1$: $\hat{F}_0 = \hat{H}(F \times 0)$ is a strong deformation retract of $\hat{M}$. Let $J_s: \hat{M} \to \hat{M}$ denote this deformation, then $k \circ J_s \circ \hat{H}$ homotopes H into F' relative $\partial(F \times [0,1])$. And so now H is a homotopy in F' between H_0 and H_1. Such a homotopy is homotopic relative $\partial(F \times [0,1])$ to an isotopy, and we are done.

2. $\hat{F}_0 \neq \hat{F}_1$: Cut $\hat{M}$ along $\hat{F}$ and let $\hat{M}'$ be the piece containing $\hat{H}(F \times [0,1])$. By Lemma 5.1 of [7] $\hat{M}' = F \times [0,1]$. And so we may homotope $\hat{H}$ relative $\partial(F \times [0,1])$ so that $\hat{H}: F \times [0,1] \to \hat{M}'$ will be a homeomorphism.

Cut M along F' and let M' be the piece covered by $\hat{M}'$. (Perhaps there is no other piece.) Then $H = k \circ \hat{H}: F \times [0,1] \to M'$ is now a covering map. Since F' was originally covered exactly twice by $\partial(F \times [0,1])$, $H: F \times [0,1] \to M'$ is a covering of degree at most 2. And so H, when thought of again as a homotopy, is a regular homotopy which has at most double points.

We now let $n > 1$ and so $0 < s_1 < 1$. If now $\hat{F}_0 = \hat{F}_1$ then (since $\hat{F}_0$ is a strong deformation retract of $\hat{M}$), we may homotope $\hat{H}$ relative $\partial(F \times [0,1])$ as to push $F \times [0, s_1]$ to the other side of $\hat{F}_0$, by that reducing n. And so we may assume $\hat{F}_0 \neq \hat{F}_1$. As before we may use Lemma 5.1 of [7], to see that $\hat{F}_1$ is parallel to $\hat{F}_0$, and so we may now homotope $\hat{H}$ relative $\partial(F \times [0,1])$ and without changing $\hat{H}^{-1}(\hat{F})$ such that $\hat{H}|_{F \times s_1}$ will be an embedding, and then that $\hat{H}|_{F \times [0,s_1]}$ will be an embedding.

We will now show that also $H|_{F \times s_1} = k \circ \hat{H}|_{F \times s_1}$ is an embedding (which is trivial if F is not a torus.) As before, $H: F \times [0, s_1] \to M'$ is a covering map. If in M', $H(F \times 0) \neq H(F \times s_1)$ then since $H|_{F \times 0}$ is an embedding, $H(F \times 0)$ is covered exactly once, and so the same must be with $H(F \times s_1)$ and so $H|_{F \times s_1}$ is an embedding. If $H(F \times 0) = H(F \times s_1)$ and $H|_{F \times s_1}$ is some non-trivial covering, then we will get via $H|_{F \times [0,s_1]}$ that $\pi_1(F)$ is conjugate in $\pi_1(M')$ to a proper subgroup of itself, which is impossible since it has finite index (since it corresponds to the covering space $F \times [0, s_1]$).

And so $H|_{F \times s_1}$ is an embedding, and we may apply the induction hypothesis to $H|_{F \times [0,s_1]}$ and $H|_{F \times [s_1, 1]}$. □

We are now ready to prove:

Theorem 3.15. *Let M be an orientable irreducible 3-manifold with $\pi_3(M) = 0$. Let F be a system of closed orientable surfaces. If $H_t: F \to M$ is any CGRH in the regular homotopy class of an embedding, then $q(H_t) = 0$.*

Proof. By the opening remark in the proof of Proposition 3.12 and by induction on Lemma 3.13, we may assume that H_0 is an embedding and that for each i either $F^i = S^2$ or $H_0|_{F^i}$ is an incompressible embedding (where $F^1, \dots, F^n$ are the connected components of F).

