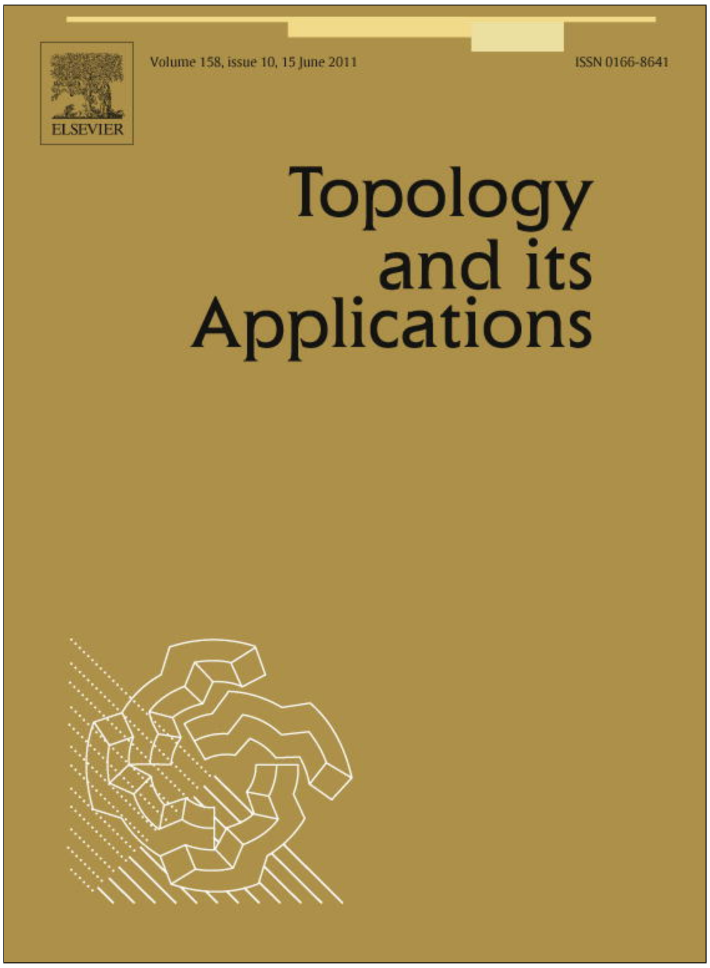
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Order one invariants of spherical curves

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ABSTRACT

We give a complete description of all order 1 invariants of spherical curves. We also identify the subspaces of all J-invariants and S-invariants, and present two equalities satisfied by any spherical curve.

1. Introduction

A spherical or plane curve is an immersion of S^1 into S^2 or $\mathbb{R}^2$ respectively. The study of plane curves dates back as far as Gauss [5]. The regular homotopy classes of plane curves have been classified by Whitney and Graustein in [28]. Smale has generalized this to a study of curves in general smooth manifolds in [22] and then continued, with Hirsch, to develop a general theory of immersions of manifolds in [23,24,8]. New interest in plane and spherical curves has arised when Arnold introduced his three basic invariants of plane and spherical curves, J^+, J^-, St, in [1–3]. Various explicit formulas for these and other invariants appear in [4,18,20,21,27].

Invariants of curves may be characterized by the way their value changes when the curve is deformed by regular homotopy. Arnold's three invariants are, in this sense, very restrictive; the change in the value of these invariants when passing a nonstable (that is, nongeneric, or singular) curve, depends only on the local configuration of the given singularity. A huge class of invariants of curves is the class of "finite order invariants", to be defined in the next section, for which the restrictions on the change in the value of the invariants are far more general. The analogous class of invariants for knots, are known as "Vassiliev invariants". But, whereas the space of Vassiliev invariants of any given order n is finite-dimensional, in the case of curves, as we shall see, the space of order 1 invariants is already infinite-dimensional.

The study of finite order invariants of curves has in general split into the study of J-invariants, as in [6,7,10,19], and S-invariants as in [9,25,26], where J-invariants are invariants which are unchanged when passing a triple point, and S-invariants are invariants which are unchanged when passing a tangency. A complete description of all order 1 invariants of plane curves appears in [16]. In the present work, the analogous classification for spherical curves is presented. These invariants are used in [17] in the study of the complexity of plane and spherical curves, namely, the invariants are used for giving lower bounds on the number of singular moves required for passing between given curves. Invariants of immersions of surfaces in 3-space are studied in [11–15], where a classification of all finite order invariants is given.

