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Games with Costly Winnings

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We introduce a new sequential game, where each player has a limited resource that he needs to spend on increasing the probability of winning each stage, but also on maintaining the assets that he has won in the previous stages. Thus, the players' strategies must take into account that winning at any given stage negatively affects the chances of winning in later stages. Whenever the initial resources of the players are not too small, we present explicit strategies for the players, and show that they are a Nash equilibrium, which is unique in an appropriate sense.

Keywords: Game theory; stochastic Blotto games; costly winnings.

1. Introduction

We present a new n-stage game. Each player starts the game with some given resource, and at the beginning of each stage, he must decide how much resource to invest in that stage. A player wins the given stage with probability corresponding to the relative investments of the players. The winner of the stage receives a payoff which may differ from stage to stage. The new feature of our game is that the winner of each stage is required to spend additional resources on the maintenance of his winning. This is a real life situation, where the winnings are some assets, and resources are required for their maintenance, as in wars, territorial contests among organisms, or in the political arena. The winner of a given stage must put aside all resources that will be required for future maintenance costs of the won asset. Thus, a fixed amount will be deducted from the resources of the winner immediately after winning, which should be thought of as the sum of all future maintenance costs for the given acquired asset.

Since the probability for winning is the relative investment of each player, a rule must be given for what happens if all players invest 0. The rule that is usually chosen in similar games is that if all players invest 0, then all players win with equal probability. This rule is, however, inappropriate in our setting where the winnings are costly. A player may feel that the maintenance costs of a given win are too high and so he does not want to win the given stage. In such case, it would be unreasonable to force the win upon him. So, our rule will be that if all players invest 0, then no player wins that stage. Since it is thus possible that certain stages will not be won by any player, this is not a fixed sum game.

The players' resources from which the investments are taken and from which the maintenance costs are paid can be thought of as money, whereas the payoffs should be thought of as a quantity of different nature, such as political gain. The two quantities cannot be interchanged, that is, the payoff cannot be converted into resources for further investment.

At each stage, the player thus needs to decide how much to invest in the given stage, where winning that stage on one hand leads to the payoff of the given stage, but on the other hand, the maintenance cost for the given winning negatively affects the probabilities for future winnings. In the present work, we analyze the case where the initial resources of the players are not too small. In this case, we present explicit strategies for the players, and show that they are a Nash equilibrium which is unique in an appropriate sense. (Theorem 3.6 for two players and Theorem 4.1 for m players.)

There are similarities between our game and the well-known Colonel Blotto game [Borel, 1921]. In Blotto games, two players simultaneously distribute forces across several battlefields. At each battlefield, the player who allocates the largest force wins. The Blotto game has been developed and generalized in many directions (see e.g., Borel [1921]; Duffy and Matros [2015]; Friedman [1958]; Hart [2008]; Lake [1979]; Roberson [2006]). Two main developments are the "asymmetric" and the "stochastic" models. The asymmetric version allows the payoffs of the battlefields to differ from each other, and in the stochastic model, the deterministic rule deciding on the winner is replaced by a probabilistic one, by which the chances of winning a battlefield depends on the size of investment. The present work adds a new feature which changes the nature of the game, in making the winnings costly. The players thus do not know before hand how much of their resources will be available for investing in winning rather than on maintenance, and so the game cannot be formulated with simultaneous investments, as in the usual Blotto games, but rather must be formulated with sequential stages.

This work was inspired by previous work of the first author with Zamir and Segev [Nowik, 2009; Nowik *et al.*, 2012] on a developmental competition that occurs in the nervous system, which we now describe. A muscle is composed of many muscle fibers. At birth, each muscle fiber is innervated by several motor neurons (MNs) that "compete" to singly innervate it. It has been found that MNs with higher activation threshold win in more competitions than MNs with lower activation thresholds. In

Nowik [2009], this competitive process is modeled as a multistage game between two groups of players: those with lower and those with higher thresholds. At each stage, a competition at the most active muscle fiber is resolved. The strategy of a group is defined as the average activity level of its members and the payoff is defined as the sum of their wins. If a MN wins (i.e., singly innervates) a muscle fiber, then from that stage on, it must continually devote resources for maintaining this muscle fiber. Hence, the MNs use their resources both for winning competitions and for maintaining previously acquired muscle fibers. It is proved in Nowik [2009] that in such circumstances, it is advantageous to win in later competitions rather than in earlier ones, since winning at a late stage will encounter less maintenance and thus will negatively affect only the few competitions that were not yet resolved. If μ is the cost of maintaining a win at each subsequent stage, then in the terminology of the present work, the fee payed by the MNs for winning the kth stage of an n stage game is $(n-k)\mu$.