Given $p^i \in F^i$ we will construct a CGRH $G^i_t: F^i \to M$ satisfying : (1) $G^i_0 = H_0|_{F^i}$. (2) The paths $t \mapsto H_t(p^i)$ and $t \mapsto G^i_t(p^i)$ are homotopic relative endpoints. (3) The highest multiplicities of G^i_t are double curves. We then define $G_t: F \to M$ to be the CGRH which performs each G^i_t in its turn, while fixing all other components. By condition (3) the highest multiplicities of G_t will be triple points, and so $q(G_t) = 0$. By conditions (2), Proposition 3.12 applies to $H_t * G_{-t}$ and so $q(H_t) = q(H_t * G_{-t}) = 0$.

And so we are left with constructing the CGRHs G^i_t satisfying (1), (2) and (3) above.

For $F^i = S^2$, G^i_t is defined as follows: $H_0(S^2)$ bounds a ball B. Shrink S^2 inside B until it is very small. Then move this tiny sphere along the loop $t \mapsto H_t(p^i)$, finally re-entering B. Since M is orientable, our S^2 will return with the same orientation it originally had. (One may think of a little ball bounded by S^2 which is moving along with it.) And so we may isotope S^2 in B, back to its original position. And so G^i_t is actually an isotopy.

For $H_0(F^i)$ incompressible, use Lemma 3.14. □

Remark 3.16. In the proof of Theorem 3.15, when we were assuming $F^i = S^2$ we did not actually need $H_0(S^2)$ to bound a ball. It would be enough that $\pi_2(M) = 0$ since then we could deform S^2 by some generic regular homotopy G_t until it is contained in a ball, inside which G_t continues until the sphere is embedded again. Then move the embedded sphere around the loop, and then return to the original position by G_{-t}. This would still contribute $q = 0$.

And so, if the embedding $f: F \to M$ is such that repeated compression of $f(F)$ will turn it into a union of spheres, then the theorem will still be true with the (perhaps) weaker assumption $\pi_2(M) = 0$ in place of irreducibility. In particular, if M is a fake open 3-cell, then the compressibility down to spheres always holds. (On the other hand, when applying Lemma 3.14 to the case $H_0(F^i)$ is incompressible, we indeed use irreducibility.)

4. Surfaces in 3-manifolds: counter-examples

Example 4.1. The most obvious example of a CGRH with $q = 1$ is the following $H_t: S^2 \to S^3$. H_0 is some embedding. H_t then isotopes S^2 in S^3 so as to reverse the sides of S^2. Finally it performs a generic eversion inside a ball $B \subseteq S^3$ so as to return S^2 to its original position. By Theorem 2.3 $q(H_t) = 1$. Furthermore, this H_t may be performed while fixing some disc in S^2. And so H_t satisfies all the conditions of Theorems 3.4 and 3.15 except $\pi_3(M) = 0$.

Example 4.2. Let $h: S^3 \to M$ be the covering map where $M = \mathbb{R}P^3$ or $L_{3,1}$, and so h is a double or triple covering. Perform $H_t: S^2 \to S^3$ of Example 4.1 such that a ball B in which we start H_t and in which the eversion takes place, is embedded into M by h. Let $G_t = h \circ H_t$. Since the covering is double or triple, the first part of H_t contributes no quadruple points, and so $q(G_t) = 1$.

Example 4.3. Lemma 3.11 provides an example of $q = 1$ if one takes F to be a surface of odd Euler characteristic. Instead of having D fixed, we let it rotate once so as to cancel the Dehn twist. And so we get a CGRH with $q = \chi(F) \bmod 2 = 1$. The condition of Theorem 3.4 which is violated is that no disc is fixed, though, we may perform this CGRH while keeping just a point fixed (some point in D around which we rotate D to cancel the Dehn twist).