In this work we give a complete presentation of all order 1 invariants of spherical curves. Since the vector space in which the invariants take their values is arbitrary, the presentation is in terms of a universal order 1 invariant. This universal order 1 invariant F is the direct sum of two invariants f^X, f^Y. The invariant f^X is analogous to the universal order 1 invariant of plane curves appearing in [16], whereas f^Y is special to spherical curves. We identify the subspaces of J-invariants and

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S-invariants, and observe that the space of invariants spanned by the order 1 J- and S-invariants is much smaller than the full space of order 1 invariants. We present formulas for Arnold's three basic invariants which stem from our presentation. In the final section we present two peculiar equalities satisfied by any spherical curve.

The structure of the paper is as follows. In Section 2 we present the basic definitions, construct our invariant F of spherical curves, and state our main result. In Section 3 we construct an abstract invariant, as opposed to the explicit invariant F, and show that the abstract invariant is universal (Theorem 3.4). In Section 4 we prove our main result, namely, that the explicit invariant F is universal (Theorem 4.5). In Section 5 we construct universal invariants for the spaces of S-invariants and J-invariants. We also present formulas for Arnold's invariants J^+, J^-, St for spherical curves. In the concluding Section 6 we discuss the way in which all our work divides between the two regular homotopy classes of spherical curves, and present two equalities satisfied by any spherical curve (Theorem 6.5).

2. Definitions and statement of results

By a *curve* we will always mean an immersion $c : S^1 \to S^2$. Let $\mathcal{C}$ denote the space of all curves. The space $\mathcal{C}$ has two connected components, that is, two regular homotopy classes, which we denote $\mathcal{C}^{od}, \mathcal{C}^{ev}$. A curve c is in $\mathcal{C}^{od}$ (respectively $\mathcal{C}^{ev}$) if when deleting a point p from S^2 not in the image of c, the curve c, as a curve in $S^2 - \{p\} \cong \mathbb{R}^2$ has odd Whitney number (respectively even Whitney number). A curve will be called *stable* if its only self intersections are transverse double points. A stable curve c is in $\mathcal{C}^{od}$ (respectively $\mathcal{C}^{ev}$) iff the number of double points of c is even (respectively odd).

The generic singularities a curve may have are either a tangency of first order between two strands, which will be called a J-type singularity, or three strands meeting at a point, each two of which are transverse, which will be called an S-type singularity. Singularities of type J and S appear in Figs. 2 and 3.

A generic singularity can be resolved in two ways, and there is a standard way defined in [1] for considering one resolution positive, and the other negative, as we now explain. For J-type singularity, the positive resolution is that where two additional double points appear. For S-type singularity, the sign of the resolution is defined as follows. When resolving an S-type singularity in either way, a small triangle appears. There are two ways for defining an orientation on each of the three edges of this triangle. The first is the orientation restricted from that of the curve itself. The second is the restriction of the orientation of the triangle, which is determined by the cyclic order in which the curve visits the three edges of the triangle. Now, the positive resolution of an S-type singularity is that where the number of edges of the triangle for which the above two orientations coincide, is even.

We denote by $\mathcal{C}_n \subseteq \mathcal{C}$ ($n \geqslant 0$) the space of all curves which have precisely n generic singularity points (the self intersection being elsewhere stable). In particular, $\mathcal{C}_0$ is the space of all stable curves. An invariant of curves is a function $f : \mathcal{C}_0 \to W$, which is constant on the connected components of $\mathcal{C}_0$, and where W in this work will always be a vector space over $\mathbb{Q}$.

Given a curve $c \in \mathcal{C}_n$, with singularities located at $p_1, \dots, p_n \in S^2$, and given a subset $A \subseteq \{p_1, \dots, p_n\}$, we define $c_A \in C_0$ to be the stable curve obtained from c by resolving all singularities of c at points of A into the negative side, and all singularities not in A into the positive side. Given an invariant $f : \mathcal{C}_0 \to W$ we define the "nth derivative" of f to be the function $f^{(n)} : \mathcal{C}_n \to W$ defined by

$$f^{(n)}(c) = \sum_{A \subseteq \{p_1, \dots, p_n\}} (-1)^{|A|} f(c_A)$$

where $|A|$ is the number of elements in A. An invariant $f : \mathcal{C}_0 \to W$ is called *of order n* if $f^{(n+1)}(c) = 0$ for all $c \in \mathcal{C}_{n+1}$. The space of all W valued invariants on $\mathcal{C}_0$ of order n is denoted $V_n = V_n(W)$. Clearly $V_n \subseteq V_m$ for $n \leqslant m$. In this work we give a full description of V_1. We construct a "universal" order 1 invariant, by which we mean the following:

Definition 2.1. An order 1 invariant $\widehat{f} : \mathcal{C}_0 \to \widehat{W}$ will be called *universal*, if for any W and any order 1 invariant $f : \mathcal{C}_0 \to W$, there exists a unique linear map $\phi : \widehat{W} \to W$ such that $f = \phi \circ \widehat{f}$. In other words, for any W, the natural map $Hom_{\mathbb{Q}}(\widehat{W}, W) \to V_1(W)$ given by $\phi \mapsto \phi \circ \widehat{f}$, is an isomorphism.

