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Gregory's Sixth Operation

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Abstract In relation to a thesis put forward by Marx Wartofsky, we seek to show that a historiography of mathematics requires an analysis of the ontology of the part of mathematics under scrutiny. Following Ian Hacking, we point out that in the history of mathematics the amount of contingency is larger than is usually thought. As a case study, we analyze the historians' approach to interpreting James Gregory's expression *ultimate terms* in his paper attempting to prove the irrationality of π. Here Gregory referred to the *last* or

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ultimate terms of a series. More broadly, we analyze the following questions: which modern framework is more appropriate for interpreting the procedures at work in texts from the early history of infinitesimal analysis? As well as the related question: what is a logical theory that is close to something early modern mathematicians could have used when studying infinite series and quadrature problems? We argue that what has been routinely viewed from the viewpoint of classical analysis as an example of an "unrigorous" practice, in fact finds close procedural proxies in modern infinitesimal theories. We analyze a mix of social and religious reasons that had led to the suppression of both the religious order of Gregory's teacher degli Angeli, and Gregory's books at Venice, in the late 1660s.

1 Introduction

Marx Wartofsky pointed out in his programmatic contribution *The Relation between Philosophy of Science and History of Science* that there are many distinct possible relations between philosophy of science and history of science, some "more agreeable" and fruitful than others (Wartofsky 1976, p. 719ff). Accordingly, a fruitful relation between history and philosophy of science requires a rich and complex *ontology* of that science. In the case of mathematics, this means that a fruitful relation between history and philosophy must go beyond offering an ontology of the domain over which a certain piece of mathematics ranges (say, numbers, functions, sets, infinitesimals, structures, etc.). Namely, it must develop the ontology of mathematics *as a scientific theory* itself (ibid., p. 723). A crucial distinction here is that between the (historically relative) *ontology* of the mathematical objects in a certain historical setting, and its *procedures*, particularly emphasizing the different roles these components play in the history of mathematics. More precisely, *procedures* serve as a representative of what Wartofsky called the *praxis* characteristic of the mathematics of a certain time period, and *ontology* in the narrow sense takes care of the mathematical entities recognized at that time. On the procedure/entity distinction, A. Robinson had this to say:

> ...from a formalist point of view we may look at our theory syntactically and may consider that what we have done is to introduce *new deductive procedures* rather than new mathematical entities. (Robinson 1966, p. 282) (emphasis in the original)

As a case study, we analyze the text *Vera Circuli* (Gregory 1667) by James Gregory.

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2 Ultimate Terms and Termination of Series

Gregory studied under Italian indivisibilists[1] and specifically Stefano degli Angeli during his years 1664–1668 in Padua. Some of Gregory's first books were published in Italy. He mathematical accomplishments include the series expansions not only for the sine but also for the tangent and secant functions (González-Velasco 2011).

The *Vera Circuli* contains a characterisation of the "termination" of a convergent *series* (i.e., *sequence* in modern terminology). This was given by Gregory in the context of a discussion of a double sequence (lower and upper bounds) of successive polygonal approximations to the area of a circle:

> & igitur imaginando hanc seriem in infinitum continuari, possimus imaginari vltimos terminos couergentes [sic] esse equales, quos terminos equales appellamus seriei terminationem. (Gregory 1667, pp. 18–19)

In the passage above, Gregory's *seriem* refers to a *sequence*, and the expression *terminus* has its usual meaning of a *term* of a sequence. The passage can be rendered in English as follows:

> And so by imagining this series [i.e., sequence] to be continued to infinity, we can imagine the ultimate convergent terms *to be equal*; and we call those equal ultimate terms the termination of the series. [emphasis added]

Lützen (2014, p. 225) denotes the lower and upper bounds respectively by I_n (for *inscribed*) and C_n (for *circumscribed*). Gregory proves the recursive formulas $I_{n+1}^2 = C_n I_n$ and $C_{n+1} = \frac{2C_n I_{n+1}}{C_n + I_{n+1}}$. Gregory states that the "ultimate convergent terms" of the sequences I_n and C_n are *equal*.

