# THE HOMOLOGY OF RANDOM SIMPLICIAL COMPLEXES IN THE MULTI-PARAMETER UPPER MODEL

BY

Michael Farber*

*School of Mathematical Sciences, Queen Mary University of London
London E1 4NS, UK
e-mail: M.Farber@qmul.ac.uk
URL: www.qmul.ac.uk/maths/profiles/farberm.html*

AND

Tahl Nowik

*Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel
e-mail: tahl@math.biu.ac.il
URL: www.math.biu.ac.il/˜tahl*

*Dedicated to our dear friend Nati Linial on his 70th birthday*

ABSTRACT

We study random simplicial complexes in the multi-parameter upper model. In this model simplices of various dimensions are taken randomly and independently, and our random simplicial complex $Y$ is then taken to be the minimal simplicial complex containing this collection of simplices.

We study the asymptotic behavior of the homology of $Y$ as the number of vertices goes to $\infty$. We observe the following phenomenon asymptotically almost surely. The given probabilities with which the simplices are taken determine a range of dimensions $\ell \leq k \leq \ell'$ with $\ell' \leq 2\ell+1$, outside of which the homology of $Y$ vanishes. Within this range, the homology in the critical dimension $\ell$ is significantly the largest, and we specify the precise rate of growth of the $\ell$th Betti number. For the remaining Betti numbers in this range we give upper bounds that strongly decrease from dimension to dimension.

## 1. Introduction

In this work we are interested in random simplicial complexes in the multi-parameter upper model. In its most general form it is described as follows. Let $\Delta^n$ be the $n$-dimensional simplex thought of as a simplicial complex. That is, $\Delta^n$ is the set of all non-empty subsets of the set $\{0, \ldots, n\}$ of $n+1$ vertices. Let $\Omega_n$ denote the set of all simplicial complexes $Y \subseteq \Delta^n$. Given an assignment of probabilities $\{p_\sigma\}_{\sigma \in \Delta^n}, 0 \leq p_\sigma \leq 1$, it induces a probability measure on $\Omega_n$ as follows. As an intermediate step we randomly select a hypergraph $X \subseteq \Delta^n$, by which we mean an arbitrary subset of $\Delta^n$, not necessarily a simplicial complex. Each simplex $\sigma \in \Delta^n$ is included in $X$ independently with probability $p_\sigma$, i.e., the probability for obtaining $X$ is $\prod_{\sigma \in X} p_\sigma \prod_{\sigma \notin X} q_\sigma$, where $q_\sigma = 1 - p_\sigma$. Now, the simplicial complex produced by this random process is the minimal simplicial complex containing $X$, which we denote by $\overline{X}$. That is, $\overline{X}$ includes all simplices of $X$ and all their faces.

Though we will not need it in this work, we present an explicit formula for the probability $\overline{\mathbf{P}}(Y)$ that a given simplicial complex $Y$ is obtained in this random process. That is, $\overline{\mathbf{P}}(Y)$ is the probability that our random hypergraph $X$ satisfies $\overline{X} = Y$. Let $M(Y)$ denote the set of all maximal simplices in $Y$, then

$$\overline{\mathbf{P}}(Y) = \prod_{\sigma \notin Y} q_\sigma \prod_{\sigma \in M(Y)} p_\sigma.$$

This is because $\overline{X} = Y$ iff $X \subseteq Y$ and $X \supseteq M(Y)$. Indeed, if $X \subseteq Y$ then $\overline{X} \subseteq Y$, since $Y$ is a simplicial complex, and if $X \supseteq M(Y)$ then $\overline{X} \supseteq Y$, since every simplex in $Y$ is contained in a maximal simplex. On the other hand, if $\overline{X} = Y$ then $X \subseteq \overline{X} = Y$ which implies $X \supseteq M(Y)$, since if $\sigma \in M(Y)$ then the only simplex in $Y$ containing $\sigma$ is $\sigma$ itself.

We will be interested in $\{p_\sigma\}$ of a very particular form. We will be given an $(r+1)$-tuple $\alpha = (\alpha_0, \ldots, \alpha_r) \in [0, \infty]^{r+1}$. The probabilities $p_\sigma$ determined by $\alpha \in [0, \infty]^{r+1}$ are as follows. Let $j = \dim \sigma$. If $j > r$ or $\alpha_j = \infty$ then $p_\sigma = 0$. Otherwise $p_\sigma = n^{-\alpha_j}$.

For a fixed $\alpha \in [0, \infty]^{r+1}$ we will be interested in the asymptotic behavior of the homologies $\tilde{H}_i(Y; \mathbb{Z})$ of the random complex $Y$ as $n \to \infty$. In the present work we show that the parameter space $[0, \infty]^{r+1}$ may be divided into domains $\mathcal{U}_\ell$, where if $\alpha \in \mathcal{U}_\ell$ then asymptotically almost surely (a.a.s.), that is, with probability converging to 1 as $n \to \infty$, the homology of $Y$ is dominated by that in dimension $\ell$. More in detail, there is $\ell' \leq 2\ell + 1$ (depending

on $\alpha \in \mathcal{U}_\ell$) such that a.a.s. the homology of $Y$ vanishes outside the range of dimensions $\ell \le k \le \ell'$. Within this range, the homology in dimension $\ell$ is significantly the largest, and we specify the precise rate of growth of the $\ell$th Betti number. For the remaining Betti numbers in this range we give upper bounds that strongly decrease from dimension to dimension. The dominant dimension $\ell$ is named the **critical dimension**. A specific special case of this phenomenon in the upper model has been studied in [FMN].

The upper model we have just described is in contrast to the multi-parameter lower model, which begins with the same random hypergraph $X$ but then produces from it the maximal simplicial complex contained in $X$, which we denote by $\underline{X}$. That is, a simplex $\sigma$ is in $\underline{X}$ if $\sigma$ and all its faces are in $X$. The upper and lower models are dual in a clear sense, and so the formula for the probability $\underline{\mathbf{P}}(Y)$, for obtaining a given simplicial complex $Y$ in the lower model, is dual to that for the upper model. Let $E(Y)$ denote the set of all minimal simplices among those not in $Y$. An equivalent and geometrically more suggestive definition is

$$E(Y) = \{\sigma \in \Delta^n : \sigma \notin Y \text{ but } \partial \sigma \subseteq Y\}.$$

(This includes the case that $\sigma$ is a vertex $v$ not in $Y$ since then $\partial v = \varnothing \subseteq Y$.) We have

$$\underline{\mathbf{P}}(Y) = \prod_{\sigma \in Y} p_\sigma \prod_{\sigma \in E(Y)} q_\sigma.$$

This is because $\underline{X} = Y$ iff $X \supseteq Y$ and $X \cap E(Y) = \varnothing$. Indeed, if $X \supseteq Y$ then $\underline{X} \supseteq Y$, since $Y$ is a simplicial complex, and if $X \cap E(Y) = \varnothing$ then $\underline{X} \subseteq Y$, since every simplex not in $Y$ contains a simplex which is minimal among those not in $Y$. On the other hand if $\underline{X} = Y$ then $X \supseteq \underline{X} = Y$ which implies $X \cap E(Y) = \varnothing$, since if $\sigma \in X \cap E(Y)$ then $Y \cup \{\sigma\}$ is a simplicial complex contained in $X$.