Definition 2.2. Let D be a 2-disc. An immersion $e : [0,1] \to D$ will be called a *simple arc* if $e(0), e(1) \in \partial D$, $e(0) \neq e(1)$, e is transverse to ∂D, and the self intersections of e are transverse double points.

Definition 2.3. Let D be an oriented 2-disc, and let e be a simple arc in D.

(1) For a double point v of e we define $i(v) \in \{1, -1\}$, where $i(v) = 1$ if the orientation at v given by the two tangents to e at v, in the order they are visited, coincides with the orientation of D. Otherwise $i(v) = -1$.
(2) We define the index of e, $i(e) \in \mathbb{Z}$ by $i(e) = \sum_v i(v)$ where the sum is over all double points v of e.

The following is easy to see, and is the reason for defining i:

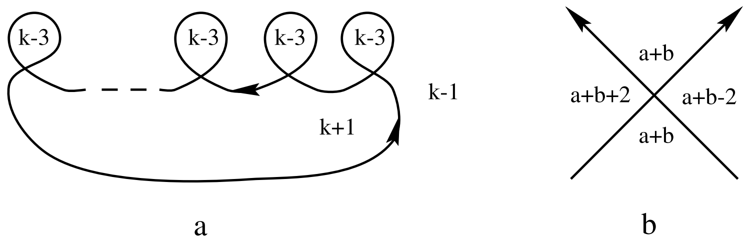

Fig. 1. a: The curve Γ_k. b: $d(R)$ in the vicinity of a double point of type (a,b).

Lemma 2.4. *If e, e' are two simple arcs in D with the same initial point and initial tangent, and the same final point and final tangent, then $i(e) = i(e')$ iff e and e' are regularly homotopic in D, keeping the initial and final points and tangents fixed.*

Define $\mathbb{X}$ to be the vector space over $\mathbb{Q}$ with basis all symbols $X_{a,b}$ where $(a,b) \in \mathbb{Z}^2$ is an ordered pair of integers. Let $\mathbb{Y}$ be the vector space over $\mathbb{Q}$ with basis all symbols Y_d where $d \in \mathbb{Z}$. We construct two invariants $f^X : \mathcal{C}_0 \to \mathbb{X}$ and $f^Y : \mathcal{C}_0 \to \mathbb{Y}$ in the following way.

For $c \in \mathcal{C}_0$ let v be a double point of c, and let u_1, u_2 be the two tangents at v ordered by the orientation of S^2. Let U be a small neighborhood of v and let $D = S^2 - U$. Now $c|_{c^{-1}(D)}$ defines two simple arcs c_1, c_2 in D, ordered so that the tangent u_i leads to c_i, $i = 1, 2$. They will be called the exterior arcs of c. We denote $a(v) = i(c_1)$ and $b(v) = i(c_2)$ where the orientation on D is that restricted from S^2. We define $f^X : \mathcal{C}_0 \to \mathbb{X}$ as follows

$$f^X(c) = \sum_v X_{a(v),b(v)}$$

where the sum is over all double points v of c.

If $c \in \mathcal{C}_0$ then the image of c divides S^2 into complementary regions (which are open discs). To each such region R we attach an integer $d(R)$ as follows. Take $p \in R$ and view c as a curve in $S^2 - \{p\} \cong \mathbb{R}^2$. Let $d(R)$ be *minus* the Whitney number of c as a curve in $S^2 - \{p\}$ where the orientation on $S^2 - \{p\}$ is that restricted from S^2. We define $f^Y : \mathcal{C}_0 \to \mathbb{Y}$ as follows

$$f^Y(c) = \sum_R Y_{d(R)}$$

where the sum is over all complementary regions R of c.

Definition 2.5. Let $F : \mathcal{C}_0 \to \mathbb{X} \oplus \mathbb{Y}$ be the invariant given by $F(c) = f^X(c) + f^Y(c)$.

Example 2.6. For each $k \geqslant 0$ let Γ_k be the curve with k double points appearing in Fig. 1a. In particular Γ_0 is the embedded circle and Γ_1 is the figure eight curve. We have $\Gamma_k \in \mathcal{C}^{\mathrm{od}}$ for k even, and $\Gamma_k \in \mathcal{C}^{\mathrm{ev}}$ for k odd. The integers marked in the figure are $d(R)$ of the various complementary regions. For each of the k double points we have $a(v) = 0$ and $b(v) = k - 1$, and so we obtain $F(\Gamma_k) = kX_{0,k-1} + kY_{k-3} + Y_{k-1} + Y_{k+1}$.

