On 3-Manifolds Having the Same Turaev-Viro Invariants

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Two 3-manifolds are called similar if they admit special spines with the same incidence relations between 2-cells and vertices. The Turaev-Viro invariants of similar 3-manifolds coincide. We produce an example of similar 3-manifolds that are not homeomorphic and prove, under certain conditions, that if two closed 3-manifolds are similar, then they are homeomorphic.

§0. INTRODUCTION

Recently V. G. Turaev and O. Ya. Viro constructed new invariants of 3-manifolds [TV]. They are based on fundamental notions of statistical mechanics and quantum group theory. Their construction uses the notion of the special spine of a 3-manifold. We say that two special spines of different 3-manifolds are similar if there exists a bijection between their 2-cells that preserves the incidence relation between 2-cells and vertices in a certain strong sense. Manifolds having similar special spines are also called similar. The Turaev-Viro invariants of similar 3-manifolds are the same. Using an idea of W. B. R. Lickorish, we produce an example of 3-manifolds that are similar but not homeomorphic, and prove that under certain conditions, if two closed 3-manifolds are similar, then they are homeomorphic.

The investigation reported here was initiated when the first-named author was visiting the Hebrew University of Jerusalem in the fall of 1991.

§1. DEFINITIONS AND STATEMENT OF RESULTS

A compact 3-dimensional polyhedron P is called *special* if it can be represented as a CW-complex in which the link of every vertex is homeomorphic to Δ, where Δ is the circle with three radii. The 1-dimensional skeleton of a special polyhedron P is denoted by SP. Recall that a *spine* of a compact 3-manifold M is a subpolyhedron $P \subset \operatorname{Int} M$ such that the manifold $M - P$ is homeomorphic to $\partial M \times (0,1]$ if $\partial M \neq \varnothing$ and to an open 3-cell if $\partial M = \varnothing$. A spine is called *special* if it is a special polyhedron. It is known that any compact 3-manifold possesses a special spine and can be uniquely reconstructed from its special spine ([C, M]).

Let P be a special spine of a 3-manifold and let $V = V(P)$ be the set of its vertices. Denote by $N(V, P)$ a regular neighborhood of V in P. It consists of some number of disjoint copies of the polyhedron $\mathrm{Con}\,\Delta$, where Con is the cone. The intersection of the union of all open 2-cells in P with each such copy consists of exactly six half-open 2-cells, which are called *wings*.

Definition. Two special polyhedra P_1 and P_2 are called *similar* if there exists a homeomorphism $\varphi : N(V(P_1), P_1) \to N(V(P_2), P_2)$ such that for any two wings w_1 and w_2 of P_1 the following condition holds: w_1 and w_2 belong to the same 2-cell of P_1 if and only if $\varphi(w_1)$ and $\varphi(w_2)$ belong to the same 2-cell of P_2. The homeomorphism φ is called a *similarity homeomorphism*.

A good way to think of it is the following: let us paint the 2-cells of P_1 in different colors and the corresponding 2-cells of P_2 with the same colors. Then the similarity homeomorphism φ is required to preserve the colors of wings.

Two 3-manifolds M_1 and M_2 are said to be *similar* if a special spine of M_1 is similar to a special spine of M_2. It follows easily from the definition of Turaev-Viro invariants that similar manifolds have the same invariants.

Theorem 1. *There exist similar closed orientable 3-manifolds M_1 and M_2 such that $H_1(M_1;\mathbb{Z}/k) \neq H_1(M_2;\mathbb{Z}/k)$ for each k, $0 \leqslant k \neq 1, 2$.*

Note that the coincidence of Turaev-Viro invariants implies the coincidence of the first homology groups with coefficients $\mathbb{Z}/2$.