The sets appearing in the formulas for $\overline{P}$ and $\underline{P}$ are $Y, M(Y), E(Y)$, which have clear geometric meaning in terms of $Y$, but make the duality slightly less apparent. From the point of view of duality one may like to add the notation $Y^c$ for the set of all simplices not in $Y$. Then $E(Y)$ is the set of minimal simplices in $Y^c$ and so may be denoted $m(Y^c)$. In terms of this notation we have $\overline{\mathbf{P}}(Y) = \prod_{\sigma \in Y^c} q_\sigma \prod_{\sigma \in M(Y)} p_\sigma$ and $\underline{\mathbf{P}}(Y) = \prod_{\sigma \in Y} p_\sigma \prod_{\sigma \in m(Y^c)} q_\sigma$. This makes the duality completely transparent, one formula is obtained from the other by everywhere exchanging $Y \leftrightarrow Y^c$, $M \leftrightarrow m$, $p \leftrightarrow q$. For more on the duality between the upper and lower models see [FMN].

The phenomenon of critical dimension also holds in the lower model. This has been established in [CF3]. The asymptotic behavior observed in the lower model resembles that of the upper model but occurs for a different division of the parameter space into domains $\mathcal{U}_\ell$. More on the lower model may be found in [CF1], [CF2], [F].

## 2. Definitions and statement of result

For a fixed integer $r \geq 0$ we study random simplicial complexes of dimension $\leq r$. We are given an $(r+1)$-tuple $\alpha = (\alpha_0, \ldots, \alpha_r) \in [0, \infty]^{r+1}$, and an integer $n \geq 0$. With this data we produce a random hypergraph $X$ by taking each simplex $\sigma$ of dimension $0 \leq i \leq r$ on the vertex set $\{0, \ldots, n\}$ independently with probability $n^{-\alpha_i}$ (where by definition $n^{-\infty} = 0$). Our random simplicial complex $Y$ is then defined to include all the simplices in $X$ and all their faces.

We are interested in the asymptotic behavior of our random simplicial complex $Y$, by which we mean the following. We fix the parameters $(\alpha_0, \ldots, \alpha_r)$, and we take $n$ to be larger and larger. The asymptotic behavior is then described in terms of the following probabilistic notion.

*Definition 2.1:* If for every $n$ we have a random object $Z = Z(n)$, and if $T$ is a property that $Z$ may or may not have, then we say $T$ holds **asymptotically almost surely** (a.a.s.) if the probability that $T$ holds converges to 1 as $n \to \infty$.

Given $\alpha = (\alpha_0, \ldots, \alpha_r) \in [0, \infty]^{r+1}$ we define $\beta_i = i + 1 - \alpha_i$ and set

$$(2.1) \qquad \beta = \beta(\alpha) = \max\{\beta_0, \ldots, \beta_r\} \leq r + 1.$$

We divide our space of parameters $\mathcal{V} = [0, \infty]^{r+1}$ into domains $\mathcal{U}_-$ and $\mathcal{U}_0, \ldots, \mathcal{U}_r$ where $\mathcal{U}_- = \{\alpha \in \mathcal{V} : \beta(\alpha) < 0\}$ and $\mathcal{U}_\ell = \{\alpha \in \mathcal{V} : \ell < \beta(\alpha) < \ell + 1\}$ for $0 \leq \ell \leq r$. The asymptotic behavior of our random simplicial complex $Y$ is as follows. If $\alpha \in \mathcal{U}_-$, i.e., $\beta < 0$, then $Y = \varnothing$ a.a.s., see Proposition 3.1. If $\alpha \in \mathcal{U}_\ell$ for $0 \leq \ell \leq r$, i.e., $\beta > 0$ is not an integer and $\ell = \lfloor \beta \rfloor$ (the integer part of $\beta$), then there is $\ell' \leq \lfloor 2\beta \rfloor$ (depending on $\alpha \in \mathcal{U}_\ell$) such that a.a.s. the homology of $Y$ vanishes outside the range of dimensions $\ell \leq k \leq \ell'$. The $\ell$th Betti number grows like $dn^\beta$, and for the remaining Betti numbers in this range we give upper bounds that strongly decrease from dimension to dimension. The precise details are stated in Theorem 2.3, for which we need one more definition.

*Definition 2.2:* For two quantities $a = a(n), b = b(n)$:

(1) We say $a \sim b$ if $\frac{a}{b} \to 1$ as $n \to \infty$.

(2) If $a$ is a random quantity then we say $a \sim b$ a.a.s. if there is a sequence $\epsilon_n \to 0$ such that $|\frac{a}{b} - 1| < \epsilon_n$ a.a.s., or equivalently, if there is a sequence $c_n$ such that $|a - b| < c_n$ a.a.s. and $\frac{c_n}{b} \to 0$.

THEOREM 2.3: *Let $Y$ be the random simplicial complex in the multi-parameter upper model determined by parameters $\alpha = (\alpha_0, \ldots, \alpha_r)$, and let $\beta = \beta(\alpha)$ be as in (2.1). Assume $\beta > 0$, $\beta \notin \mathbb{Z}$, and define the critical dimension $\ell = \lfloor \beta \rfloor$. We further define the following quantities:*

$$(2.2) \qquad d = \sum_{i, \beta_i = \beta} \frac{1}{(i+1)\ell!(i-\ell)!},$$

$$(2.3) \qquad \nu_k = 2\gamma_k - k \quad \text{with } \gamma_k = \max\{\beta_k, \ldots, \beta_r\},$$

$$(2.4) \qquad \ell' = \max\{k : \nu_k \geq 0\}.$$

*Then the following holds a.a.s. (Definition 2.1):*

- *$Y$ has full $(\ell - 1)$-skeleton.*
- *$Y$ may be collapsed into its $\ell'$-skeleton, having $\ell \leq \ell' \leq \lfloor 2\beta \rfloor \leq 2\ell + 1$.*

*Let $b_k(Y)$ denote the $k$th Betti number of $Y$, then furthermore, for every sequence $\omega = \omega(n) \to \infty$ the following holds a.a.s.:*

- *For $k < \ell$, $\tilde{H}_k(Y; \mathbb{Z}) = 0$.*
- *For $k = \ell$, $b_\ell(Y) \sim dn^\beta$.*
- *For $\ell < k \leq \ell'$, $b_k(Y) \leq \omega n^{\nu_k}$ having $\nu_{\ell+1} < \beta$ and $\nu_{k+1} \leq \nu_k - 1$ for all $k$.*
- *For $k > \ell'$, $\tilde{H}_k(Y; \mathbb{Z}) = 0$.*

We make the following remarks:

(1) The constant $d$ depends on $\alpha$, but it attains only finitely many different values. Indeed, $d$ is determined by the (non-empty) set of indices $i$ for which $\beta_i = \beta$. By Remark 3.2 below if $\beta_i = \beta$ then $i \geq \ell$. So we have a stratification of $\mathcal{U}_\ell$ into $2^{r-\ell+1} - 1$ strata, on each of which $d$ is constant. These strata are convex (for this to be meaningful we need to exclude the values $\alpha_i = \infty$). The domain $\mathcal{U}_\ell$ itself is connected, but in general it is not convex.

(2) We think of the piecewise linear hypersurfaces $\{\alpha \in \mathcal{V} : \beta(\alpha) = \ell\}$ that separate between the domains $\mathcal{U}_\ell$ in the parameter space $\mathcal{V}$ as multi-parameter thresholds for passing from one typical behavior to another. For example, on one side of the piecewise linear hypersurface $\{\alpha \in \mathcal{V} : \beta(\alpha) = \ell + 1\}$, in $\mathcal{U}_\ell$, we have $\tilde{H}_\ell(Y; \mathbb{Z})$ very large, namely $b_\ell(Y) \sim dn^\beta$ a.a.s. and on the other side of this hypersurface, in $\mathcal{U}_{\ell+1}$, we have $\tilde{H}_\ell(Y; \mathbb{Z}) = 0$ a.a.s.