This example stresses another point: Let $H_t: F \to M$ denote the CGRH we have just constructed, and let $G_t: F \to M$ be defined by $G_t = H_0$ for all t. It is easy to see that the maps $H, G: F \times [0,1] \to M$ defined by H_t, G_t are homotopic relative $\partial(F \times [0,1])$. This shows that the assumption in Lemma 3.2 that H_t, G_t are equivalent, may not be replaced by the weaker condition, that H, G are homotopic relative $\partial(F \times [0,1])$.

Example 4.4. We will now show how to extend Lemma 3.11 to the case where $i: F \to M$ is a 1-sided embedding. This will provide more examples of CGRHs with $q = 1$, as in Example 4.3 above. Identify F with its image $i(F)$ and let N be a regular neighborhood of F in M, then N is a 1-sided I-bundle over F. Let $p: N \to F$ be the projection. There exists a circle $c \subseteq F$ such that cutting N along $p^{-1}(c)$ results in a 2-sided I-bundle $p': N' \to F'$. Furthermore, $p^{-1}(c)$ is an annulus (resp. Mobius band) iff a regular neighborhood U of c in F is an annulus (resp. Mobius band) iff F' has two (resp. one) boundary components.

We apply the construction of Lemma 3.11 to F', where the Morse function will be chosen to have a constant maximum on $\partial F'$. We let the rings follow the increasing level curves, until right before the maximum, and so $\chi(F') \bmod 2$ quadruple points occur. Looking back in N, we have rings approaching c from both sides: In case U is an annulus these are two distinct rings facing opposite sides of U in N. In case U is a Mobius band, this is one ring going twice around, such that after once around, it is on the other side of c in F and facing the other side of F in N. For either case, it is easy to construct an explicit regular homotopy that gets rid of the rings/ring and has no quadruple points at all. Since $\chi(F') = \chi(F)$ we have the right number mod 2 of quadruple points.

Example 4.5. In any non-orientable 3-manifold there is a CGRH with $q = 1$. Take a little sphere inside a ball. Move the little sphere along an orientation reversing loop until it returns to itself with opposite orientation. Then perform an eversion inside the ball to return to the original position.

Example 4.6. Let F_1, F_2 be two surfaces such that there exists an immersion $g: F_2 \to F_1 \times S^1$ with an odd number of triple points. (e.g. such an immersion exists whenever F_2 has odd Euler characteristic.) Let $F = F_1 \cup F_2$ and $M = F_1 \times S^1$. Let $H_0: F \to M$ be g on F_2 and the inclusion $F_1 \to F_1 \times * \subseteq F_1 \times S^1$ on F_1. Let H_t move F_1 once around M while fixing F_2, then $q(H_t) = 1$.

5. Embeddings of tori in $\mathbb{R}^3$

Let T be the standard torus in $\mathbb{R}^3$. For the sake of definiteness say T is the surface obtained by rotating the circle $\{y=0,(x-2)^2+z^2=1\}$ about the z-axis. Let $m \subseteq T$ be the meridian, e.g. the loop $\{y=0,(x-2)^2+z^2=1\}$. m bounds a disc in the compact side of T. Let $l \subseteq T$ be the longitude, e.g. the loop $\{z=0,x^2+y^2=1\}$. l bounds a disc in the non-compact side of T. We choose m,l (with some orientation) as the basis of $H_1(T,\mathbb{Z})$, and this induces an identification between $H_1(T,\mathbb{Z})$ and $\mathbb{Z}^2$. The group $M(T)$ of self diffeomorphisms of T up to homotopy, is then identified with $GL_2(\mathbb{Z})$, and so $f \in M(T)$ will be thought of both as a map $T \to T$ and as a 2×2 matrix. Let τ denote reduction mod 2, so we have $\tau : \mathbb{Z}^2 \to (\mathbb{Z}/2)^2$ and $\tau : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/2)$. We will use round brackets for objects over $\mathbb{Z}$, and square brackets for objects over $\mathbb{Z}/2$. Let $H \subseteq GL_2(\mathbb{Z}/2)$ be the subgroup $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$.