It is not hard to show that the pair of indices $a(v), b(v)$ of a double point v, determine $d(R)$ for the four regions adjacent to v, as stated in Fig. 1b (recall that $d(R)$ is *minus* the corresponding Whitney number).

We define the following six linear maps $\psi_i : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}$, $i = 1, \ldots, 6$ by stating their value on the basis $\{X_{a,b}, Y_d\}$, and where $\psi_i = 0$ on each basis element which is not explicitly mentioned in its definition.

(1) $\psi_1(Y_d) = 1$ for d odd. $\psi_1(X_{a,b}) = -1$ for $a + b$ odd.
(2) $\psi_2(Y_d) = 1$ for d even. $\psi_2(X_{a,b}) = -1$ for $a + b$ even.
(3) $\psi_3(Y_d) = d$ for d odd. $\psi_3(X_{a,b}) = -(a + b)$ for $a + b$ odd.
(4) $\psi_4(Y_d) = d$ for d even. $\psi_4(X_{a,b}) = -(a + b)$ for $a + b$ even.
(5) $\psi_5(Y_d) = \frac{1}{d}$ for d odd. $\psi_5(X_{a,b}) = \frac{4(a-b+1)-(a+b)^2}{(a+b)((a+b)^2-4)}$ for $(a + b)$ odd.
(6) $\psi_6(Y_0) = 1$. $\psi_6(X_{a,b}) = b - a - 1$ for $a + b = 0$. $\psi_6(X_{a,b}) = \frac{a-b}{2}$ for $a + b = \pm 2$.

Definition 2.7. Let W be a vector space over $\mathbb{Q}$, and let $\varphi_1, \ldots, \varphi_n : W \to \mathbb{Q}$ be linear maps. We denote by $W^{\varphi_1,\ldots,\varphi_n}$ the subspace of W determined by the n equations $\varphi_1 = 0, \ldots, \varphi_n = 0$.

Our main result, Theorem 4.5, will be proved in Section 4, stating: $F(\mathcal{C}_0) \subseteq (\mathbb{X} \oplus \mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$, and $F : \mathcal{C}_0 \to (\mathbb{X} \oplus \mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$ is a universal order 1 invariant.

In Section 5 we will construct universal invariants for the class of order 1 S-invariants, J-invariants, and for their combinations which we name SJ-invariants. In Section 6 we will explain how all our constructions split between $\mathcal{C}^{\mathrm{od}}$ and $\mathcal{C}^{\mathrm{ev}}$, and will also note the two equalities expressing the fact that ψ_5, ψ_6 vanish on the image of F.

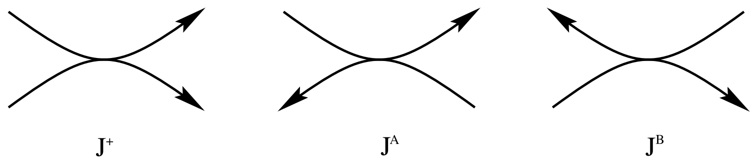

Fig. 2. Singularities of type J^+, J^A, J^B.

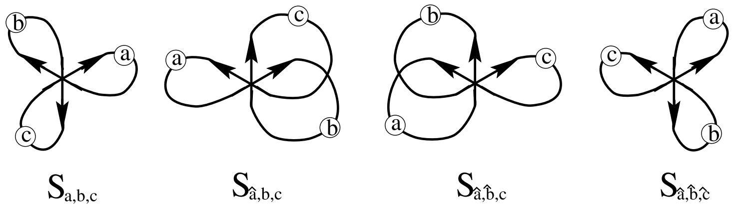

Fig. 3. Singularities of type $S_{a,b,c}$, $S_{\widehat{a},b,c}$, $S_{\widehat{a},\widehat{b},c}$, $S_{\widehat{a},\widehat{b},\widehat{c}}$.

3. An abstract universal order one invariant

In this section we construct an "abstract" universal order 1 invariant, as opposed to the "concrete" invariant described in the previous section. This will be an intermediate step in proving that the concrete invariant is universal. In addition, the existence and properties of the concrete invariant will aid us in proving properties of the abstract invariant.