(3) We have noted that $\beta \leq r + 1$. The boundary case where $\beta = r + 1$ is easily understood. In this case $\alpha_r = 0$, so every $r$-simplex is included in our random hypergraph $X$ with probability 1, and so $Y$ is the full $r$-skeleton on $\{0, \ldots, n\}$ with probability 1.

(4) If $\nu_{\ell+1} < 0$ then $\ell' = \ell$, meaning that the case $\ell < k \leq \ell'$ of the theorem is empty. That is, all the homology of $Y$ appears only in dimension $\ell$. In general we have $\gamma_{\ell+1} \leq \beta < \ell + 1$, whereas the present case $\nu_{\ell+1} < 0$ means $\gamma_{\ell+1} < \frac{1}{2}(\ell + 1)$.

The plan of the paper is as follows. In Section 3 we are interested in the number $f_k$ of $k$-simplices in $Y$ for $k \geq \ell$, and give the asymptotic behavior of $f_k$ in Proposition 3.5. In Section 4 we are interested in the Betti numbers $b_k(Y)$ for $k \geq \ell$. The asymptotic behavior of these Betti numbers is given in Propositions 4.6, 4.7, 4.9. This is achieved by collapsing $Y$ onto a smaller subcomplex $Y'$. In Section 5 we are interested in $\tilde{H}_k(Y; \mathbb{Z})$ for $k < \ell$, showing in Propositions 5.5, 5.6 that $\tilde{H}_k(Y; \mathbb{Z}) = 0$ a.a.s. This is achieved by a modification of our random model that reduces it to that of Linial–Meshulam. Propositions 4.6, 4.7, 4.9, 5.5, 5.6 together constitute Theorem 2.3.

## 3. Counting simplices

Let $g_i$ denote the number of $i$-simplices in our random hypergraph $X$. Then $g_i$ is a binomial random variable with parameters $\binom{n+1}{i+1}$, $n^{-\alpha_i}$, so

$$(3.1) \qquad \mathbf{E}g_i = \binom{n+1}{i+1} n^{-\alpha_i} \sim \frac{1}{(i+1)!} n^{i+1-\alpha_i} = \frac{1}{(i+1)!} n^{\beta_i}$$

where $\mathbf{E}g_i$ denotes the expectation of $g_i$. Our first domain $\mathcal{U}_- = \{\beta < 0\}$ is easily understood:

PROPOSITION 3.1: *If $\beta < 0$ then a.a.s. $X = \varnothing$ and so $Y = \varnothing$.*

*Proof.* If $\beta < 0$ then $\beta_i < 0$ for all $0 \leq i \leq r$. Markov's inequality gives

$$\mathbf{P}(g_i \geq 1) \leq \mathbf{E}(g_i) \to 0$$

by (3.1), i.e., $g_i = 0$ a.a.s. for each $0 \leq i \leq r$, so $X = \varnothing$ a.a.s.   ∎

For the rest of this work we fix an integer $0 \leq \ell \leq r$ and an $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathcal{U}_\ell$. That is, for our fixed $\ell$ we have

(3.2)                            $$\ell < \beta < \ell + 1.$$

This may also be stated as follows: We assume $0 < \beta < r + 1$, $\beta \notin \mathbb{Z}$, and we set $\ell = \lfloor \beta \rfloor$. For $0 \leq k \leq r$ let $f_k = f_k(Y)$ denote the number of $k$-simplices in $Y$. Our first goal is to approximate $f_k$ for $k \geq \ell$. Since for every $i \geq k$ each $i$-simplex of $X$ contributes $\binom{i+1}{k+1}$ $k$-simplices to $Y$, we have $f_k \leq \sum_{i=k}^{r} \binom{i+1}{k+1} g_i$. It is only an inequality since different $i$-simplices may contribute the same $k$-simplex. This sum will be central in our computations so we denote $\widehat{g}_k = \sum_{i=k}^{r} \binom{i+1}{k+1} g_i$ and we have

(3.3)                            $$f_k \leq \widehat{g}_k.$$

By (3.1) we have

(3.4)
$$\mathbf{E}\widehat{g}_k = \sum_{i=k}^{r} \binom{i+1}{k+1}\binom{n+1}{i+1} n^{-\alpha_i} = \binom{n+1}{k+1} \sum_{i=k}^{r} \binom{n-k}{i-k} n^{-\alpha_i}$$
$$\sim \sum_{i=k}^{r} \frac{n^{\beta_i}}{(k+1)!(i-k)!}.$$

Equality holds by the identity $\binom{i+1}{k+1}\binom{n+1}{i+1} = \binom{n+1}{k+1}\binom{n-k}{i-k}$ which is true since both sides count the number of pairs of simplices $(\sigma, \tau)$ with $\dim \sigma = k$, $\dim \tau = i$, $\sigma \subseteq \tau$.

*Remark 3.2:* If $i$ is such that $\beta_i = \beta$ then $i \geq \ell$. Indeed, by (3.2) we have $\ell < \beta = \beta_i = i + 1 - \alpha_i \leq i + 1$, so $\ell \leq i$. Note that there may be more than one $i$ such that $\beta_i = \beta$.

Recall from (2.3) that we define $\gamma_k = \max\{\beta_k, \ldots, \beta_r\}$. By Remark 3.2 we have

(3.5)                            $$\gamma_\ell = \beta.$$

We have $\gamma_k \leq \beta < \ell + 1$ by (3.2), so for every $k \geq \ell$ we have

(3.6)                            $$\gamma_k < k + 1.$$

In Lemma 3.3 we give a bound on the difference $\widehat{g}_k - f_k$ for $k \geq \ell$. In the present section it will be used for evaluating $f_k$ via an evaluation of $\widehat{g}_k$. In the next section it will be used for estimating the extent to which the simplices of $X$ overlap. Note for example that $\widehat{g}_k - f_k = 0$ iff every two simplices of $X$ of dimension $\geq k$ intersect in dimension $< k$.

LEMMA 3.3: *Let $k \geq \ell$. For any sequence $\omega \to \infty$ we have $\widehat{g}_k - f_k < \omega n^{2\gamma_k - k - 1}$ a.a.s.*

*Proof.* For a given $k$-simplex $\sigma$ and $i \geq k$, there are $\binom{n-k}{i-k}$ $i$-simplices that contain $\sigma$, and so we have

$$\mathbf{P}(\sigma \in Y) = 1 - \prod_{i=k}^{r}(1 - n^{-\alpha_i})^{\binom{n-k}{i-k}}.$$

Let $N_i = \binom{n-k}{i-k}$, $u_i = n^{-\alpha_i}$. We have

$$\mathbf{P}(\sigma \in Y) = 1 - \prod_{i=k}^{r}(1 - u_i)^{N_i} = 1 - \prod_{i=k}^{r}\left(1 - N_i u_i + \binom{N_i}{2}u_i^2 - \binom{N_i}{3}u_i^3 + \cdots\right)$$

$$\geq 1 - \prod_{i=k}^{r}\left(1 - N_i u_i + \binom{N_i}{2}u_i^2\right) \geq 1 - \prod_{i=k}^{r}(1 - N_i u_i + (N_i u_i)^2).$$