Theorem 2. *Let M_1 and M_2 be similar closed 3-manifolds. Suppose M_1 does not contain closed incompressible surfaces with nonnegative Euler characteristics. Then M_1 and M_2 are homeomorphic.*

§2. MOVES ON MANIFOLDS

Let M be a 3-manifold and let $F \subset \mathrm{Int}\, M$ be a closed connected surface such that F is two-sided in M and $\chi(F) \geqslant 0$. The last condition implies that F is homeomorphic to S^2, RP^2, $T^2 = S^1 \times S^1$, or to the Klein bottle K^2. Let us choose a homeomorphism $R : F \to F$ such that:

(1) if $F \approx S^2$, then r reverses the orientation;
(2) if $F \approx RP^2$, then r is identical;
(3) if $F \approx T^2$ or $F \approx K^2$, then r induces multiplication by (-1) in $H_1(F;\mathbb{Z})$.

It is clear that r is unique up to isotropy.

Now cut M along F and repaste the two copies of F thus obtained according to the homeomorphism r. We shall say the new 3-manifold arising in such a way is obtained from M by a *manifold move* along F. Note that if F is compressible, then $M_1 \approx M$.

§3. MOVES ON SPINES

Let G be a connected (multi)graph with two vertices such that each of the vertices has valency 3. There exist two such graphs: a theta-curve (a circle with a diameter) and the "eyeglass" graph (two circles joined by a segment). Choose a homeomorphism $\rho : G \to G$ such that:

(1) if G is a theta-curve, then $\rho = \rho_1$, where $\rho_1 : G \to G$ permutes the vertices and maps each of the three edges into itself;
(2) if G is the eyeglass graph, then $\rho = \rho_2$, where $\rho_2 : G \to G$ leaves the joining segment fixed and inverses both loops.

Definition. A 1-dimensional subpolyhedron G of a special polyhedron P is called *proper* if a regular neighborhood $N(G, P)$ of G in P is a twisted or untwisted I-bundle over G. If $N(G, P) \approx G \times I$, then G is called *two-sided*.

Let $G \subset P$ be a two-sided theta-curve or eyeglass graph in a special spine P. Cut P along G and repaste the two copies of G thus obtained according to the homeomorphism ρ. We shall say that the resulting new special polyhedron P_1 is obtained from P by a *spine move* along G.

Remark. Since ρ maps each edge of G to itself, P_1 is similar to P.

Thus we have two types of spine moves: the theta-move σ_1 and the eyeglass-move σ_2. It is convenient to introduce a third move σ_3.

Let $G \subset P$ be a proper theta-curve with edges l_1, l_2, l_3 such that

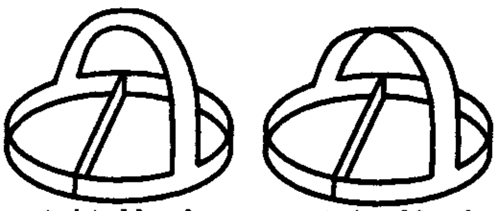
(1) G separates P;
(2) l_1 and l_2 belong to the same 2-cell C of P;
(3) there exists a simple arc $l \subset C$ such that l connects l_1 and l_2 and $l \cap G = \partial l$.

Choose a homeomorphism $\rho_3 : G \to G$ such that ρ_3 leaves l_3 fixed and permutes l_1 and l_2. Cut P along G and repaste the two copies of G thus obtained according to ρ_3. We shall say that the new special polyhedron P_1 arising in such a way is obtained from P by the move σ_3.

Lemma 1. *σ_3 can be expressed through σ_1 and σ_2.*

Proof. Consider a regular neighborhood $N = N(G \cup l, P)$ of $G \cup l$ in P. Since F separates P, the set N can be presented as $G \times [0,1]$ with twisted or untwisted band attached to $G \times \{1\}$, see Fig. 1.

Figure 1.

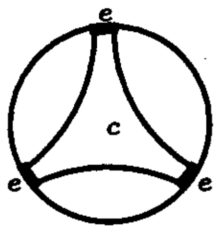


If the band is untwisted, then N is bounded by G_1 and G_2, where G_1 is a theta-curve isotopic to G and G_2 is the eyeglass graph. There exists a homeomorphism $h : N \to N$ such that $h\big|_{G_1} = \rho_3$ and $h\big|_{G_2} = \rho_2$. It follows that the move σ_3 along G_1 (and along G) is equivalent to the move σ_2 along G_2.