After having defined the two series of inscribed and circumscribed polygons, Gregory notes:

> atque in infinitum illam [=hanc polygonorum seriem] continuando, manifestum est tandem exhiberi quantitatem sectori circulari, elliptico vel hyperbolico ABEIOP æquale[m]; differentia enim polygonorum complicatorum in seriei continuatione semper diminuitur, ità vt omni exhibita quantitate fieri possit minor, & vt in sequenti theorematis Scholio demonstrabimus: si igitur prædicta polygonorum series terminari posset, hoc est, si inueniretur vltimum illud polygonum inscriptum (si ità loquì liceat) æquale vltimo illi polygono circumscripto, daretur infallibiliter circuli & hyperbolæ quadratura: sed quoniam difficile est, & in geometria omninò fortasse inauditu[m] tales series terminare; præmitte[n]dæ sunt quæ dam propositiones è quibus inueniri possit huiusmodi aliquot serierum terminationes, & tandem (si fieri possit) generalis methodus inueniendi omnium serierum co[n]uergentium terminationes.

This can be translated as follows:

> and that [series of polygons] being continued to infinity, it is clear that a quantity equal to a circular, elliptic, or hyperbolic sector ABEIOP will be produced. The

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[1] Today scholars distinguish carefully between indivisibles (i.e., codimension one objects) and infinitesimals (i.e., of the same dimension as the entity they make up); see e.g., Koyré (1954). However, in the 17th century the situation was less clearcut. The term *infinitesimal* itself was not coined until the 1670s; see Katz and Sherry (2013).

difference between [two *n*-th terms] in the continuation of the series of complicated polygons always diminishes so that it can become less than any given quantity indeed, as we will prove in the Scholium to the theorem. Thus, if the abovementioned series of polygons can be terminated, that is, if that ultimate inscribed polygon is found to be equal (so to speak) to that ultimate circumscribed polygon, it would undoubtedly provide the quadrature of a circle as well as a hyperbola. But since it is difficult, and in geometry perhaps unheard-of, for such a series to come to an end [lit.: be terminated], we have to start by showing some Propositions by means of which it is possible to find the terminations of a certain number of series of this type, and finally (if it can be done) a general method of finding terminations of all convergent series.

The passage clearly shows that Gregory is using the term "ultimate (or last) circumscribed polygon" in a figurative sense, as indicated by

- his parenthetical 'so to speak,' which indicates that he is not using the term literally;
- his insistence that "in geometry it is unheard-of" for a sequence to come to be terminated.

He makes it clear that he is using the word 'termination' in a new sense, which is precisely his sixth operation, as discussed below.

One possible interpretation of *ultimate terms* would be the following. This could refer to those terms that are all closer than epsilon to one another. If ordinary terms are *further* than epsilon, that would make them different. The difficulty for this interpretation is that, even if ordinary terms are *closer* than epsilon, they are still *different*, contrary to what Gregory wrote about their being *equal*. M. Dehn and E. Hellinger attribute to Gregory

> a very general, new analytic process which he coordinates as the "sixth" operation along with the five traditional operations (addition, subtraction, multiplication, division, and extraction of roots). In the introduction, he proudly states "ut hae c nostra inventio addat arithmeticae aliam operationem et geometriae aliam rationis speciem, ante incognitam orbi geometrico." This operation is, as a matter of fact, our modern limiting process. (Dehn and Hellinger 1943, pp. 157–158)

We will have more to say about what this sixth operation could be *as a matter of fact* (see Sect. 4 on *shadow*-taking). A. Malet expressed an appreciation of Gregory's contribution to analysis in the following terms:

> Studying Gregorie's work on "Taylor" expansions and his analytical method of tangents, which has passed unnoticed so far, [we argue] that Gregorie's work is a counter-example to the standard thesis that geometry and algebra were opposed forces in 17th-century mathematics. (Malet 1989, p. 1)

What is, then, Gregory's *sixth operation* mentioned by Dehn and Hellinger, and how is it related to convergence?

3 Law of Continuity

The use of infinity was not unusual for this period. As we mentioned in the introduction, Gregory fit naturally in the proud Italian tradition of the method of indivisibles, and was a student of Stefano degli Angeli at Padua between 1664 and 1668. Degli Angeli published

sharp responses to critiques of indivisibles penned by jesuits Mario Bettini and André Tacquet. Bettini's criticisms were extensions of earlier criticisms by jesuit Paul Guldin. Degli Angeli defended the method of indivisibles against their criticisms.