We start by analyzing the two-person game in Secs. 2 and 3, and then generalize to the m-person setting in Sec. 4.

2. The Two-Person Game

The initial data for the two-person version of our game is the following:

(1) The number n of stages of the game.
(2) Fixed payoffs $w_k > 0,\ 1 \leq k \leq n$, to be received by the winner of the kth stage.
(3) The initial resources $A, B \geq 0$ of players I, II, respectively.
(4) Fixed fees $c_k \geq 0,\ \ 1 \leq k \leq n-1$, to be deducted from the resources of the winner after the kth stage.

The rules of the game are as follows: At the kth stage of the game, the two players, which we name PI, PII, each has some remaining resource A_k, B_k, where $A_1 = A, B_1 = B$. PI, PII each needs to decide his investment x_k, y_k for that stage, respectively, with $0 \leq x_k \leq A_k - c_k,\ \ 0 \leq y_k \leq B_k - c_k$, and where if $A_k < c_k$, then PI may only invest 0, and similarly for PII. The probability for PI, PII of winning this stage is, respectively, $\frac{x_k}{x_k+y_k}$ and $\frac{y_k}{x_k+y_k}$, where if $x_k = y_k = 0$, then no player wins. These rules ensure that the winner of the given stage will have the resources for paying the given fee c_k. The resource of the winner of the kth stage is then reduced by an additional c_k, that is, if PI wins the kth stage, then $A_{k+1} = A_k - x_k - c_k$ and $B_{k+1} = B_k - y_k$, and if PII wins, then $A_{k+1} = A_k - x_k$ and $B_{k+1} = B_k - y_k - c_k$. The role of c_k is in determining A_{k+1}, B_{k+1}, thus there is no c_n. It will however be convenient in the sequel to formally define $c_n = 0$. The payoff received by the winner of the kth stage is w_k. Since it is possible that no player wins certain stages, this game is not a fixed sum game.

As already mentioned, the resource quantities A_k, B_k, x_k, y_k, c_k used for the investments and fees are of different nature than that of the payoffs w_k. The two quantities cannot be interchanged and should be thought of as having different "units". Note that all expressions below are unit consistent, that is, if say, we divide resources by payoff, then such expression has units of $\frac{\text{resources}}{\text{payoff}}$, and may only be added or equated to expressions of the same units.

If A and B are too small in comparison to $c_1, \ldots, c_{n-1}$ then the players' strategies are strongly influenced by the possibility of running out of resources before the end of the game. In the present work, we analyze the game when A, B are not too small. Namely, we introduce a quantity M depending on $c_1, \ldots, c_{n-1}$ and $w_1, \ldots, w_n$, and for $A, B > M$, we introduce explicit strategies for $P\mathrm{I}$ and $P\mathrm{II}$, and show that they are a Nash equilibrium for the game, which is unique in a sense to be explained.

For $k = 1, \ldots, n$ let $W_k = \sum_{i=k}^{n} w_i$ and $W = W_1$. We now show that if $A > M$, then if $P\mathrm{I}$ always chooses to invest $x_k \leq \frac{w_k}{W_k} A_k$ (as holds for our strategy $\sigma_{n,A,B}$ presented in Definition 3.1 below), then whatever the random outcomes of the game are, his resources will not run out before the end of the game. We in fact give a specific lower bound on A_k for every k, which will be used repeatedly in the sequel.