Let the band be twisted. Then N is bounded by two theta-curves G_1 and G_2, where G_1 is isotopic to G. There exists a homeomorphism $h : N \to N$ such that $h\big|_{G_1} = \rho_1\rho_3$ and $h\big|_{G_2} = \rho_1$. Hence, the superposition of the moves σ_1 and σ_3 along G_1 is equivalent to the move σ_1 along G_2. Taking into account that $\rho_1^2 = 1$, we can conclude that the move σ_3 along G is equivalent to the superposition of the move σ_1 along G_1 and the move σ_1 along G_2.

Suppose the boundary curve of a 2-cell C of a special spine P passes along an edge e of P three times. Choose two points on e and join them by three arcs in C as is shown in Fig. 2. The union G of the arcs is a proper two-sided theta-curve or an eyeglass graph in P. One can consider the spine move along G. To distinguish this special type of spine moves we shall call it a *spine move across e*.

Figure 2.



§4. RELATIONS BETWEEN MANIFOLD MOVES AND SPINE MOVES

Lemma 2. *Let G be a proper theta-curve or eyeglass graph in a special spine P of a closed 3-manifold M. Then there exists a closed connected surface $F \subset M$ such that $\chi(F) \geqslant 0$, $F \cap P = G$, and F is transverse to SP.*

Proof. Let $N = N(P, M)$ be a regular neighborhood of P in M. Since $N - P$ is homeomorphic to $\partial N \times (0, 1]$, one can easily construct a surface $F_1 \subset M$ such that

(1) $F_1 \cap \partial M - \partial F_1$, $F_1 \cap P = G$, and F_1 is transverse to SP;
(2) G is a spine of F_1.

To obtain F, attach disjoint 2-cells in the 3-cell $M - N$ to the boundary components of F_1. Since $\chi(F_1) = \chi(G) = -1$, we have $\chi(F) \geqslant 0$.

Proposition 1. *Let P be a special spine of a closed 3-manifold M, and let $G \subset P$ be an admissible theta-curve or eyeglass graph. Denote by P_1 the special polyhedron obtained from P by the spine move along G. Then*

(1) *P_1 is a spine of a closed 3-manifold M_1;*
(2) *M_1 can be obtained from M by a manifold move.*

Proof. Let $F \subset M$ be the surface constructed in Lemma 1. Since G is two-sided, F is also two-sided. The homeomorphism $\rho : G \to G$ can be extended to a homeomorphism $r : F \to F$. It is clear that r satisfies conditions (1)–(3) of the definition of a manifold move. Denote by M_1 the 3-manifold obtained from M by the manifold move along F. Since $r\big|_G = \rho$, P_1 is a spine of M_1.

Proposition 2. *Let a closed 3-manifold M_1 be obtained from a closed 3-manifold M by a manifold move along a surface $F \subset M$. Then M and M_1 are similar.*

Proof. One can easily construct a special spine P of M such that $G = P \cap F$ is a proper two-sided theta-curve. Apply to P the spine move along G. From Proposition 1 it follows that the special polyhedron P_1 thus obtained is a spine of M_1. Since P and P_1 are similar, the same is true for M and M_1.

Proof of Theorem 1. Let $M_1 = S^1 \times S^1 \times S^1$ and $F = S^1 \times S^1 \times \{*\} \subset M$. To construct M_2, perform the manifold move on M_1 along F. By Proposition 2, M_2 is similar to M_1. A simple calculation shows that $H_1(M_2; \mathbb{Z}/k) = \mathbb{Z}/k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ if k is even, and $H_1(M_2; \mathbb{Z}/k) = \mathbb{Z}/k \oplus \mathbb{Z}/k \oplus \mathbb{Z}/k$ if k is odd.