Proposition 5.1. *Let $T \subseteq \mathbb{R}^3$ be the standard torus, and denote the inclusion map by $i: T \to \mathbb{R}^3$. Let $M(T)$ be identified with $GL_2(\mathbb{Z})$ via m,l as above and let $H \subseteq GL_2(\mathbb{Z}/2)$ be as above.*

1. *For $f \in M(T)$, $i \circ f$ is regularly homotopic to i iff $\tau(f) \in H$.*
2. (a) *If $\tau(f) = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ then any generic regular homotopy $H_t : T \to \mathbb{R}^3$ between $i\circ f$ and i will have $q(H_t)=0$.*
   (b) *If $\tau(f) = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ then any generic regular homotopy $H_t : T \to \mathbb{R}^3$ between $i\circ f$ and i will have $q(H_t)=1$.*

Proof. Let V_1,V_2 be the standard 2-frame on T, i.e. V_1 (resp. V_2) is everywhere tangent to the translates of m (resp. l). Let $K(v_1,v_2)$ be the function defined in the beginning of the proof of Proposition 2.1. For an immersion $g: T \to \mathbb{R}^3$, let $h_g : T \to SO_3$ be defined by $h_g = K(dg(V_1), dg(V_2))$. By the Smale–Hirsch Theorem, $i\circ f$ will be regularly homotopic to i iff $h_{i\circ f}$ is homotopic to h_i. It is easy to see that $h_{i\circ f}$ is homotopic to $h_i \circ f$, and so $i\circ f$ will be regularly homotopic to i iff $h_i \circ f$ is homotopic to h_i. Let $k: H_1(T,\mathbb{Z}) \to H_1(SO_3,\mathbb{Z}) = \mathbb{Z}/2$ be the homomorphism induced by h_i. Since $\pi_2(SO_3)=0$, $h_i\circ f$ will be homotopic to h_i iff $k \circ f = k$. (Recall that we denote by f both the map $T\to T$ and the automorphism it induces on $H_1(T,\mathbb{Z})$.) Let $k' : H_1(T,\mathbb{Z}/2) \to \mathbb{Z}/2$ be defined by the relation $k = k' \circ \tau$, then $k\circ f = k$ iff $k'\circ\tau(f) = k'$. It is easy to see that $k\binom{1}{0} = k\binom{0}{1} = 1$ and so $k'\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = k'\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = 1$ and $k'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = 0$. And so $k'\circ\tau(f) = k'$ iff $\tau(f)$ maps the set $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$ into itself. Assertion 1 follows.

It is easy to verify, by means of row and column operations, that the matrices $\begin{pmatrix}1&2\\0&1\end{pmatrix}$, $\begin{pmatrix}1&0\\2&1\end{pmatrix}$, $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ and $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ generate $\tau^{-1}\begin{bmatrix}1&0\\0&1\end{bmatrix}$. By Theorem 3.9, in order to prove 2(a), we only need to construct one regular homotopy for each one of the four generators, and see that it has $q=0$.

$\begin{pmatrix}1&2\\0&1\end{pmatrix}$ is a double Dehn twist along the meridian. The regular homotopy will be as follows: Perform two A moves along parallel meridians, each giving a Dehn twist of the required orientation, and such that the rings are formed on the same side. Then perform a B move to merge the two rings. We now have a ring bounding a disc in T. We may get rid of it with one A^{-1} move. And so we are left with precisely the two Dehn twists we needed. As to q, we had two A moves, one B move and one A^{-1} move and so by Lemma 3.10 we indeed have $q=0$. (Actually, if we would have formed the rings on opposite sides of T then we could have completed the regular homotopy with no additional quadruple points, as in Example 4.4. So we would have just the two A moves which would indeed also give $q=0$. Moreover, we can construct the whole regular homotopy with no quadruple points at all, via the well known "belt trick".)

$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is a double Dehn twist along the longitude. Since for the standard torus, the longitude is also disc bounding (on the non-compact side,) we can proceed exactly as in the previous case.

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ may be achieved by rigidly rotating T around the x-axis through an angle of π, and so again $q=0$.