Two curves $c, c' \in \mathcal{C}_1$ will be called *equivalent* if there is an ambient isotopy of S^2 bringing a neighborhood U of the singular point of c onto a neighborhood U' of the singular point of c', such that the configuration near the singular point precisely matches and such that the exterior arcs in $D = S^2 - U$ are regularly homotopic in D. (When saying the configuration precisely matches this includes the orientation of the strands, and the cyclic order in which they are visited.) By Lemma 2.4 a pair of exterior arcs are regularly homotopic in D iff their indices are the same. So we have:

Proposition 3.1. *Two curves $c, c' \in \mathcal{C}_1$ are equivalent iff they have the same singularity configuration and the same corresponding indices of exterior arcs.*

The following is clear from the definition of order 1 invariant:

Lemma 3.2. *Let $f : \mathcal{C}_0 \to W$ be an invariant, then f is of order 1 iff for any two equivalent $c, c' \in \mathcal{C}_1$, $f^{(1)}(c) = f^{(1)}(c')$.*

We will attach a symbol to each equivalence class of curves in $\mathcal{C}_1$ as follows. For J type singularities there are three distinct configurations which we name J^+, J^A, J^B, see Fig. 2. The J^A, J^B singularities are symmetric with respect to a π rotation, which interchanges the two exterior arcs, and so, by Proposition 3.1, the equivalence class of a J^A or J^B singularity is characterized by a symbol $J^A_{a,b}$ and $J^B_{a,b}$ ($a, b \in \mathbb{Z}$), where a, b is an *unordered* pair, registering the indices of the two exterior arcs. The J^+ configuration, on the other hand, is not symmetric, and so characterized by a symbol $J^+_{a,b}$ with a, b an *ordered* pair, with say, a corresponding to the lower strand in Fig. 2. (We will shortly see that this ordering may however be disregarded.)

As to S type singularities, there are four types, as seen in Fig. 3. The distinction between them will be incorporated into the way we register the indices of the exterior arcs. We will have a cyclicly ordered triple of integers a, b, c registering the indices of the three exterior arcs, in the cyclic order they are visited, and each may appear with or without a hat, according to the following rule. For given exterior arc e with index a, let u_1 be the initial tangent of e and u_2 the initial tangent of the following segment. Then if u_2 is pointing to the right of u_1 then a will appear with a hat, and if u_2 is pointing to the left of u_1 then a will appear unhatted. Since the ordering is cyclic, this gives four types of S symbols, $S_{a,b,c}$, $S_{\widehat{a},b,c}$, $S_{\widehat{a},\widehat{b},c}$, $S_{\widehat{a},\widehat{b},\widehat{c}}$, which correspond to the four types of S singularities.

Let $\mathbb{F}$ denote the vector space over $\mathbb{Q}$ with basis all the above symbols, $J^+_{a,b}$, $J^A_{a,b}$, $J^B_{a,b}$, $S_{a,b,c}$, $S_{\widehat{a},b,c}$, $S_{\widehat{a},\widehat{b},c}$, $S_{\widehat{a},\widehat{b},\widehat{c}}$. Let γ be a generic path in $\mathcal{C}$, that is, a path (regular homotopy) $\gamma : [0,1] \to \mathcal{C}$, whose image lies in $\mathcal{C}_0 \cup \mathcal{C}_1$ and which is transverse with respect to $\mathcal{C}_1$. We denote by $v(\gamma) \in \mathbb{F}$ the sum of symbols of the singularities γ passes, each added with $+$ or $-$ sign according to whether we pass it from its negative side to its positive side, or from its positive side to its negative side, respectively. Let $N \subseteq \mathbb{F}$ be the subspace generated by all elements $v(\gamma)$ obtained from all possible generic *loops* γ in $\mathcal{C}$ (i.e. *closed* paths), and let $\mathbb{G} = \mathbb{F}/N$.

For an order 1 invariant $f : \mathcal{C}_0 \to W$, since $f^{(1)}$ coincides on equivalent curves in $\mathcal{C}_1$, it induces a well-defined linear map $f^{(1)} : \mathbb{F} \to W$.

The following is clear:

Lemma 3.3. *If γ is a generic path in $\mathcal{C}$, from c_1 to c_2, then $f^{(1)}(v(\gamma)) = f(c_2) - f(c_1)$.*

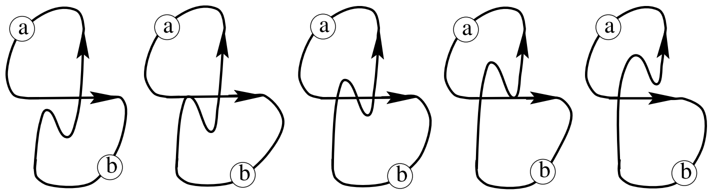

Fig. 4. $J^{+}_{a,b} - J^{+}_{b,a} = 0$.