Proposition 2.1. *Let*

$$M = W \cdot \max_{1 \leq k \leq n} \left(\frac{c_k}{w_k} + \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}} \right).$$

If $A > M$, and if $P\mathrm{I}$ plays $x_k \leq \frac{w_k A_k}{W_k}$ for all k, then $A_k > \frac{W_k c_k}{w_k}$ for all $1 \leq k \leq n$. In particular, $A_k > 0$ for all $1 \leq k \leq n$. (And similarly for $P\mathrm{II}$.)

Proof. For every $1 \leq k \leq n$, we have $\frac{A}{W} > \frac{M}{W} \geq \frac{c_k}{w_k} + \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}}$, so

$$\frac{A}{W} - \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}} > \frac{c_k}{w_k}.$$

Thus, it is enough to show that $\frac{A_k}{W_k} \geq \frac{A}{W} - \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}}$ for all $1 \leq k \leq n$. We show this by induction on k. For $k = 1$, the sum is empty and we get equality. Assuming

$$\frac{A_k}{W_k} \geq \frac{A}{W} - \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}},$$

we get

$$\begin{aligned} \frac{A_{k+1}}{W_{k+1}} &\geq \frac{1}{W_{k+1}} \left(A_k - \frac{w_k A_k}{W_k} - c_k \right) = \frac{1}{W_{k+1}} \left(\frac{W_{k+1} A_k}{W_k} - c_k \right) \\ &= \frac{A_k}{W_k} - \frac{c_k}{W_{k+1}} \geq \frac{A}{W} - \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}} - \frac{c_k}{W_{k+1}} = \frac{A}{W} - \sum_{i=1}^{k} \frac{c_i}{W_{i+1}}. \end{aligned}$$

□

We consider two simple examples of $c_1, \ldots, c_{n-1}$, $w_1, \ldots, w_n$, for which M may be readily identified.

(1) Let $c_k = n - k,\ w_k = 1$ for all k. These fees and payoffs are as in the biological game described in the introduction. For every $1 \le k \le n$, we have $n\big(c_k + \sum_{i=1}^{k-1} \frac{c_i}{n-i}\big) = n\big((n-k) + (k-1)\big) = n(n-1)$, so $M = n(n-1)$.

(2) Let $c_k = 1$ for all $1 \le k \le n-1,\ \ w_k = 1$ for all k. Using the inequality $\sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n$, we get for every $1 \le k \le n$ that $n\big(c_k + \sum_{i=1}^{k-1} \frac{c_i}{n-i}\big) < n(2 + \ln n)$, so $M < n(2 + \ln n)$.

We note that an obvious necessary condition for A to satisfy Proposition 2.1 is $A \ge \sum_{k=1}^{n-1} c_k$, since in case PI wins all stages, he will need to pay all fees c_k. We see that M in the two examples above is not much larger. Namely, in (1), $\sum_{k=1}^{n-1} c_k = \frac{n(n-1)}{2}$ and $M = n(n-1)$, and in (2), $\sum_{k=1}^{n-1} c_k = n - 1$ and $M < n(2 + \ln n)$.

3. Nash Equilibrium Strategies

We define the following two strategies $\sigma_{n,A,B}$ and $\tau_{n,A,B}$ for PI, PII, respectively. We prove that for $A, B > M$ as given in Proposition 2.1, this pair of strategies is a Nash equilibrium, which is unique in a sense to be explained, and these strategies guarantee the given payoffs.

Definition 3.1. At the kth stage of the game, let

$$a_k = \frac{w_k A_k}{W_k} - \frac{A_k c_k}{A_k + B_k} \quad \text{and} \quad b_k = \frac{w_k B_k}{W_k} - \frac{B_k c_k}{A_k + B_k},$$

where as mentioned, we formally define $c_n = 0$. The strategy $\sigma_{n,A,B}$ for PI is the following: At the kth stage, PI invests a_k if it is allowed by the rules of the game. Otherwise, he invests 0. The strategy $\tau_{n,A,B}$ for PII is similarly defined with b_k.