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a reflection with respect to the xy-plane. We can achieve it by a regular homotopy as follows: Perform two A moves on two longitudes, say the circles $\{z=0, x^2+y^2=1\}$ and $\{z=0, x^2+y^2=3\}$, with the rings both facing the non-compact side, and with the orientation of the Dehn twists chosen so that they cancel each other. We may then continue with just double curves, to exchange the upper and lower halves of T until we arrive at the required reflection. We had two A moves and so $q=0$.

And so we have proved 2(a). Since $\tau^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has index 2 in $\tau^{-1}(H)$, we need in order to prove 2(b), just to check for one element. We construct the following regular homotopy. Start with an A move on some small circle that bounds a disc in T, and with the ring facing the non-compact side. Let u be a circle on the ring, which is parallel to the intersection circle. It bounds a disc U of T. Keeping u fixed, push and expand U all the way to the other side of T so that it encloses all of T, so the torus is now embedded again. (This last move required only double curves.) A disc that spanned m is now on the non-compact side, and a disc that spanned l is now on the compact side. And so if we now isotope T until it's image again coincides with itself, we will have an f which interchanged m and l, and so $\tau(f)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This regular homotopy required one A move, then some double curves and finally an isotopy, and so $q=1$. □

We continue to think of T as the standard torus contained in $\mathbb{R}^3$, with inclusion map i. Now let $f: T \to \mathbb{R}^3$ be any embedding, then $f(T)$ again separates $\mathbb{R}^3$ into two pieces, one compact and the other non-compact. Denote the compact piece by C_f, and the non-compact piece by N_f.

For a moment, compactify $\mathbb{R}^3$ with a point ∞, so that $\mathbb{R}^3 \cup \{\infty\} = S^3$. $f(T) \subseteq S^3$ always bounds a solid torus on (at least) one side. The other side is then a knot complement. There are now two cases: (a) ∞ is in the knot complement side, and so C_f is a solid torus (Fig. 3a). (b) ∞ is in the solid torus side, and so C_f is a knot complement (Fig. 3b). It is of course possible to have a solid torus on both sides, in which case the knot in question is the unknot (Fig. 3c).

f maps T into both C_f and N_f and so we have maps $H_1(T, \mathbb{Z}/2) \to H_1(C_f, \mathbb{Z}/2)$ and $H_1(T, \mathbb{Z}/2) \to H_1(N_f, \mathbb{Z}/2)$. Denote the unique non-zero element of the kernel of each of these maps by c_f and n_f, respectively. By the above observation that $f(T)$ may be thought of as the boundary of

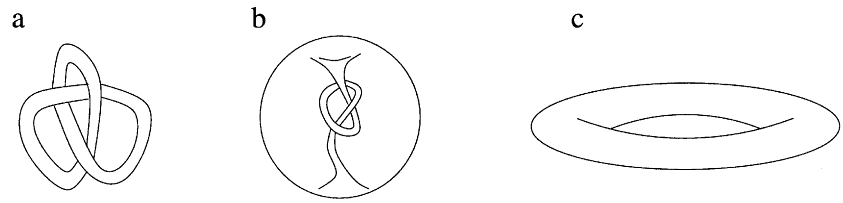

Fig. 3. Embeddings of T in $\mathbb{R}^3$.

a regular neighborhood of some knot $k \subseteq S^3$, we see that c_f and n_f generate $H_1(T, \mathbb{Z}/2)$, which means simply that they are two distinct elements of the set $\{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}], [\begin{smallmatrix}1\\1\end{smallmatrix}]\}$. Let us now choose once and for all some arbitrary ordering of this set, say $[\begin{smallmatrix}1\\0\end{smallmatrix}] < [\begin{smallmatrix}0\\1\end{smallmatrix}] < [\begin{smallmatrix}1\\1\end{smallmatrix}]$. Define $Q(f) = 0 \in \mathbb{Z}/2$ if $c_f < n_f$ and $Q(f) = 1 \in \mathbb{Z}/2$ if $c_f > n_f$. Now let $F = T^1 \cup \ldots \cup T^n$ be a union of n copies of T. If $f: F \to \mathbb{R}^3$ is an embedding, then define $Q(f) = \sum_{i=1}^{n} Q(f|_{T^i}) \in \mathbb{Z}/2$.