The first inequality holds since for each $i$, $\binom{N_i}{j}u_i^j$ is decreasing in $j$. Indeed, for $j < N_i$

$$\frac{\binom{N_i}{j+1}u_i^{j+1}}{\binom{N_i}{j}u_i^j} = \frac{N_i - j}{j+1}u_i \leq N_i u_i = \binom{n-k}{i-k}n^{-\alpha_i}$$

$$\leq n^{i-k-\alpha_i} = n^{\beta_i - k - 1} \leq n^{\gamma_k - k - 1} < 1$$

by (3.6). (For the first factor $i = k$ we have $N_k = 1$ and the inequality for this factor is seen directly.) So we have $\mathbf{P}(\sigma \in Y) \geq \sum_{i=k}^{r} N_i u_i - \sum_j T_j$ where each term $T_j$ is a product of at least two factors of the form $N_i u_i$, and there are less than $3^{r+1}$ such terms. Again using $N_i u_i \leq n^{\gamma_k - k - 1} < 1$ we get $\mathbf{P}(\sigma \in Y) \geq (\sum_{i=k}^{r} N_i u_i)(1 - cn^{\gamma_k - k - 1})$ for a constant $c > 0$. Thus

$$\mathbf{E}f_k = \binom{n+1}{k+1}\mathbf{P}(\sigma \in Y) \geq \binom{n+1}{k+1}\left(\sum_{i=k}^{r} N_i u_i\right)(1 - cn^{\gamma_k - k - 1})$$

$$= \mathbf{E}\widehat{g}_k(1 - cn^{\gamma_k - k - 1})$$

by (3.4), which may be rewritten as $\mathbf{E}(\widehat{g}_k - f_k) \leq cn^{\gamma_k - k - 1}\mathbf{E}\widehat{g}_k$. Now, by (3.4) we have $\mathbf{E}\widehat{g}_k \leq c'n^{\gamma_k}$ for some $c' > 0$, so together

$$\mathbf{E}(\widehat{g}_k - f_k) \leq cc'n^{2\gamma_k - k - 1}.$$

Since $\widehat{g}_k - f_k \geq 0$ we may use Markov's inequality

$$\mathbf{P}(\widehat{g}_k - f_k \geq \omega n^{2\gamma_k - k - 1}) \leq \frac{\mathbf{E}(\widehat{g}_k - f_k)}{\omega n^{2\gamma_k - k - 1}} \to 0. \qquad \blacksquare$$

We will have two occasions to use the following lemma, with different choices of coefficients.

LEMMA 3.4: *Given* $a_k, \ldots, a_r$ *with* $a_i > 0$, *let* $g^a = \sum_{i=k}^r a_i g_i$. *We have:*

(1) *If* $\gamma_k > 0$ *then* $g^a \sim A^a n^{\gamma_k}$ *a.a.s. with* $A^a = \sum_{i \geq k, \beta_i = \gamma_k} \frac{a_i}{(i+1)!}$.

(2) *If* $\gamma_k = 0$ *then for every sequence* $\omega \to \infty$ *we have* $g^a \leq \omega$ *a.a.s.*

(3) *If* $\gamma_k < 0$ *then* $g^a = 0$ *a.a.s.*

*Proof.* We prove in the opposite order:

(3) As in Proposition 3.1.

(2) If there are $k \leq i \leq r$ with $\beta_i < 0$ then (3) applies to them. For $i$ with $\beta_i = 0$, i.e., $\alpha_i = i + 1$, $g_i$ is a binomial random variable with parameters $\binom{n+1}{i+1}$, $n^{-(i+1)}$. We have

$$\binom{n+1}{i+1} n^{-(i+1)} \to \frac{1}{(i+1)!},$$

so the distribution of $g_i$ converges to a Poisson distribution, the claim follows.

(1) If there are $k \leq i \leq r$ with $\beta_i \leq 0$ then (3) or (2) apply to them, taking $\omega = n^\epsilon$ with $0 < \epsilon < \gamma_k$ when (2) applies. For $i$ with $\beta_i > 0$, $g_i$ is a binomial random variable with parameters $\binom{n+1}{i+1}$, $n^{-\alpha_i}$, so by Chebyshev's inequality we have

$$\mathbf{P}(|g_i - \mathbf{E}g_i| \geq n^{\frac{2}{3}\beta_i}) \leq \frac{\mathbf{Var}(g_i)}{n^{\frac{4}{3}\beta_i}} = \frac{\binom{n+1}{i+1} n^{-\alpha_i}(1 - n^{-\alpha_i})}{n^{\frac{4}{3}\beta_i}} \leq \frac{(n+1)^{\beta_i}}{n^{\frac{4}{3}\beta_i}} \to 0.$$

By (3.1) we have

$$\mathbf{E}g_i \sim \frac{1}{(i+1)!} n^{\beta_i}$$

so $\frac{n^{\frac{2}{3}\beta_i}}{\mathbf{E}g_i} \to 0$ so $g_i \sim \mathbf{E}g_i$ a.a.s. so $g_i \sim \frac{1}{(i+1)!} n^{\beta_i}$ a.a.s. The claim follows. $\qquad \blacksquare$

We arrive at the main result of this section.

PROPOSITION 3.5: *Let $f_k$ be the number of $k$-simplices in $Y$. For $k \geq \ell$ we have:*

    (1) *If $\gamma_k > 0$ then $f_k \sim D_k n^{\gamma_k}$ a.a.s. with $D_k = \sum_{i \geq k, \beta_i = \gamma_k} \frac{1}{(k+1)!(i-k)!}$.*

    (2) *If $\gamma_k = 0$ then for every sequence $\omega \to \infty$ we have $f_k \leq \omega$ a.a.s.*

    (3) *If $\gamma_k < 0$ then $f_k = 0$ a.a.s.*

*Proof.* Take $a_i = \binom{i+1}{k+1}$ in Lemma 3.4, giving $g^a = \widehat{g}_k$.

    (1) We have $A^a = D_k$ so by Lemma 3.4(1) we have $\widehat{g}_k \sim D_k n^{\gamma_k}$ a.a.s. Now let $\omega = n^{\frac{1}{2}(k+1-\gamma_k)}$ then $\omega \to \infty$ by (3.6), and $\widehat{g}_k - f_k < \omega n^{2\gamma_k - k - 1} = \frac{1}{\omega} n^{\gamma_k}$ a.a.s. by Lemma 3.3. This gives $f_k \sim D_k n^{\gamma_k}$ a.a.s.

    For (2) and (3) use $f_k \leq \widehat{g}_k$ and Lemma 3.4(2),(3). ∎

We mention that $f_k$ for $k < \ell$ is given by Proposition 5.6 below (which is part of Theorem 2.3). Namely, for $k < \ell$ we have $f_k = \binom{n+1}{k+1}$ a.a.s.

## 4. Collapsing simplices

We recall the definition of a collapse of a simplicial complex $Y$. If $\rho \subsetneq \sigma$ are two simplices in $Y$ such that $\sigma$ is the unique simplex in $Y$ strictly containing $\rho$ (so $\sigma$ is maximal in $Y$ and $\dim \sigma = \dim \rho + 1$) then the removal of $\rho$ and $\sigma$ from $Y$ is called an **elementary collapse** of $Y$. A **collapse** of $Y$ is a sequence of elementary collapses. If $Y' \subseteq Y$ is the subcomplex remaining after a collapse of $Y$ then $Y'$ is homotopy equivalent to $Y$.