Recall that $a_k \neq 0$ is allowed by the rules of the game if $0 \le a_k \le A_k - c_k$, whereas $a_k = 0$ is always allowed, even when $A_k - c_k < 0$. We interpret the quantities a_k, b_k as follows. PI first divides his remaining resource A_k to the remaining stages in proportion to the payoff for each remaining stage, which gives $\frac{w_k}{W_k} A_k$. From this, he subtracts $\frac{A_k}{A_k + B_k} c_k$ which is the expected fee he will pay for this stage, since $\frac{a_k}{a_k + b_k} = \frac{A_k}{A_k + B_k}$. Note that $W_n = w_n$ and formally $c_n = 0$, so $a_n = A_n,\ \ b_n = B_n$, i.e., at the last stage, the two players invest all their remaining resources.

Depending on A and B and on the random outcomes of the game, it may be that PI indeed reaches a stage where a_k is not allowed. In this regard, we make the following definition.

Definition 3.2. The triple (n, A, B) is PI-*effective* if when PI and PII use $\sigma_{n,A,B}$ and $\tau_{n,A,B}$, then with probability 1, a_k will always be allowed for PI. Similarly PII-effectiveness is defined for PII with b_k.

Proposition 3.3. *Let M be as in Proposition* 2.1. *If $A > M$ and B is arbitrary, then (n, A, B) is PI-effective. Furthermore, $a_k > 0$ for all k.* (*And similarly for PII when $B > M$.*)

Proof. We need to show that necessarily $0 < a_k \leq A_k - c_k$ for all $1 \leq k \leq n$. We have $a_k = \frac{w_k A_k}{W_k} - \frac{A_k c_k}{A_k + B_k} \leq \frac{w_k A_k}{W_k}$, so by Proposition 2.1, $\frac{w_k}{W_k} > \frac{c_k}{A_k} \geq \frac{c_k}{A_k + B_k}$ and $A_k > 0$, so $\frac{w_k A_k}{W_k} > \frac{A_k c_k}{A_k + B_k}$ giving $a_k > 0$.

For the inequality $a_k \leq A_k - c_k$, we first consider $k \leq n-1$. We have from the proof of Proposition 2.1 that $\frac{A_k}{W_k} - \frac{c_k}{W_{k+1}} \geq \frac{A}{W} - \sum_{i=1}^{k} \frac{c_i}{W_{i+1}} > \frac{c_{k+1}}{w_{k+1}} \geq 0$, so $\frac{A_k}{W_k} > \frac{c_k}{W_{k+1}}$, and so

$$\left(1 - \frac{A_k}{A_k + B_k}\right) c_k \leq c_k < \frac{W_{k+1} A_k}{W_k} = \left(1 - \frac{w_k}{W_k}\right) A_k.$$

This gives $c_k - \frac{A_k c_k}{A_k + B_k} < A_k - \frac{w_k A_k}{W_k}$, so $a_k = \frac{w_k A_k}{W_k} - \frac{A_k c_k}{A_k + B_k} < A_k - c_k$. For $k = n$, we note that $c_n = 0$ by definition, and $W_n = w_n$, so $a_n = A_n = A_n - c_n$. □

In general, an inductive characterization of PI-effectiveness will also involve induction regarding PII. But if we assume that $B > M$, and so by Proposition 3.3, all b_k are known to be allowed and positive, then the notion of PI-effectiveness becomes simpler, and may be characterized inductively as follows. When saying that a triple $(n-1, A', B')$ is PI-effective, we refer to the $n-1$ stage game with fees $c_2, \ldots, c_{n-1}$ and payoffs $w_2, \ldots, w_n$. Starting with $n = 1$, $(1, A, B)$ is always PI-effective. For $n \geq 2$, if a_1 is not allowed, then (n, A, B) is not PI-effective. If $a_1 = 0$, then it is allowed, and PI surely loses the first stage, and so (n, A, B) is PI-effective if and only if $(n-1, A, B - b_1 - c_1)$ is PI-effective. Finally, if $a_1 > 0$ and it is allowed, then (n, A, B) is PI-effective if and only if both $(n-1, A - a_1 - c_1, B - b_1)$ and $(n-1, A - a_1, B - b_1 - c_1)$ are PI-effective.

The crucial step in proving Theorem 3.6 below, on $\sigma_{n,A,B}$, $\tau_{n,A,B}$ being a Nash equilibrium, is the following Theorem 3.5. We point out that in Theorem 3.6, we will assume that $A > M$, in which case (n, A, B) is PI-effective, by Proposition 3.3. But, here in Theorem 3.5, we must consider arbitrary $A \geq 0$ in order for an induction argument to carry through.