Both indivisibles and degli Angeli himself appear to have been controversial at the time in the eyes of the jesuit order, which banned indivisibles from being taught in their colleges on several occassions. Thus, in 1632 (the year Galileo was summoned to stand trial over heliocentrism) the Society's Revisors General led by Jacob Bidermann banned teaching indivisibles in their colleges (Festa 1990, 1992, p. 198). Indivisibles were placed on the Society's list of *permanently* banned doctrines in 1651 (Hellyer 1996).

It seems that Gregory's 1668 departure from Padua was well timed, for his teacher degli Angeli's jesuat order[2] was suppressed by papal brief in the same year, cutting short degli Angeli's output on indivisibles. Gregory's own books were suppressed at Venice, according to a letter from John Collins to Gregory dated 25 november 1669, in which he writes:

> One Mr. Norris a Master's Mate recently come from Venice, saith it was there reported that your bookes were suppressed, not a booke of them to be had anywhere, but from Dr. Caddenhead to whom application being made for one of them, he presently sent him one (though a stranger) refusing any thing for it. (Turnbull 1939, p. 74)

In a 1670 letter to Collins, Gregory writes:

> I shall be very willing ye writ to Dr Caddenhead in Padua, for some of my books. In the mean time, I desire you to present my service to him, and to inquire of him if my books be suppressed, and the reason thereof. (Gregory to Collins, St Andrews, March 7, 1670, in Turnbull p. 88)

In a letter to Gregory, written in London, 29 september 1670, Collins reported as follows: "Father Bertet[3] sayth your Bookes are in great esteeme, but not to be procured in Italy." (Turnbull p. 107).

The publishers' apparent reluctance to get involved with Gregory's books may also explain degli Angeli's silence on indivisibles following the suppression of his order, but it is hard to say anything definite in the matter until the archives at the Vatican dealing with the suppression of the jesuats are opened to independent researchers. Certainly one can understand Gregory's own caution in matters infinitesimal (of course, the latter term wasn't coined until later).

John Wallis introduced the symbol ∞ for an infinite number in his *Arithmetica Infinitorum* (Wallis 1656) and exploited an infinitesimal number of the form $\frac{1}{\infty}$ in area calculations (Scott 1981, p. 18), over a decade before the publication of Gregory's *Vera Circuli*. At about the same time, Isaac Barrow "dared to explore the logical underpinnings of infinitesimals," as Malet put it:

> Barrow, who dared to explore the logical underpinnings of infinitesimals, was certainly modern and innovative when he publicly defended the new mathematical methods against Tacquet and other mathematical "classicists" reluctant to abandon the Aristotelian continuum. And after all, to use historical hindsight, it was the non-

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[2] This was an older order than the jesuits. Cavalieri had also belonged to the jesuat order.

[3] Jean Bertet (1622–1692), jesuit, quit the Order in 1681. In 1689 Bertet conspired with Leibniz and Antonio Baldigiani in Rome to have the ban on Copernicanism lifted (Wallis 2012).

Archimedean structure of the continuum linked to the notion of infinitesimal and advocated by Barrow that was to prove immensely fruitful as the basis for the Leibnizian differential calculus. (Malet 1989, p. 244).

We know that G. W. Leibniz was an avid reader of Gregory; see e.g., Leibniz (1672). To elaborate on the link to Leibniz mentioned by Malet, note that Leibniz might have interpreted Gregory's definition of convergence as follows. Leibniz's *law of continuity* (Leibniz 1702, pp. 93–94) asserts that whatever succeeds in the finite, succeeds also in the infinite, and vice versa; see Katz and Sherry (2013) for details. Thus, if one can take terms of a sequence corresponding to a finite value of the index n, one should also be able to take terms corresponding to infinite values of the index n. What Gregory refers to as the "ultimate" terms would then be the terms I_n and C_n corresponding to an infinite index n.

Leibniz interpreted equality as a relation in a larger sense of equality *up to* (negligible terms). This was codified as his *transcendental law of homogeneity* (Leibniz 1710); see Bos (1974, p. 33) for a thorough discussion. Thus, Leibniz wrote:

> Caeterum aequalia esse puto, non tantum quorum differentia est omnino nulla, sed et quorum differentia est incomparabiliter parva; et licet ea Nihil omnino dici non debeat, non tamen est quantitas comparabilis cum ipsis, quorum est differentia. (Leibniz 1695, p. 322)

This can be translated as follows:

> "Furthermore I think that not only those things are equal whose difference is absolutely zero, but also whose difference is incomparably small. And although this [difference] need not absolutely be called Nothing, neither is it a quantity comparable to those whose difference it is."