Recall $X$ is the random hypergraph that produces our random simplicial complex $Y$, and let $k \geq \ell$. Let $Y_k = \{\sigma \in Y : \dim \sigma = k\}$. Let

$$X_{k+} = \{\tau \in X : \dim \tau \geq k\},$$

which are all the simplices in $X$ that contribute $k$-simplices to $Y$.

*Definition 4.1:* Let $\tau \in X$ and $k \geq \ell$. We say that $\tau$ is $k$-**good** if $\tau \in X_{k+}$ and for any other $\tau' \in X_{k+}$ we have $\dim(\tau \cap \tau') < k$. We say that $\tau$ is $k$-**bad** if $\tau \in X_{k+}$ and $\tau$ is not $k$-good.

To avoid confusion in the sequel, we emphasize that being $k$-good or $k$-bad is a property of simplices in $X$, not in $Y$, and it is a property of simplices of dimension at least $k$, not just $k$. We also note that if $\ell \leq k \leq i$ and $\tau$ is a $k$-good $i$-simplex, then $\tau$ is also $j$-good for every $k \leq j \leq i$.

*Definition 4.2:* Let $\tau \in X$.

(1) We say $\tau$ is **good** if $\tau$ is $k$-good for some $k \geq \ell$.
(2) If $\tau$ is good then we denote by $G(\tau)$ the minimal $k \geq \ell$ such that $\tau$ is $k$-good.

LEMMA 4.3: *The simplicial complex $Y$ may be collapsed onto a subcomplex $Y' \subseteq Y$ such that for every good simplex $\tau$, if $i = \dim \tau$ and $k = G(\tau)$ then:*

(1) *All $j$-faces of $\tau$ with $j > k$ are removed.*
(2) *All $j$-faces of $\tau$ with $j < k$ remain.*
(3) *Precisely $\binom{i}{k}$ of the $k$-faces of $\tau$ remain.*

*Proof.* We describe the collapse corresponding to each good simplex. Let $\tau = \{v_0, \dots, v_i\}$ be a good $i$-simplex with $G(\tau) = k$. By definition $i \geq k$ and assume first that $k > 0$. Let $\hat{\tau}$ denote the subcomplex of $Y$ consisting of $\tau$ and all its faces, and let $\tau_0 = \{v_1, \dots, v_i\}$ be the $(i-1)$-face of $\tau$ opposite to $v_0$. For $j \leq i-1$ let $\Delta_j$ denote the $j$-skeleton of $\hat{\tau}_0$, and let $v_0 * \Delta_j$ denote the cone over $\Delta_j$ with vertex $v_0$. We collapse $\hat{\tau}$ onto $v_0 * \Delta_{k-1}$ doing it step by step

$$\hat{\tau} = v_0 * \Delta_{i-1} \longrightarrow v_0 * \Delta_{i-2} \longrightarrow \cdots \longrightarrow v_0 * \Delta_{k-1}.$$

For the collapse $v_0 * \Delta_j \to v_0 * \Delta_{j-1}$ we go over all $j$-simplices $\rho \in \Delta_j$, and for each such $\rho$ we remove the pair of simplices $\rho, \{v_0\} \cup \rho$. See Figure 1 where $i = 3$, $k = 1$. For all this to be a collapse, the following needs to hold at each stage. If $\hat{\tau}$ has already been collapsed onto $v_0 * \Delta_j$ with $j \geq k$, and $\rho \in \Delta_j$ is a $j$-simplex, then $\{v_0\} \cup \rho$ is the only simplex that strictly contains $\rho$. This is indeed true since $\rho$ is a maximal simplex in $\Delta_j$, and since $\tau$ was a $k$-good simplex and $\dim \rho = j \geq k$.
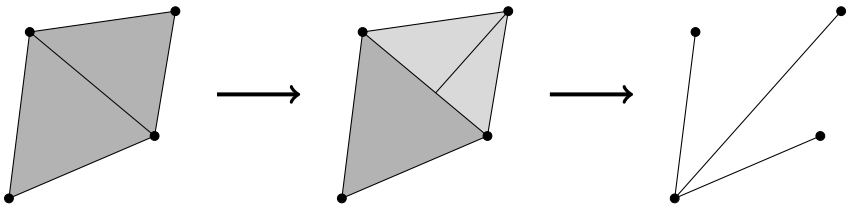


Figure 1. $\hat{\tau} = v_0 * \Delta_2 \longrightarrow v_0 * \Delta_1 \longrightarrow v_0 * \Delta_0$

We have collapsed $\hat\tau$ onto $v_0 * \Delta_{k-1}$. We note that $\Delta_{k-1}$ includes $\binom{i}{k}$ $(k-1)$-simplices, and so $v_0 * \Delta_{k-1}$ includes $\binom{i}{k}$ $k$-simplices as claimed in (3), completing the case $k > 0$. If $k = 0$ then $\tau$ is disjoint from all other simplices of $X$, so $\hat\tau$ may be collapsed to $\{v_0\}$ and the claim holds as well. ∎

In view of Lemma 4.3, for each $k \geq \ell$ we would like to have a bound on the number of $k$-bad simplices.

*Definition 4.4:* Let $B_k = B_k(X)$ denote the number of $k$-bad simplices.

LEMMA 4.5: *Let $k \geq \ell$. For any sequence $\omega \to \infty$ we have*

$$B_k \leq \omega n^{2\gamma_k - k - 1} \quad \text{a.a.s.}$$

*Proof.* We show $B_k \leq 2(\widehat{g}_k - f_k)$ which together with Lemma 3.3 establishes our claim. For $\sigma \in Y_k$, let $M_\sigma = \{\tau \in X_{k^+} : \tau \supseteq \sigma\}$ and let $m_\sigma = |M_\sigma|$ (the number of elements in $M_\sigma$), so $m_\sigma \geq 1$. We claim $\widehat{g}_k = \sum_{\sigma \in Y_k} m_\sigma$. Indeed, both sides of the equality count the number of pairs $(\sigma, \tau) \in Y_k \times X_{k^+}$ with $\sigma \subseteq \tau$. On the other hand $f_k = \sum_{\sigma \in Y_k} 1$. So

$$\widehat{g}_k - f_k = \sum_{\sigma \in Y_k} (m_\sigma - 1) = \sum_{\sigma \in Y_k, m_\sigma \geq 2} (m_\sigma - 1).$$

Now if $\tau \in X_{k^+}$ is a $k$-bad simplex and $\tau' \in X_{k^+}$, $\tau' \neq \tau$, is such that $\dim(\tau \cap \tau') \geq k$ then $\tau \cap \tau'$ contains a $k$-simplex $\sigma$. Thus, a simplex $\tau \in X_{k^+}$ is $k$-bad iff $\tau$ contains a $k$-simplex $\sigma$ with $m_\sigma \geq 2$, that is, the set of all $k$-bad simplices is $\bigcup_{\sigma \in Y_k, m_\sigma \geq 2} M_\sigma$. So we have

$$B_k = \left| \bigcup_{\sigma \in Y_k, m_\sigma \geq 2} M_\sigma \right| \leq \sum_{\sigma \in Y_k, m_\sigma \geq 2} m_\sigma \leq \sum_{\sigma \in Y_k, m_\sigma \geq 2} 2(m_\sigma - 1) = 2(\widehat{g}_k - f_k). \quad ∎$$

Recall $b_k(Y)$ denotes the $k$th Betti number of $Y$.