We will now prove:

Theorem 5.2. *Let F and Q be as above, and let $f, g: F \to \mathbb{R}^3$ be two embeddings. If f and g are regularly homotopic, then any generic regular homotopy H_t between them will satisfy $q(H_t) = Q(f) - Q(g)$.*

Proof. By Theorem 3.9 any such H_t will have the same q. And so we need to verify the theorem for just one H_t.

Since f is an embedding, if we rigidly move one of the T^is while keeping the others fixed, only double curves will appear. And so we may move them one by one until they are contained in disjoint balls, and this will contribute nothing to q. Since Q is defined by computing it for each component separately, $Q(f)$ is also unchanged. We may do the same with g, and so we actually need to deal with each component separately, and so we may assume from now on that F is the one torus T.

As mentioned, $f(T)$ is either (a) a torus which bounds a solid torus, or (b) a sphere with a tube running inside it. See Fig. 3a and b, respectively. We now begin our regular homotopy by having the solid torus, in case (a), or the inner tube, in case (b), pass across itself until we eliminate all knotting. This creates only double curves and so contributes nothing to q, and we claim that it also does not change c_f and n_f. This follows from the fact that the $\mathbb{Z}/2$-meridian and the $\mathbb{Z}/2$-longitude of a knot, do not change under such crossing moves. (In case (a) c_f is the $\mathbb{Z}/2$-meridian of a knot and n_f is the $\mathbb{Z}/2$-longitude, and in case (b) it is the other way around.) And so since c_f and n_f remain unchanged, Q is unchanged. We may now continue with an isotopy until the image of T coincides with T itself.

As we may do the same for g, we may assume from now on that the image of both f and g is T itself, and so $f = i \circ f'$ and $g = i \circ g'$ for some $f', g': T \to T$. If H_t is a regular homotopy from f to g, then $H_t \circ f'^{-1}$ is a regular homotopy from i to $i \circ (g' \circ f'^{-1})$. By Proposition 5.1(1) $\tau(g' \circ f'^{-1})$ is either $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ or $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. By 5.1(2), if $\tau(g' \circ f'^{-1}) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ (i.e. $\tau(g') = \tau(f')$,) then $q(H_t \circ f'^{-1}) = 0$ and if $\tau(g' \circ f'^{-1}) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (i.e. $\tau(g') = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \circ \tau(f')$) then $q(H_t \circ f'^{-1}) = 1$.

If $\tau(f')=\tau(g')$ then $c_f=c_g$ and $n_f=n_g$ and so $Q(f)=Q(g)$ and so $Q(f)-Q(g)=0=q(H_t\circ f'^{-1})=q(H_t)$ and we are done.

Assume now $\tau(g')=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\circ\tau(f')$. By definition of c_f and n_f we must have in $\mathbb{Z}/2$ homology: $\tau(f')(c_f)=\tau(g')(c_g)=[m]$ and $\tau(f')(n_f)=\tau(g')(n_g)=[l]$. So $\tau(g')(c_f)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\circ\tau(f')(c_f)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}[m]=[l]=\tau(g')(n_g)$, and so $c_f=n_g$. Similarly $\tau(g')(n_f)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\circ\tau(f')(n_f)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}[l]=[m]=\tau(g')(c_g)$ and so $n_f=c_g$. That is, the pairs c_f, n_f and c_g, n_g are the same pair of elements of $H_1(T,\mathbb{Z}/2)$ just with opposite order. So the order of one pair matches the chosen order iff the order of the other pair does not. And so $Q(f)-Q(g)=1=q(H_t\circ f'^{-1})=q(H_t)$. □

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