§5. MOVES ON COLORED GRAPHS

Let Γ be a finite (multi)graph. Fix a finite set A. By a *coloring* of Γ we mean a map $c : E(\Gamma) \to A$, where $E(\Gamma)$ is the set of all open edges of Γ. Denote by $V(\Gamma)$ the set of vertices of Γ and by $N(V, \Gamma)$ a regular neighborhood of V in Γ. The intersection of the union of all open edges with each connected component of $N(V, \Gamma)$ consists of half-open 1-cells, which are called *thorns*.

Definition. Two colored graphs Γ_1 and Γ_2 are called *similar*, if there exists a homeomorphism $\varphi : N(V(\Gamma_1), \Gamma_1) \to N(V(\Gamma_2), \Gamma_2)$ preserving the colors of thorns. The homeomorphism φ is called a *similarity homeomorphism*.

Let Γ be a colored graph. Choose two edges e_1 and e_2 of the same color and cut each of them in the middle. Repaste the four "half edges" thus obtained into two new edges which do not coincide with the initial ones. We shall say that the new colored graph Γ_1 arising in this way is obtained from Γ by a *graph move* along e_1 and e_2. The graph move is called *admissible* if Γ and Γ_1 are connected. Note that for given e_1 and e_2 there exist two different graph moves along e_1 and e_2; if Γ is connected, then precisely one of them is admissible.

Lemma 2. *Let Γ_1 and Γ_2 be similar graphs. If they are connected, then one can pass from Γ_1 to Γ_2 by a sequence of admissible graph moves.*

Proof. By definition, there exists a homeomorphism $\varphi : N(V(\Gamma_1), \Gamma_1) \to N(V(\Gamma_2), \Gamma_2)$ preserving the colors of thorns. We call an edge e in Γ_1 *correct* if φ maps the two thorns t_1 and t_2 contained in it into the same edge f in Γ_2. The thorns t_1 and t_2, the edge f and the thorns $\varphi(t_1), \varphi(t_2) \subset f$ are also called *correct*. The homeomorphism φ can be extended to an edge e if and only if e is correct, and thus, to prove Lemma 3 it is sufficient to show that the number of correct edges can be increased by admissible graph moves on Γ_1 and Γ_2.

Let t_1 be an incorrect thorn in Γ_1 and let $t_2, t_3, \ldots, t_{2n}$ be all other incorrect thorns of the same color (say, red). We shall say that a thorn t_i, $2 \leqslant i \leqslant 2n$, is *good*, if t_1 and t_i belong to the same edge or if they can be transferred to the same edge by an admissible graph move on Γ_1. Denote by T the set $\{\tau_i = \varphi(t_i),\ 1 \leqslant i \leqslant 2n\}$ of all red incorrect thorns in Γ_2. We shall say that a thorn τ_i, $2 \leqslant i \leqslant 2n$, is *good*, if τ_1 and τ_i belong to the same edge or if they can be transferred to the same edge by an admissible graph move on Γ_2.

Consider two subsets A_1 and A_2 of the set T. The subset $A_1 \subset T$ consists of the images of good thorns in Γ_1, the subset $A_2 \subset T$ is the set of all good thorns in Γ_2. Let $\#X$ denote the number of elements in X. Since any red incorrect edge in Γ_1 and Γ_2 contains at least one good thorn, we have $\#A_1 \geqslant n$ and $\#A_2 \geqslant n$. Note that $\#T = 2n$ and, because t_1 and $\tau_1 = \varphi(t_1)$ are not good, τ_1 does not belong to $A_1 \cup A_2$. Hence, $\#(A_1 \cup A_2) < 2n$, and $A_1 \cap A_2 \neq \varnothing$. We can conclude that there exist i and j, $2 \leqslant i, j \leqslant 2n$, such that t_i and τ_j are good. By the definition of good edges, we can perform graph moves such that after these moves, t_1 and t_2 belong to the same edge and τ_1 and τ_2 also belong to the same edge. The moves are performed along incorrect edges. Hence, all correct edges are preserved, but now a new correct edge has appeared (namely, the one containing t_1 and t_2).