Definition 3.4. Two strategies σ, σ' of PI are said to be equivalent with respect to a strategy τ of PII, if when PII uses τ, then with probability 1, σ and σ' will always dictate the exact same moves. Equivalence of strategies τ, τ' of PII with respect to σ of PI, is similarly defined.

Theorem 3.5. *Given $c_1, \ldots, c_{n-1}$ and $w_1, \ldots, w_n$, let M be as in Proposition 2.1, and assume that $B > M$ and PII plays the strategy $\tau_{n,A,B}$.*

For $A \geq 0$, if (n, A, B) is PI-effective, and PI plays according to $\sigma_{n,A,B}$, then his expected payoff is $\frac{AW}{A+B}$. On the other hand, if (n, A, B) is not PI-effective, or if PI uses a strategy that is not equivalent to $\sigma_{n,A,B}$ with respect to $\tau_{n,A,B}$, then the expected payoff of PI is strictly less than $\frac{AW}{A+B}$.

Proof. By induction on n. We note that throughout the present proof, we do not use the condition $B > M$ directly, but rather only through the statements of

Propositions 3.3 and 2.1 saying that (n, A, B) is PII-effective, $b_k > 0$ and $c_k < \frac{w_k B_k}{W_k}$ for all $1 \le k \le n$, which indeed continue to hold along the induction process.

If $A = 0$, then $a_k = 0$ for all k, which is the only possible investment, and its payoff is $0 = \frac{AW}{A+B}$, so the statement holds. We thus assume from now on that $A > 0$. For $n = 1$, we have $b_1 = B$. The allowed investment for PI is $0 \le s \le A$ with expected payoff $\frac{s}{s+B} w_1 = \frac{s}{s+B} W$ which indeed attains a strict maximum $\frac{A}{A+B} W$ at $s = A = a_1$.

For $n \ge 2$, let s be the investment of PI in the first stage. Assume first that $s = 0$. In this case, PII surely wins the first stage and so following this stage, we have $A_2 = A$ and $B_2 = B - b_1 - c_1$. The moves for PII dictated by $\tau_{n,A,B}$ for the remaining $n-1$ stages of the game are $\tau_{n-1,A,B-b_1-c_1}$, and so by the induction hypothesis, the expected total payoff of PI is at most $\frac{AW_2}{A+B-b_1-c_1}$. Since Proposition 2.1 holds for PII, we have $c_1 < \frac{w_1 B}{W} \le \frac{w_1(A+B)}{W}$, that is, $\frac{w_1}{W} - \frac{c_1}{A+B} > 0$, and since $A > 0$, we get $a_1 = A(\frac{w_1}{W} - \frac{c_1}{A+B}) > 0$. This means that $s = 0 \ne a_1$, so we must verify the strict inequality $\frac{AW_2}{A+B-b_1-c_1} < \frac{AW}{A+B}$. This is readily verified, using $A > 0$, $c_1 < \frac{w_1 B}{W}$, $W_2 = W - w_1$, and $b_1 + c_1 = \frac{w_1 B}{W} - \frac{Bc_1}{A+B} + c_1 = \frac{w_1 B}{W} + \frac{Ac_1}{A+B}$.

We now assume $s > 0$. This is allowed only if $A > c_1$ and $0 < s \le A - c_1$. The moves for PII dictated by $\tau_{n,A,B}$ for the remaining $n-1$ stages of the game are τ_{n-1,A_2,B_2}. By the induction hypothesis, if PI wins the first stage, which happens with probability $\frac{s}{s+b_1} > 0$, then his expected payoff in the remaining $n-1$ stages of the game is at most $\frac{(A-s-c_1)W_2}{A+B-s-b_1-c_1}$. Similarly, if he loses the first stage, which happens with probability $\frac{b_1}{s+b_1} > 0$, then his expected payoff in the remaining $n-1$ stages is at most $\frac{(A-s)W_2}{A+B-s-b_1-c_1}$. Thus, the expected payoff of PI for the whole n stage game is at most $F(s)$, where

$$F(s) = \frac{s}{s+b_1}\left(w_1 + \frac{(A-s-c_1)W_2}{A+B-s-b_1-c_1}\right) + \frac{b_1}{s+b_1} \cdot \frac{(A-s)W_2}{A+B-s-b_1-c_1}$$

with $b_1 = \frac{w_1 B}{W} - \frac{Bc_1}{A+B}$.