PROPOSITION 4.6: *For $k > \ell$ let $\nu_k$ be as in (2.3), then the following holds:*

(1) *Given any sequence $\omega \to \infty$ we have $b_k(Y) \leq \omega n^{\nu_k}$ a.a.s.*
(2) *If $\nu_k < 0$ then $Y$ is collapsible into its $(k-1)$-skeleton a.a.s. so*

$$\tilde{H}_k(Y; \mathbb{Z}) = 0 \quad \text{a.a.s.}$$

*Proof.* Let $Y'$ denote the collapsed complex given by Lemma 4.3 and let $f'_k$ denote the number of $k$-simplices in $Y'$.

(1) Let $\sigma \in Y' \subseteq Y$ be a $k$-simplex, then there is $\tau \in X_{k^+} \subseteq X_{(k-1)^+}$ such that $\tau \supseteq \sigma$, and we claim that $\tau$ is $(k-1)$-bad. Indeed, otherwise $\tau$ is $(k-1)$-good so $G(\tau) \leq k-1$ and so by Lemma 4.3(1) $\sigma$ has been removed during the collapse. This gives $f'_k \leq \binom{r+1}{k+1} B_{k-1}$ since an $i$-simplex contains $\binom{i+1}{k+1}$ $k$-simplices and $\binom{i+1}{k+1} \leq \binom{r+1}{k+1}$. By Lemma 4.5 we get $f'_k \leq \omega n^{2\gamma_{k-1}-k}$ a.a.s.

To obtain (1) we need to replace $\gamma_{k-1}$ with $\gamma_k$ in the last inequality. To achieve this we also look at the hypergraph $\widetilde{X}$ obtained from $X$ by deleting all $(k-1)$-simplices. Let $\widetilde{Y}, \widetilde{Y}', \widetilde{f}'_k$ etc. be the corresponding objects. The possibilities for collapsing simplices of dimension $\geq k$ in $Y$ and $\widetilde{Y}$ are identical, so we look at the collapse assigned to $\widetilde{Y}$ by Lemma 4.3 and apply it to the simplices of dimension $\geq k$ in $Y$. (Some $(k-1)$-simplices of $Y$ are also removed in this process.) We thus obtain a collapse of $Y$ after which $\widetilde{f}'_k$ $k$-simplices remain, so $b_k(Y) \leq \widetilde{f}'_k$. The random model that starts with $X$ and then deletes all $(k-1)$-simplices to obtain $\widetilde{X}$ is equivalent to our usual model only with a different assigned probability in dimension $k-1$, namely, $n^{-\alpha_{k-1}}$ is replaced with 0, i.e., $\widetilde{\alpha}_{k-1} = \infty$. This gives $\widetilde{\beta}_{k-1} = -\infty$ so $\widetilde{\gamma}_{k-1} = \gamma_k$. Being equivalent to the usual model, the bound concluding the previous paragraph applies, and we get

$$b_k(Y) \leq \widetilde{f}'_k \leq \omega n^{2\widetilde{\gamma}_{k-1}-k} = \omega n^{2\gamma_k-k} = \omega n^{\nu_k} \quad \text{a.a.s.}$$

(2) We continue looking at the collapse of $Y$ induced by that of $\widetilde{Y}$. We have $\widetilde{f}'_k \leq \omega n^{\nu_k}$ a.a.s. and if $\nu_k < 0$ then we can take $\omega = n^\epsilon$ with $\epsilon > 0$ such that $\nu_k + \epsilon < 0$, so $\omega n^{\nu_k} \to 0$. But $\widetilde{f}'_k$ is a sequence of non-negative integers so in fact $\widetilde{f}'_k = 0$ a.a.s.  ∎

We remark about the proof above, that the difference between $f'_k$ and $\widetilde{f}'_k$ is only due to our specific definition of $(k-1)$-good, which in turn determines the specific collapse of Lemma 4.3. There may be an $i$-simplex $\tau$ with $i \geq k$ which is $(k-1)$-good in $\widetilde{X}$, but there is a $(k-1)$-simplex $\sigma$ in $X$ with $\sigma \subseteq \tau$ so $\tau$ is $(k-1)$-bad in $X$. In our modified collapse of $Y$ using $\widetilde{X}$, such $\tau$ gets to be collapsed. The collapse of Lemma 4.3 with no modification will be used in the proof of Proposition 4.9, followed by a discussion of its efficiency.

We now show that the exponents $\nu_k$ and the dimension $\ell'$ satisfy the properties stated in Theorem 2.3.

PROPOSITION 4.7: *The exponents $\nu_k$ defined in (2.3) and $\ell'$ defined in (2.4) satisfy the following:*

(1) $\nu_{\ell+1} < \beta$.
(2) $\nu_{k+1} \leq \nu_k - 1$ for all $k$.
(3) $\ell \leq \ell' \leq \lfloor 2\beta \rfloor$.

*Proof.* (1) $\nu_{\ell+1} = 2\gamma_{\ell+1} - \ell - 1 \leq 2\beta - \ell - 1 < \beta$ by (3.2).

(2) $\nu_{k+1} = 2\gamma_{k+1} - k - 1 \leq 2\gamma_k - k - 1 = \nu_k - 1$.

(3) We have $\nu_\ell = 2\gamma_\ell - \ell = 2\beta - \ell > \beta > 0$ by (3.5) and (3.2), so $\ell \leq \ell' \leq r$. (We use $\nu_\ell$ only here, otherwise only $\nu_k$ with $k > \ell$ is of interest.) Now assume $\lfloor 2\beta \rfloor < r$, otherwise we are done. If $\ell < \beta < \ell + \frac{1}{2}$ then $\lfloor 2\beta \rfloor = 2\ell$ and $\nu_{\ell+1} \leq 2\beta - \ell - 1 < \ell$, so by iterating (2) $\ell$ times we have $\nu_{2\ell+1} < 0$ so

$$\ell' \leq 2\ell = \lfloor 2\beta \rfloor.$$

($\nu_{2\ell+1}$ is indeed defined since $2\ell = \lfloor 2\beta \rfloor < r$.) Similarly, if $\ell + \frac{1}{2} \leq \beta < \ell + 1$ then $\lfloor 2\beta \rfloor = 2\ell + 1$ and $\nu_{\ell+1} \leq 2\beta - \ell - 1 < \ell + 1$ so by iterating (2) $\ell + 1$ times we have $\nu_{2\ell+2} < 0$ so

$$\ell' \leq 2\ell + 1 = \lfloor 2\beta \rfloor.$$

($\nu_{2\ell+2}$ is indeed defined since $2\ell + 1 = \lfloor 2\beta \rfloor < r$.) ∎

We now evaluate $b_\ell(Y)$, again via the collapsed complex $Y'$. By Lemma 4.3(3) every $\ell$-good $i$-simplex $\tau$ contributes $\binom{i}{\ell}$ $\ell$-simplices after being collapsed, as opposed to the $\binom{i+1}{\ell+1}$ $\ell$-simplices contained in $\tau$ before the collapse. We would thus like to have a "collapsed version" of our quantity $\widehat{g}_\ell$ where we replace the coefficients $\binom{i+1}{\ell+1}$ by $\binom{i}{\ell}$.