§6. CORRECTION OF EDGES

Lemma 4. *Let P be a special spine of a closed 3-manifold M. Suppose that M does not contain closed incompressible surfaces with a nonnegative Euler characteristic. Then each proper theta-curve or eyeglass graph $G \subset P$ is two-sided and separates P.*

Proof. Let $F \subset M$ be the surfaces constructed in Lemma 2. From the assumptions of Lemma 4 it follows that F is compressible and M is irreducible. It remains to note that each compressible sphere, torus, or Klein bottle in an irreducible 3-manifold M separates M and that compressible projective planes do not exist.

Suppose P is a special spine of a closed 3-manifold M. Let us color the 2-cells of P different colors. In each edge of P, three 2-cells meet, and so to each edge there corresponds some unoriented triplet of colors (possibly with multiplicity). We call this triplet the *tricolor* of the edge. Thus, we may treat SP as a colored graph. Note that each spine move on P induces an admissible graph move on SP. It turns out that under certain conditions the converse is also true.

Lemma 5. *If M does not contain incompressible surfaces with a nonnegative Euler characteristic, then each graph move γ on SP is induced by a spine move on P.*

Proof. Let the move γ be performed along edges e_1 and e_2. Then e_1 and e_2 have the same tricolor. Connect the middle points of e_1 and e_2 by three disjoint arcs $l_j \subset P$ $(j = 1, 2, 3)$ in such a way that $G = l_1 \cup l_2 \cup l_3$ in a proper theta-curve (this is also possible if the tricolor has multiplicity). By Lemma 4, G is two-sided and separates P. Denote by σ_1 the spine move along G. Then σ_1 induces an admissible graph move along e_1 and e_2. Since such a move is unique, it coincides with γ.

Proposition 3. *Let M_1 and M_2 be similar closed 3-manifolds. Suppose M_1 does not contain closed incompressible surfaces with a nonnegative Euler characteristic. Then there exists special spines P_i of M_i $(i = 1, 2)$ and a homeomorphism $\psi : N_1 \cup SP_1 \to N_2 \cup SP_2$ such that $\psi\big|_{N_1} : N_1 \to N_2$ is a similarity homeomorphism, where $N_i = N(V(P_i), P_i)$, $i = 1, 2$.*

Proof. Let $\varphi : N_1 \to N_2$ be a similarity homeomorphism, where P_1 and P_2 are special spines of M_1 and M_2, respectively. We imagine the 2-cells of P_1 and P_2 as being painted in different colors such that φ preserves the colors of wings. As above, we also paint each edge in the corresponding tricolor. Then φ induces a similarity homeomorphism between SP_1 and SP_2. If all edges of SP_1 are correct, then φ can be extended to a homeomorphism ψ satisfying the conclusion of the proposition. If not, we use Lemma 3 to correct them by a sequence of graph moves. By Lemma 5, this sequence can be realized by a sequence of spine moves on P_1. It remains to note that each move on a spine of M_1 produces a spine of the same manifold.

§7. CORRECTION OF 2-CELLS

Let P_1 and P_2 be special spines of M_1 and M_2, and let a homeomorphism $\psi : N_1 \cup SP_1 \to N_2 \cup SP_2$ induce a similarity homeomorphism ψ' between $N_1 = N(V(P_1), P_1)$ and $N_2 = N(V(P_2), P_2)$. Identify $N_1 \cup SP_1$ and $N_2 \cup SP_2$ via ψ. We obtain two special spines P_1 and P_2 such that their singular graphs and wings coincide.

Let e be an edge of P_1. It contains two thorns t_1, t_2. Let $\omega_1^{(i)}, \omega_2^{(i)}, \omega_3^{(i)}$ be the wings adjacent to t_i, $i = 1, 2$. A regular neighborhood $N(e - \mathrm{Int}(t_1 \cup t_2), P_1)$ of a middle part of e in P_1 is homeomorphic to $Y \times I$, where Y is a wedge of three segments. Hence, we have a natural bijection $a_{1e} : \{\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}\} \to \{\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}\}$. In the same way a direct product structure on $N(e - \mathrm{Int}(t_1 \cup t_2), P_2)$ determines a natural bijection a_{2e} : $\{\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}\} \to \{\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}\}$. Denote by β_e the permutation $a_{2e}^{-1} a_{1e}$.