In the 17th century, such a generalized notion of equality was by no means unique to Leibniz. Indeed, Leibniz himself cites an antecedent in Pierre de Fermat's technique (known as the method of *adequality*), in the following terms:

> Quod autem in aequationibus Fermatianis abjiciuntur termini, quos ingrediuntur talia quadrata vel rectangula, non vero illi quos ingrediuntur simplices lineae infinitesimae, ejus ratio non est quod hae sint aliquid, illae vero sint nihil, sed quod termini ordinarii per se destruuntur.[4] (Leibniz 1695, p. 323)

On this page, Leibniz describes Fermat's method in a way similar to Leibniz's own. On occasion Leibniz used the notation " ⊓ " for the relation of equality. Note that Leibniz also used our symbol "=" and other signs for equality, and did not distinguish between "=" and " ⊓ " in this regard. To emphasize the special meaning *equality* had for Leibniz, it may be helpful to use the symbol ⊓ so as to distinguish Leibniz's equality from the modern notion of equality "on the nose." Then Gregory's comment about the equality of the ultimate terms translates into

$$I_n \sqcap C_n \tag{1}$$

when n is infinite.

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[4] Translation: "But the fact that in Fermat's equations those terms into which such things enter as squares or rectangles [i.e., multiplied by themselves or by each other] are eliminated but not those into which simple infinitesimal lines [i.e., segments] enter—the reason for that is not because the latter are something whereas the former are really nothing [as Nieuwentijt maintained], but because ordinary terms cancel each other out."

From the viewpoint of the modern Weierstrassian framework, it is difficult to relate to Gregory's insight. Thus, G. Ferraro translates Gregory's "vltimos terminos conuergentes" as "last convergent terms" (Ferraro 2008, p. 21), and goes on a few pages later to mention

> Gregory's reference to *the last term*, p. 21. ... In Leibniz they appear in a clearer way. (Ferraro 2008, p. 27, note 41) (emphasis added)

Ferraro may have provided an accurate translation of Gregory's comment, but Ferraro's assumption that there is something unclear about Gregory's comment because of an alleged "last term", is unjustified. Note that Ferraro's use of the singular "last term" (note 41) is not consistent with Gregory's use of the plural *terminos* (terms) in his book. One may find it odd for a mathematician of Gregory's caliber to hold that there is literally a *last* term in a sequence. Dehn and Hellinger mention only the plural "last convergent terms" (Dehn and Hellinger 1943, p. 158).

4 The Unguru Controversy

There is a debate in the community of historians whether it is appropriate to use modern theories and/or modern notation in interpreting mathematical texts of the past, with S. Unguru a staunch opponent, whether with regard to interpreting Euclid, Apollonius, or Fermat (Unguru 1976). See Corry (2013) for a summary of the debate. Note that Ferraro does not follow Unguru in this respect. Indeed, Ferraro exploits the modern notation

$$\sum_{i=1}^{\infty} a_i \tag{2}$$

for the sum of the series, already on page 5 of his book, while discussing late 16th (!) century texts of Viète. We note the following two aspects of the notation (2):

(1) It presupposes the modern epsilontic notion of limit, where $S = \sum_{i=1}^{\infty} a_i$ means $\forall \epsilon > 0 \, \exists N \in \mathbb{N} \big(n > N \implies \big| S - \sum_{i=1}^{n} a_i \big| < \epsilon \big)$, in the context of a Weierstrassian framework involving a strictly Archimedean punctiform continuum;

(2) The symbol "∞" occurring in Ferraro's formula has no meaning other than a reminder that a limit was taken in the construction. In particular, this usage of the symbol ∞ is distinct from its original 17th century usage by Wallis, who used it to denote a specific infinite number, and proceeded to work with infinitesimal numbers like $\frac{1}{\infty}$ (see Sect. 3).

We will avoid choosing sides in the debate over Unguru's proposal.[5] However, once one resolves to exploit modern frameworks involving punctiform continua/number systems, as Ferraro does, to interpret 17th century texts, one still needs to address the following important question:

> *Which modern framework is more appropriate for interpreting the said historical texts?*

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[5] The sources of such a proposal go back (at least) to A. Koyré who wrote: "Le problème du langage à adopter pour l'exposition des oeuvres du passé est extrêmement grave et ne comporte pas de solution parfaite. En effet, si nous gardons la langue (la terminologie) de l'auteur étudié, nous risquons de le laisser incompréhensible, et si nous lui substituons la nôtre, de le trahir." (Koyré 1954, p. 335, note 3).