LEMMA 4.8: *Let $g' = \sum_{i=\ell}^r \binom{i}{\ell} g_i$, then $g' \sim dn^\beta$ a.a.s. with $d$ given in (2.2).*

*Proof.* Take $k = \ell$ and $a_i = \binom{i}{\ell}$ in Lemma 3.4, then $g^a = g'$. By (3.5) we have $\gamma_\ell = \beta > 0$, so case (1) of Lemma 3.4 applies. Finally, by Remark 3.2, $i \geq \ell$ for every $i$ such that $\beta_i = \beta$, so $A^a = d$. ∎

PROPOSITION 4.9: $b_\ell(Y) \sim dn^\beta$ *a.a.s. with $d$ given in (2.2).*

*Proof.* As before, let $Y'$ denote the collapsed complex given by Lemma 4.3 and let $f'_k$ denote the number of $k$-simplices in $Y'$. For $i \geq \ell$ denote by $g_i^G$ the number of $\ell$-good $i$-simplices. By definition of $G(\tau)$, if $\tau$ is $\ell$-good then $G(\tau) = \ell$, and so by Lemma 4.3(3) every $\ell$-good $i$-simplex contributes $\binom{i}{\ell}$ $\ell$-simplices to $Y'$.

By definition of $\ell$-good simplices there is no overlap in these contributions, so we have

$$(4.1) \qquad \sum_{i=\ell}^{r} \binom{i}{\ell} g_i^G \leq f_\ell' \leq \sum_{i=\ell}^{r} \binom{i}{\ell} g_i^G + \binom{r+1}{\ell+1} B_\ell.$$

We further note that $g_i - B_\ell \leq g_i^G \leq g_i$ for every $i$. Substituting this into (4.1) gives $g' - cB_\ell \leq f_\ell' \leq g' + cB_\ell$ for some $c > 0$, where by Lemma 4.8 we have $g' \sim dn^\beta$ a.a.s. By (3.2) we have $2\beta - \ell - 1 < \beta$ and take $\epsilon > 0$ so that $\delta = 2\beta - \ell - 1 + \epsilon < \beta$. Taking $\omega = n^\epsilon$ in Lemma 4.5 we get $B_\ell \leq n^\delta$ a.a.s. since $\gamma_\ell = \beta$ by (3.5). Together we get that $f_\ell' \sim dn^\beta$ a.a.s.

Now, in the proof of Proposition 4.6 we noted that $f_{\ell+1}' \leq cB_\ell$ for some $c > 0$ so $f_{\ell+1}' \leq cn^\delta$ a.a.s. Furthermore $f_{\ell-1}' \leq \binom{n+1}{\ell} \leq (n+1)^\ell$ and $\ell < \beta$ by (3.2). So using $f_\ell' - f_{\ell+1}' - f_{\ell-1}' \leq b_\ell(Y) \leq f_\ell'$ we get $b_\ell(Y) \sim dn^\beta$ a.a.s. ∎

Our collapse pattern of Lemma 4.3 involves certain choices that may seem arbitrary and perhaps not as efficient as possible. We can now see that in dimension $\ell$ only negligible further collapse may be possible. Indeed, by the proof of Proposition 4.9 the number of $\ell$-simplices in our particular collapse satisfies $f_\ell' \sim dn^\beta$ a.a.s. and let $f_\ell''(Y)$ denote the minimal number of $\ell$-simplices in any collapse of $Y$. Then $b_\ell(Y) \leq f_\ell''(Y) \leq f_\ell'$ and so also $f_\ell''(Y) \sim dn^\beta$ a.a.s.

We would like to compare the behavior of $b_k(Y)$ described in Propositions 4.6 and 4.9 to that of $f_k(Y)$ described in Proposition 3.5. For $k = \ell$ we have the same exponent $\beta$ by (3.5). As to the coefficient, in general $d < D_\ell$ since each term in the sum for $d$ is $\frac{\ell+1}{i+1}$ times the corresponding term in the sum for $D_\ell$. This reflects the fact that when collapsing an $\ell$-good $i$-simplex $\tau$, a fraction

$$\frac{\binom{i}{\ell}}{\binom{i+1}{\ell+1}} = \frac{\ell+1}{i+1}$$

of the $\ell$-faces of $\tau$ survive the collapse.

For $k > \ell$, the difference $e_k = \nu_k - \gamma_k$ between the corresponding exponents is $e_k = \nu_k - \gamma_k = \gamma_k - k$ which is negative, and drastically more so from dimension to dimension. Indeed, as in the proof of Proposition 4.7(1),(2) we get $e_{\ell+1} < 0$ and $e_{k+1} \leq e_k - 1$ for all $k$. This reflects the increasing proportion of collapse that takes place as we go up in the dimensions.

This completes our analysis regarding the homologies of $Y$ in dimensions $k \geq \ell$. The homologies $\tilde{H}_k(Y; \mathbb{Z})$ for $k < \ell$ are addressed in the next section. At this point the collapsed complex $Y'$ has completed its role in our computations and we return to our original random complex $Y$.

## 5. The homology $\tilde{H}_k(Y;\mathbb{Z})$ for $k < \ell$

In case $\ell = 0$ our analysis is already complete, so we assume $\ell > 0$. We analyze $\tilde{H}_{\ell-1}(Y;\mathbb{Z})$ by reduction to the $\ell$-dimensional Linial–Meshulam model appearing in [MW]. We start with the full $(\ell - 1)$-skeleton $K$ on the vertex set $\{0,\ldots,n\}$ and use our random hypergraph $X$ to add $\ell$-simplices to $K$ by a certain rule presented below. This modified model for a random complex produces an $\ell$-complex that we denote $\widehat{Y}$. We will make sure that the $\ell$-simplices are added independently with probabilities bounded below by $cn^a$ with $a > -1$. It then follows from Theorem 1.1 of [HKP] that $\tilde{H}_{\ell-1}(\widehat{Y};\mathbb{Z}) = 0$ a.a.s. We will use this to deduce our desired results regarding our original random complex $Y$.

For this construction, choose one index $i$ such that $\beta_i = \beta$ and fix it for the rest of this section. We have $i \geq \ell$ by Remark 3.2, and let

$$X_i = \{\tau \in X : \dim \tau = i\}.$$

In our modified random model we use $X_i$ for adding $\ell$-simplices to $K$. But note that if we add to $K$ all $\ell$-faces of the simplices in $X_i$ then, if $i > \ell$, the $\ell$-simplices will not be added independently (since for example we would have $\mathbf{P}(f_\ell = 1) = 0$ and $\mathbf{P}(f_\ell = \binom{i+1}{\ell+1})) > 0$). To circumvent this problem, we will add only one $\ell$-face of each $\tau \in X_i$. In order that the $\ell$-faces will be added with sufficiently large probability, we wish to have a function that chooses an $\ell$-face from each $i$-simplex in a way that every $\ell$-simplex is chosen by sufficiently many $i$-simplices. Let $S_j = S_j(n)$ denote the set of all $j$-simplices on our set $\{0,\ldots,n\}$ of vertices.

LEMMA 5.1: *For every sufficiently large $n$ there exists a function $h : S_i \to S_\ell$ satisfying the following two properties:*

(1) $h(\tau) \subseteq \tau$ for every $\tau \in S_i$.
(2) $|h^{-1}(\sigma)| \geq \dfrac{\binom{n-\ell}{i-\ell}}{2\binom{i+1}{\ell+1}}$ for every $\sigma \in S_\ell$.

*Proof.* Assume first that $i > \ell$. We prove existence of a function $h$ satisfying (1) and (2) using the probabilistic method. For each $\tau \in S_i$ we choose $h(\tau)$ randomly from among the $\binom{i+1}{\ell+1}$ $\ell$-faces of $\tau$, with equal probabilities and independently. The function $h$ satisfies property (1) by definition. If we show that there is a positive probability that $h$ satisfies property (2), then there must exist at least one such function $h$.