Definition. An edge e is called *even* (*odd*) if β_e is an even (odd) permutation.

Let C be a 2-cell of P_1. Denote by E_C the collection of edges incident to C. We admit multiplicity, so that if the boundary curve of C passes along an edge e two or three times, then e is included in E_C respectively two or three times. Note that E_C coincides with the set of edges incident to the 2-cell of P_2 having the same color.

Lemma 6. *For any 2-cell C of P_1 the collection E_C contains an even number of odd edges.*

Proof. Regular neighborhoods $N(V(P_i), M_i)$ $(i = 1, 2)$ consist of 3-balls. Choose orientations of the 3-balls such that the similarity homeomorphism $\psi' : N(V(P_1), P_1) \to N(V(P_2), P_2)$ is extensible to an orientation-preserving homeomorphism between $N(V(P_1), M_1)$ and $N(V(P_2), M_2)$. The orientations induce a cyclic order on the set $\{\omega_1^{(j)}, \omega_2^{(j)}, \omega_3^{(j)}\}$ of wings adjacent to each thorn of P_1 or P_2. We shall say that an edge e is ***orientation-reversing*** with respect to P_i, if the corresponding bijection $a_{ie} : \{\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}\} \to \{\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}\}$ preserves the cyclic order, $i = 1, 2$. Since the boundary curve of each 2-cell in a 3-manifold is orientation-preserving, E_C contains an even number of orientation-reversing edges with respect to P_1 and an even number of orientation-reversing edges with respect to P_2. It remains to note that e is odd if and only if e is orientation-reversing with respect to one of spines P_1 and P_2 and orientation-preserving with respect to the other.

Proof of Theorem 2. According to proposition 3, there exist special spines P_i of M_i $(i = 1, 2)$ and a homeomorphism $\psi : N_1 \cup SP_1 \to N_2 \cup SP_2$ such that $\psi\big|_{N_1} : N_1 \to N_2$ is a similarity homeomorphism, where $N_i = N(V(P_i), P_i)$. As above, let us identify $N_1 \cup SP_1$ with $N_2 \cup SP_2$ via ψ. We define an edge e of P_1 as ***strongly correct*** (SC) if the corresponding permutation β_e is trivial. In the other words, e is SC if and only if the identification ψ can be extended to a neighborhood of e in P_1. Note that if all edges are SC, then ψ can be extended to a homeomorphism between P_1 and P_2 and to a homeomorphism between M_1 and M_2. We assert that one can perform spine moves on P_1 until all edges become SC. This will prove Theorem 2, because each spine move can be extended to a manifold move on M_1 that does not change its homeomorphism type.

As above, we paint the 2-cells of P_1 and P_2 in different colors and the edges in tricolors. Note that if the tricolor of an edge e consists of three different colors, then e is obviously SC. Suppose that the tricolor of e is bichromatic (that is, it has the form (x, y, y), $x \neq y$), and that e is not SC. Then e is odd. It follows from Lemma 6 that there is another non-SC edge e' of tricolor (x, z, z) (possibly $z = x$ or $z = y$). Assuming first that $z \neq y$, we construct a proper eyeglass graph G with the vertices on e and e' (this is also possible when $z = x$). By Lemma 4, G is two-sided, and the spine move σ_2 along G can be performed. The edge e will now be SC. If $z = y$, there are two possibilities for the relative displacement of e and e' along the boundary curve of the y-colored 2-cell, namely, the displacement (e, e, e', e') and the displacement (e, e', e, e'). In the first case we can still construct an eyeglass graph with vertices on e and e' and perform σ_2. In the second case we construct a proper theta-curve G with the vertices on e and e'. The move σ_3 along G makes e strongly correct.