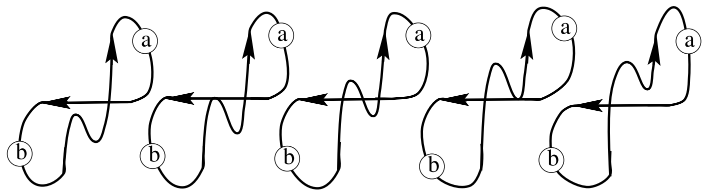

Fig. 5. $J^{B}_{a+1,b} - J^{A}_{a,b-1} = 0$.

From Lemma 3.3 it follows that $f^{(1)}$ vanishes on the generators of N, so it also induces a well-defined linear map $f^{(1)} : \mathbb{G} \to W$.

Let $\widehat{\mathbb{G}} = \mathbb{G} \oplus \mathbb{Q}a_0 \oplus \mathbb{Q}a_1$, where a_0, a_1 are two new vectors. We define an order 1 invariant $\widehat{f} : \mathcal{C}_0 \to \widehat{\mathbb{G}}$ as follows: Denote $\mathcal{C}^{\mathrm{od}}_n = \mathcal{C}_n \cap \mathcal{C}^{\mathrm{od}}$ and $\mathcal{C}^{\mathrm{ev}}_n = \mathcal{C}_n \cap \mathcal{C}^{\mathrm{ev}}$, and recall $\Gamma_0 \in \mathcal{C}^{\mathrm{od}}_0$, $\Gamma_1 \in \mathcal{C}^{\mathrm{ev}}_0$, are the embedded circle, and figure eight curve, respectively. For any $c \in \mathcal{C}^{\mathrm{od}}_0$ there is a generic path γ from Γ_0 to c. Let $\widehat{f}(c) = a_0 + v(\gamma) \in \widehat{\mathbb{G}}$. Similarly for $c \in \mathcal{C}^{\mathrm{ev}}_0$ there is a generic path γ from Γ_1 to c. Let $\widehat{f}(c) = a_1 + v(\gamma) \in \widehat{\mathbb{G}}$. By definition of N, $v(\gamma) \in \mathbb{G}$ is indeed independent of the choice of path γ. From Lemma 3.2 it is clear that $\widehat{f}$ is an order 1 invariant, and we will now see that it is *universal*:

Theorem 3.4. *$\widehat{f} : \mathcal{C}_0 \to \widehat{\mathbb{G}}$ is a universal order* 1 *invariant.*

Proof. For an order 1 invariant $f : \mathcal{C}_0 \to W$, define the linear map $\phi_f : \widehat{\mathbb{G}} \to W$ by $\phi_f|_{\mathbb{G}} = f^{(1)}$ and $\phi_f(a_i) = f(\Gamma_i)$, $i = 0, 1$. We claim $\phi_f \circ \widehat{f} = f$ and that ϕ_f is the unique linear map satisfying this property. Indeed, let $c \in \mathcal{C}^{\mathrm{od}}_0$ and let γ be a generic path from Γ_0 to c. Then by Lemma 3.3 $f(c) = f(\Gamma_0) + f^{(1)}(v(\gamma)) = \phi_f(a_0) + \phi_f(v(\gamma)) = \phi_f(\widehat{f}(c))$. Similarly this is shown for $c \in \mathcal{C}^{\mathrm{ev}}_0$.

For uniqueness, it is enough to show that $\widehat{f}(\mathcal{C}_0)$ spans $\widehat{\mathbb{G}}$. We have $a_i = \widehat{f}(\Gamma_i)$ for $i = 0, 1$ so it remains to show that $\mathbb{G} \subseteq \mathit{span}\, \widehat{f}(\mathcal{C}_0)$. Indeed for any generating symbol T of $\mathbb{G}$, T is the difference $\widehat{f}(c) - \widehat{f}(c')$ for two curves $c, c' \in \mathcal{C}_0$, namely, the two resolutions of a curve in $\mathcal{C}_1$ whose symbol is T. $\square$

The following is also clear:

Lemma 3.5. *Let $f : \mathcal{C}_0 \to W$ be an order* 1 *invariant. If $\phi_f : \widehat{\mathbb{G}} \to W$* (*appearing in the proof of Theorem* 3.4) *is an isomorphism, then $f : \mathcal{C}_0 \to W$ is also a universal order* 1 *invariant.*

We now present six specific subfamilies of the set of generators of N. We will eventually see (concluding paragraph of Section 4) that these elements in fact span N.