By the induction hypothesis, we know furthermore, that in case PI wins the first stage, he will attain the maximal expected payoff $\frac{(A-s-c_1)W_2}{A+B-s-b_1-c_1}$ in the remaining stages of the game only if $(n-1, A-s-c_1, B-b_1)$ is PI-effective, and he always plays according to the instructions of $\sigma_{n-1,A-s-c_1,B-b_1}$. Similarly, if he loses the first stage, he will attain the maximal expected payoff $\frac{(A-s)W_2}{A+B-s-b_1-c_1}$ only if $(n-1, A-s, B-b_1-c_1)$ is PI-effective and he plays according to $\sigma_{n-1,A-s,B-b_1-c_1}$. If not, then since both alternatives occur with positive probability, his expected total payoff for the whole n stage game will be strictly less than $F(s)$.

To analyze $F(s)$, we make a change of variable $s = a_1 + x$, that is, we define $\widehat{F}(x) = F(a_1 + x) = F(\frac{w_1 A}{W} - \frac{Ac_1}{A+B} + x)$. After some manipulations, we get

$$\widehat{F}(x) = \frac{AW}{A+B} - \frac{BW^3x^2}{(A+B)(W_2(A+B) - Wx)(Wx - Wc_1 + w_1(A+B))}.$$

Under this substitution, $s = a_1$ corresponds to $x = 0$, and the allowed domain $0 < s \leq A - c_1$ corresponds to

$$\frac{Ac_1}{A+B} - \frac{w_1 A}{W} < x \leq \frac{W_2 A}{W} - \frac{Bc_1}{A+B}.$$

Using $c_1 < \frac{w_1 B}{W}$, one may verify that in the above expression for $\widehat{F}$ the two linear factors appearing in the denominator of the second term are both strictly positive in this domain. It follows that $\widehat{F}$ in the given domain is at most $\frac{AW}{A+B}$, and this maximal value is attained only for $x = 0$ (if it is in the domain), which corresponds to $s = a_1$ for the original F. Finally, as mentioned, unless $(n-1, A_2, B_2)$ is PI-effective and PI plays according to σ_{n-1,A_2,B_2}, his expected payoff will be strictly less than $F(s)$, and so strictly less than $\frac{AW}{A+B}$. □

We may now prove our main result.

Theorem 3.6. *Given $c_1, \ldots, c_{n-1}$ and $w_1, \ldots, w_n$, let M be as in Proposition* 2.1, *and assume $A, B > M$. Then*

(1) *The pair of strategies $\sigma_{n,A,B}, \tau_{n,A,B}$ is a Nash equilibrium for the game, with expected total payoffs $\frac{AW}{A+B}$, $\frac{BW}{A+B}$, which are guaranteed by these strategies.*
(2) *Any other Nash equilibrium pair σ, τ satisfies that σ is equivalent to $\sigma_{n,A,B}$ with respect to $\tau_{n,A,B}$, and τ is equivalent to $\tau_{n,A,B}$ with respect to $\sigma_{n,A,B}$.*

Proof. Denote $\sigma_0 = \sigma_{n,A,B}$ and $\tau_0 = \tau_{n,A,B}$, and for any pair of strategies σ, τ, let $S_1(\sigma, \tau)$, $S_2(\sigma, \tau)$ be the expected payoffs of PI, PII, respectively. Theorem 3.5 and Proposition 3.3 applied to both PI and PII imply that σ_0, τ_0 are a Nash equilibrium with the given payoffs. Since $B > M$, if PII plays τ_0, then by Proposition 3.3, we have $b_k > 0$ for all k, and so there is a winner to each stage of the game, and thus the total combined payoff of PI and PII is necessarily W. Thus, again by Theorem 3.5, for any strategy σ of PI we have $S_2(\sigma, \tau_0) = W - S_1(\sigma, \tau_0) \geq \frac{BW}{A+B}$, i.e., the payoff $\frac{BW}{A+B}$ is guaranteed by τ_0. Similarly, for any strategy τ of PII, $S_1(\sigma_0, \tau) \geq \frac{AW}{A+B}$.