Suppose now that e is a monochromatic non-SC edge of tricolor (x, x, x), and suppose further that there is another edge e' with the same tricolor. Denote by C_x the x-colored 2-cell of P_1. We shall say that e and e' are *linked* if the boundary curve of C_x cannot be decomposed into two arcs d and d', such that d passes three times along l and d' passes three times along l'. Suppose that l and l' are linked. In order to make l strongly correct, we use spine moves σ_3 along theta-curves with vertices on l and l'. Each such move changes β_e by some permutation. It is sufficient to show that each transposition τ of wings can be achieved. In essence, there are two possibilities for the relative displacement of e and e' on the boundary curve of C_x. It is clear that in both cases τ can be realized by a move σ_3 along the theta-curve $G = l_1 \cup l_2 \cup l_3$, see Fig. 3 on the next page.

Suppose now that each two non-SC edges of tricolor (x, x, x) are unlinked. If e is an odd edge with tricolor (x, x, x), then there is another odd edge e' with the same tricolor. We use the eyeglass-move along $G = l_1 \cup l_2 \cup l_3$ to make e and e' even; see Fig. 4 on the next page.

Figure 3.

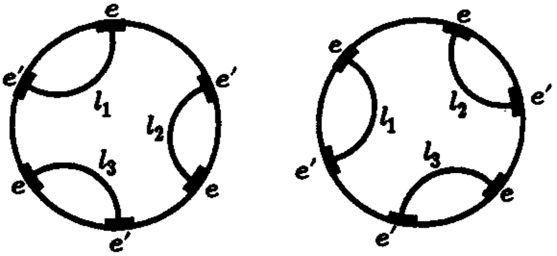
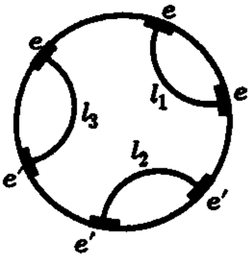


Figure 4.



In conclusion, we consider the situation in which all non-SC edges are monochromatic and even, and there are no linked edges among them. Let e be a non-SC edge with tricolor (x, x, x). Denote by P_3 the spine obtained from P_1 by the spine move across e. Let t_1 and t_2 be the thorns in e and let $w_1^{(i)}, w_2^{(i)}, w_3^{(i)}$ be the wings adjacent to t_i, $i = 1, 2$. The direct product structures on regular neighborhoods of $e - \mathrm{Int}(t_1 \cup t_2)$ in P_i determine natural bijections $a_{ie} : \{w_1^{(1)}, w_2^{(2)}, w_3^{(i)}\} \to \{w_1^{(2)}, w_2^{(2)}, w_3^{(2)}\}$, $i = 1, 2, 3$. It is sufficient to prove that $a_{2e} \equiv a_{3e}$, because this means that the spine move across e makes e strongly correct.

Consider a regular neighborhood N of $SP_1 - e$ in P_1. The difference $N - SP_1$ consists of some number of half-open annuli and precisely three x-colored half-open discs. Each of the discs contains two wings from the set $W = \{w_j^{(i)},\ 1 \leqslant j \leqslant 3,\ i = 1, 2\}$. Thus we have a decomposition of the set W into three pairs. Taking P_2 or P_3 instead of P_1, we obtain two other decompositions. A very important observation is that since all nonmonochromatic edges are SC and e is not linked with any other edge, these three decompositions do coincide.

At least one pair resulting from the decomposition contains a wing adjacent to the pair, $1 \leqslant j, k \leqslant 3$. Since each of the spines P_1, P_2, P_3 contains only one x-colored 2-cell, we have $a_{ie}(w_j^{(1)}) \neq w_k^{(2)}$, $1 \leqslant i \leqslant 3$. Hence, among $a_{1e}(w_j^{(1)}), a_{2e}(w_j^{(1)}), a_{3e}(w_j^{(1)})$ at least two wings coincide. Taking into account that any two of different bijections a_{1e}, a_{2e}, a_{3e} differ on an even permutation, we can conclude that at least two of them do coincide. Since e is not SC and since the spine move across e changes the corresponding bijection, we have $a_{1e} \not\equiv a_{2e}$ and $a_{1e} \not\equiv a_{3e}$. It follows that $a_{2e} \equiv a_{3e}$.

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