The first two families of loops appear in Figs. 4, 5. In each white disc in a figure, there is assumed to be a simple arc of index given by the label on the disc. The element obtained from Fig. 4 is $J^{+}_{a,b} - J^{+}_{b,a}$, that is, in $\mathbb{G}$ we have the relation $J^{+}_{a,b} - J^{+}_{b,a} = 0$, and so from now on we can simply regard the indices of $J^{+}_{a,b}$ as being *un-ordered* (as is true by definition for the J^A and J^B symbols, and as opposed to the indices of $X_{a,b}$ which are ordered). The relation obtained from Fig. 5 is $J^{B}_{a+1,b} - J^{A}_{a,b-1} = 0$, or after index shift: $J^{B}_{a,b} = J^{A}_{a-1,b-1}$.

Our next two families of loops appear in Figs. 6, 7. The relations obtained are: $J^{+}_{c+a-1,b} - S_{\widehat{a},b,c} + S_{\widehat{a},\widehat{b},c} - J^{+}_{c+a+1,b} = 0$ and $J^{A}_{c+a-2,b} + S_{\widehat{a},\widehat{b+1},\widehat{c}} - S_{\widehat{a},b,\widehat{c}} - J^{A}_{c+a,b} = 0$.

When taking the mirror image of such figure, the effect is that the indices are negated, J^A and J^B are interchanged, and the "hat status" of each index is reversed. So, the mirror image of the last two relations gives (after replacing $-a, -b, -c$ by a, b, c): $J^{+}_{c+a+1,b} - S_{a,\widehat{b},\widehat{c}} + S_{a,b,\widehat{c}} - J^{+}_{c+a-1,b} = 0$ and $J^{B}_{c+a+2,b} + S_{a,b-1,c} - S_{a,\widehat{b},c} - J^{B}_{c+a,b} = 0$. After replacing each $J^{B}_{a,b}$ with $J^{A}_{a-1,b-1}$ in the last relation, and an index shift, we obtain: $J^{A}_{c+a+1,b} + S_{a,b,c} - S_{a,\widehat{b+1},c} - J^{A}_{c+a-1,b} = 0$. We write these last four relations (coming from Figs. 6, 7 and their mirror images) in the following more convenient form:

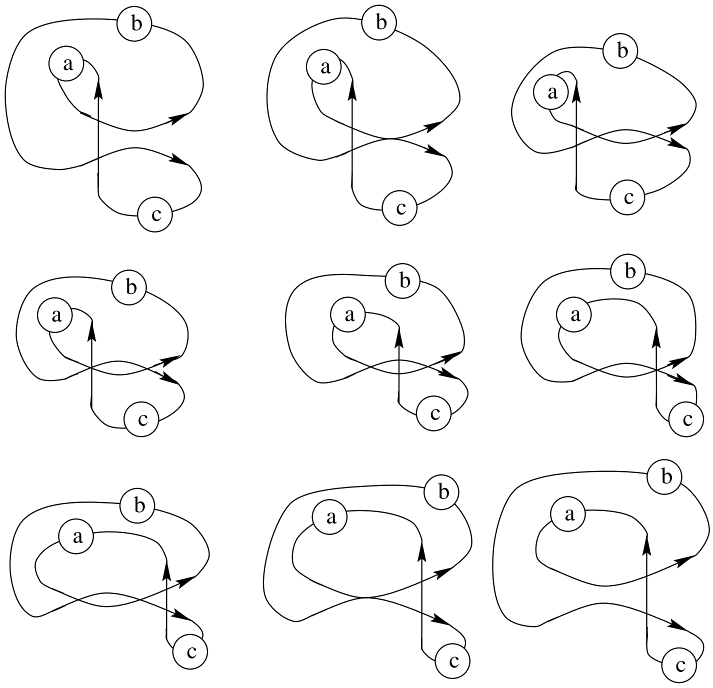

Fig. 6. $J^+_{c+a-1,b} - S_{\widehat{a},b,c} + S_{\widehat{a},\widehat{b},c} - J^+_{c+a+1,b} = 0$.