Fix one $\sigma \in S_\ell$. Each $\tau \in S_i$ that contains $\sigma$ will choose $\sigma$ to be $h(\tau)$ with probability $\frac{1}{\binom{i+1}{\ell+1}}$, independently. So $F = |h^{-1}(\sigma)|$ is a binomial random variable

with parameters $\binom{n-\ell}{i-\ell}$, $\frac{1}{\binom{i+1}{\ell+1}}$, and we have

$$\mathbf{E}F = \frac{\binom{n-\ell}{i-\ell}}{\binom{i+1}{\ell+1}}.$$

By Chernoff's bound (see, e.g., Theorem 2.1 of [JLR]),

$$\mathbf{P}(F < \mathbf{E}F - R) \leq \exp\Big(-\frac{R^2}{2\mathbf{E}F}\Big).$$

Taking $R = \frac{1}{2}\mathbf{E}F$ we get

$$\mathbf{P}\Big(F < \frac{1}{2}\mathbf{E}F\Big) \leq \exp\Big(-\frac{1}{8}\mathbf{E}F\Big) \leq \exp(-cn^{i-\ell}).$$

This is true for every $\sigma \in S_\ell$, thus the probability that there exists some $\sigma \in S_\ell$ with $|h^{-1}(\sigma)| < \frac{\binom{n-\ell}{i-\ell}}{2\binom{i+1}{\ell+1}}$ is at most $\binom{n+1}{\ell+1}\exp(-cn^{i-\ell})$. We assumed here that $i > \ell$, so for sufficiently large $n$ this probability is strictly less than 1, and so for each such $n$ there must exist a function $h$ with the desired property.

In case $i = \ell$ take $h$ to be the identity, giving $|h^{-1}(\sigma)| = 1 \geq \frac{1}{2}$.  ∎

For each sufficiently large $n$ we choose one function $h_n$ provided by Lemma 5.1. We will use this sequence of functions $h_n$ to define our modified random model. To avoid confusion we emphasize that the probabilistic argument in the proof of Lemma 5.1 was only a method for proving that functions $h_n$ with the desired properties exit. But once the sequence of functions $h_n$ is chosen, they are fixed once and for all and are not random objects in our modified random model. Accordingly, the sets $h_n^{-1}(\sigma)$ are fixed beforehand once and for all.

Finally, our modified random complex $\widehat{Y}$ is defined as follows. Recall $X_i = \{\tau \in X : \dim \tau = i\}$ where $X$ is the random hypergraph that produces our random simplicial complex $Y$. We start with the full $(\ell-1)$-skeleton $K$ on the vertex set $\{0, \dots, n\}$, and for each $\tau \in X_i$ we add to $K$ the $\ell$-simplex $h_n(\tau)$.

LEMMA 5.2: $\tilde{H}_{\ell-1}(\widehat{Y}; \mathbb{Z}) = 0$ a.a.s.

*Proof.* An $\ell$-simplex is included in $\widehat{Y}$ iff one of the $i$-simplices in $h_n^{-1}(\sigma)$ is chosen in the random process defining $X$. Since the sets $h_n^{-1}(\sigma)$ are disjoint, it follows that the $\ell$-simplices are included in $\widehat{Y}$ independently. We now evaluate the probability that a given $\ell$-simplex $\sigma$ is included in $\widehat{Y}$. Denote $N = \binom{n-\ell}{i-\ell}$, $u = n^{-\alpha_i}$, and $N_\sigma = |h_n^{-1}(\sigma)|$, then $eN \leq N_\sigma \leq N$ with $e = \frac{1}{2\binom{i+1}{\ell+1}}$, by

Lemma 5.1. We have

$$\mathbf{P}(\sigma \in \widehat{Y}) = 1 - (1 - u)^{N_\sigma} \geq N_\sigma u - (N_\sigma u)^2$$

since the terms in the alternating binomial sum are decreasing as in the proof of Lemma 3.3, using $N_\sigma u \leq N u \leq n^{i-\ell-\alpha_i} \leq n^{\beta-\ell-1} < 1$. (Here again, in case $i = \ell$ we have $N_\sigma = 1$ and the inequality is seen directly.) Thus we have for sufficiently large $n$

$$\mathbf{P}(\sigma \in \widehat{Y}) \geq e N u - (N u)^2 = (e - N u) N u \geq c N u \geq c' n^{i-\ell-\alpha_i} = c' n^{\beta-\ell-1}$$

since $i$ was chosen such that $\beta_i = \beta$. By (3.2) we have $\beta - \ell - 1 > -1$, so as mentioned in the opening paragraph of this section, it follows from Theorem 1.1 of [HKP] that $\tilde{H}_{\ell-1}(\widehat{Y}; \mathbb{Z}) = 0$ a.a.s. ∎

Returning to our original random complex $Y$ we get the following.

COROLLARY 5.3: $\tilde{H}_{\ell-1}(K \cup Y; \mathbb{Z}) = 0$ a.a.s.

*Proof.* We have that the $(\ell-1)$-skeleton of $\widehat{Y}$ coincides with that of $K \cup Y$, and the set of $\ell$-simplices of $\widehat{Y}$ is contained in that of $K \cup Y$. Thus $\tilde{H}_{\ell-1}(K \cup Y; \mathbb{Z})$ is a quotient of $\tilde{H}_{\ell-1}(\widehat{Y}; \mathbb{Z})$ and so it follows from Lemma 5.2 that $\tilde{H}_{\ell-1}(K \cup Y; \mathbb{Z}) = 0$ a.a.s. ∎

LEMMA 5.4: *If* $\tilde{H}_{\ell-1}(K \cup Y; \mathbb{Z}) = 0$ *then* $Y \supseteq K$, *so* $K \cup Y = Y$.

*Proof.* Assume on the contrary that there exists an $(\ell-1)$-simplex $\rho \notin Y$. Let $\sigma$ be an $\ell$-simplex such that $\rho \in \partial\sigma$. Then $\partial\sigma$ is an $(\ell-1)$-cycle in $K \cup Y$ which cannot be a boundary in $K \cup Y$ since $\rho$ is not contained in any $\ell$-simplex of $K \cup Y$. (Note that if $\ell - 1 = 0$ then $\partial\sigma$ is indeed a reduced cycle in $K \cup Y$.) ∎

This leads us to our two concluding propositions.

PROPOSITION 5.5: $\tilde{H}_{\ell-1}(Y; \mathbb{Z}) = 0$ a.a.s.

*Proof.* By Corollary 5.3 we have $\tilde{H}_{\ell-1}(K \cup Y; \mathbb{Z}) = 0$ a.a.s. but then by Lemma 5.4 we have $K \cup Y = Y$ so in fact $\tilde{H}_{\ell-1}(Y; \mathbb{Z}) = 0$ a.a.s. ∎

PROPOSITION 5.6: *$Y$ contains the full $(\ell-1)$-skeleton a.a.s. and so* $\tilde{H}_k(Y; \mathbb{Z}) = 0$ *for all* $k < \ell - 1$ *a.a.s.*

*Proof.* By Corollary 5.3 we have $\tilde{H}_{\ell-1}(K \cup Y; \mathbb{Z}) = 0$ a.a.s. so by Lemma 5.4 we have $Y \supseteq K$ a.a.s. ∎

For a different proof of Proposition 5.6 see Lemma 11.6 of [FMN].

Finally, Propositions 4.6, 4.7, 4.9, 5.5, 5.6 together constitute our desired Theorem 2.3. This is complemented by Proposition 3.1 that covers the case $\beta < 0$.

# References

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