Now, let σ, τ be any other Nash equilibrium and assume that σ is not equivalent to σ_0 with respect to τ_0. By Theorem 3.5, we have $S_1(\sigma, \tau_0) < S_1(\sigma_0, \tau_0)$ and since playing τ_0 guarantees a combined total payoff of W, we have $S_2(\sigma, \tau_0) > S_2(\sigma_0, \tau_0)$. Since the pair σ, τ is a Nash equilibrium, we also have $S_2(\sigma, \tau) \geq S_2(\sigma, \tau_0)$, and so together $S_2(\sigma, \tau) > S_2(\sigma_0, \tau_0)$, and so $S_1(\sigma, \tau) \leq W - S_2(\sigma, \tau) < W - S_2(\sigma_0, \tau_0) = S_1(\sigma_0, \tau_0)$. Again, since σ, τ is a Nash equilibrium, we have $S_1(\sigma_0, \tau) \leq S_1(\sigma, \tau)$, so together $S_1(\sigma_0, \tau) < S_1(\sigma_0, \tau_0) = \frac{AW}{A+B}$, contradicting the conclusion of the previous paragraph. □

4. The m-Person Game

The generalization of our game and results to an m-person setting is straightforward. Player P_i, $1 \leq i \leq m$, starts with resource $A^i \geq 0$, and we are again given fixed fees $c_1, \ldots, c_{n-1} \geq 0$, and payoffs $w_1, \ldots, w_n > 0$. At stage $1 \leq k \leq n$, player

P_i has remaining resource A^i_k with $A^i_1 = A^i$. At stage k, each player decides to invest $0 \le x^i_k \le A^i_k - c_k$, or 0 if $A^i_k < c_k$. The probability for P_i to win is $\frac{x^i_k}{\sum_{j=1}^m x^j_k}$, and if $x^i_k = 0$ for all i, then no player wins. The winner of the kth stage receives payoff w_k and pays the maintenance fee c_k.

We define M as before, $M = W \cdot \max_{1\le k\le n}(\frac{c_k}{w_k} + \sum_{i=1}^{k-1} \frac{c_i}{W_{i+1}})$.

The strategy $\sigma^i_{n,A^1,\dots,A^m}$ of P_i is the straightforward generalization of the strategies $\sigma_{n,A,B}, \tau_{n,A,B}$ of the two-person game, namely, at the kth stage P_i invests

$$a^i_k = \frac{w_k A^i_k}{W_k} - \frac{A^i_k c_k}{\sum_{j=1}^m A^j_k},$$

if it is allowed, and 0 otherwise.

The generalization of Theorem 3.6 is the following:

Theorem 4.1. *If $A^1,\dots,A^m > M$, then the m strategies $\sigma^1_{n,A^1,\dots,A^m},\dots,\sigma^m_{n,A^1,\dots,A^m}$ are a Nash equilibrium for the m-person game. Each of the strategies is unique up to equivalence with respect to all others. The expected total payoffs are $\frac{A^1 W}{\sum_{i=1}^m A^i},\dots,\frac{A^m W}{\sum_{i=1}^m A^i}$, which are guaranteed by the given strategies.*

Proof. We prove for P_1. Let $B_k = \sum_{i=2}^m A^i_k$ and $b_k = \sum_{i=2}^m a^i_k$. Since $a^1_k = \frac{w_k A^1_k}{W_k} - \frac{A^1_k c_k}{A^1_k + B_k}$ and $b_k = \frac{w_k B_k}{W_k} - \frac{B_k c_k}{A^1_k + B_k}$, our player P_1 can imagine that he is playing a two-person game against one joint player whose resource is B_k, and whose strategy dictates investing b_k. Thus, the claim follows from Theorem 3.6. □

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