Here appropriateness could be gauged in terms of providing the best proxies for the *procedural* moves found in the great 17th century masters.

Hacking (2014) points out that there is a greater amount of contingency in the historical evolution of mathematics than is generally thought. Hacking proposes a *Latin model* of development (of a natural language like Latin, with the attendant contingencies of development due to social factors) to the usual *butterfly model* of development (of a biological organism like a butterfly, which is genetically predetermined inspite of apparently discontinuous changes in its development). This tends to undercut the apparent inevitability of the Weierstrassian model.

We leave aside the *ontological* or foundational questions of how to justify the entities like points or numbers (in terms of modern mathematical foundations), and focus instead of the *procedures* of the historical masters, as discussed in Sect. 1.

More specifically, is a modern Weierstrassian framework based on an Archimedean continuum more appropriate for interpreting their procedures, or is a modern infinitesimal system more appropriate for this purpose?

Note that in a modern infinitesimal framework such as Robinson's, sequences possess terms with infinite indices. Gregory's relation can be formalized in terms of the standard part principle in Robinson's framework (Robinson 1966). This principle asserts that every finite hyperreal number is infinitely close to a unique real number.

In more detail, in a hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ one considers the set ${}^b\mathbb{R} \subseteq {}^*\mathbb{R}$ of *finite* hyperreals. The set ${}^b\mathbb{R}$ is the domain of the standard part function (also called the *shadow*) $\mathbf{st} : {}^b\mathbb{R} \to \mathbb{R}$ rounding off each finite hyperreal number to its nearest real number.

In the world of James Gregory, if each available term with an infinite index n is indistinguishable (in the sense of being infinitely close) from some standard number, then we "terminate the series" (to exploit Gregory's terminology) with this number, meaning that this number is the limit of the sequence. Gregory's definition corresponds to a relation of infinite proximity in a hyperreal framework. Namely we have

$$I_n \approx C_n, \tag{3}$$

where $\approx$ is the relation of being infinitely close (i.e., the difference is infinitesimal), and the common standard part of these values is the limit of the sequence. Equivalently, $\mathbf{st}(I_n) = \mathbf{st}(C_n)$. Mathematically speaking, this is equivalent to a Weierstrassian epsilontic paraphrase along the lines of item (1) above.

Recently Robinson's framework has become more visible thanks to high-profile advocates like Terry Tao; see e.g., his work Tao (2014, 2016). The field has also had its share of high-profile detractors like Errett Bishop and Alain Connes. Their critiques were critically analyzed in Katz and Katz (2011), Katz and Leichtnam (2013), and Kanovei et al. (2013). Further criticisms by J. Earman, K. Easwaran, H. M. Edwards, Ferraro, J. Gray, P. Halmos, H. Ishiguro, G. Schubring, and Y. Sergeyev were dealt with respectively in the following recent texts: Katz and Sherry (2013), Bascelli et al. (2014, 2016), Kanovei et al. (2015), Bair et al. (2017), Błaszczyk et al. (2016, 2017a, b), Gutman et al. (2016). In Borovik and Katz (2012) we analyze the Cauchy scholarship of Judith Grabiner. For a fresh look at Simon Stevin see Katz and Katz (2012).

5 Conclusion

We note a close fit between Gregory's procedure (1) and procedure (3) available in a modern infinitesimal framework. The claim that "[Gregory's] definition is rather different from the modern one" (Ferraro 2008, p. 20) is only true with regard to a *Weierstrassian* modern definition. Exploiting the richer syntax available in a modern infinitesimal framework where Gregory's procedure acquires a fitting proxy, it is possible to avoid the pitfalls of attributing to a mathematician of Gregory's caliber odd beliefs in an alleged "last" term in a sequence.

An infinitesimal framework also enables an interpretation of the notion of "ultimate terms" as proxified by terms with infinite index, and "termination of the series" as referring to the assignable number infinitely close to a term with an infinite index, by Leibniz's transcendental law of homogeneity (or the standard part principle of Robinson's framework).

While some scholars seek to interpret Gregory's procedures in a default modern post-Weierstrassian framework, arguably a modern infinitesimal framework provides better proxies for Gregory's procedural moves than a modern Weierstrassian one.

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