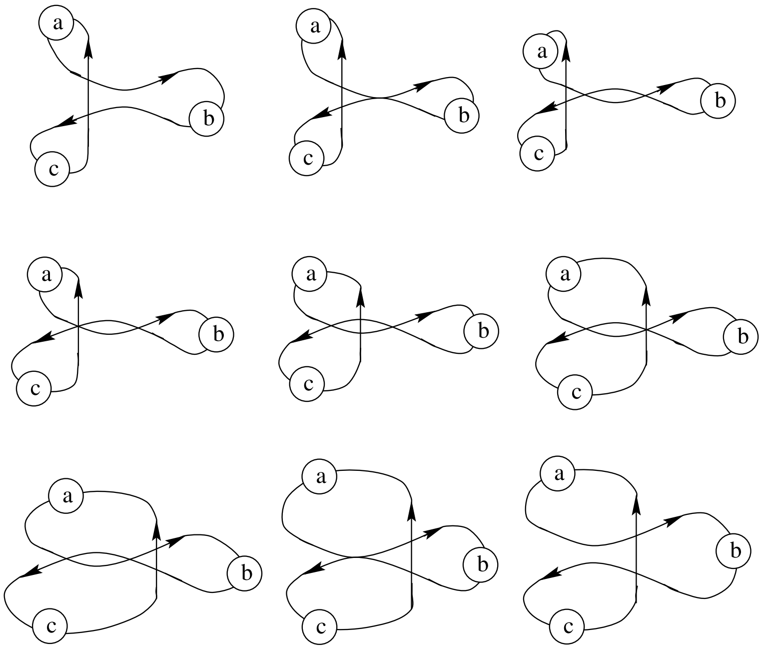

Fig. 7. $J^A_{c+a-2,b} + S_{\widehat{a},\widehat{b+1},\widehat{c}} - S_{\widehat{a},b,\widehat{c}} - J^A_{c+a,b} = 0$.

Proposition 3.6. *The following relations hold in* $\mathbb{G}$:

(1) $S_{\widehat{a},b,c} - S_{\widehat{a},\widehat{b},c} = J^+_{c+a-1,b} - J^+_{c+a+1,b}$.
(2) $S_{a,b,\widehat{c}} - S_{a,\widehat{b},\widehat{c}} = J^+_{c+a-1,b} - J^+_{c+a+1,b}$.
(3) $S_{\widehat{a},b,\widehat{c}} - S_{\widehat{a},\widehat{b+1},\widehat{c}} = J^A_{c+a-2,b} - J^A_{c+a,b}$.
(4) $S_{a,b,c} - S_{a,\widehat{b+1},c} = J^A_{c+a-1,b} - J^A_{c+a+1,b}$.

We will have several occasions to use the following observation:

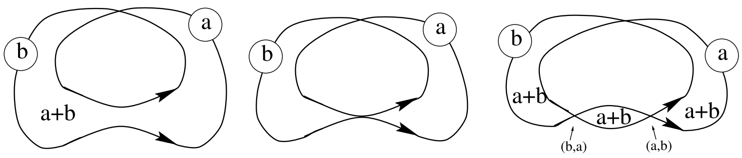

Fig. 8. $F^{(1)}(J^+_{a,b}) = X_{a,b} + X_{b,a} + 2Y_{a+b}$.

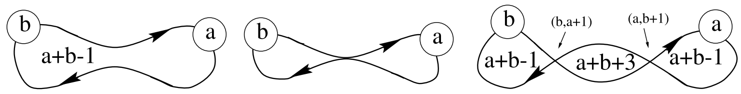

Fig. 9. $F^{(1)}(J^A_{a,b}) = X_{a,b+1} + X_{b,a+1} + Y_{a+b-1} + Y_{a+b+3}$.

Lemma 3.7. *The following two sets of relations are equivalent*:

(1) (a) $S_{\widehat{a},b,c} = S_{\widehat{a},\widehat{b},c}$.
  (b) $S_{a,b,\widehat{c}} = S_{a,\widehat{b},\widehat{c}}$.
  (c) $S_{\widehat{a},b,\widehat{c}} = S_{\widehat{a,b+1,c}}$.
  (d) $S_{a,b,c} = S_{a,\widehat{b+1,c}}$.
(2) *Denoting* $k = a + b + c$:
  (a) $S_{a,b,c} = S_{\widehat{k+1},0,0}$.
  (b) $S_{\widehat{a},b,c} = S_{\widehat{a},\widehat{b},c} = S_{\widehat{k},0,0}$.
  (c) $S_{\widehat{a},\widehat{b},\widehat{c}} = S_{\widehat{k-1},0,0}$.

Proof. Relations (2) clearly imply relations (1). For the converse, alternatingly using (1)(a), (1)(b) we have $S_{\widehat{a},b,c} = S_{\widehat{a},\widehat{b},c} = S_{a,\widehat{b},c} = S_{a,\widehat{b},\widehat{c}} = S_{a,b,\widehat{c}} = S_{\widehat{c},\widehat{a},b}$, that is, for given cyclic triple a, b, c, one or two hats at any location all give equal symbols. From this and from (1)(d) we have $S_{\widehat{a},b,c} = S_{a,\widehat{b},c} = S_{a,b-1,c} = S_{\widehat{a+1},b-1,c}$. From all the above we obtain (2)(b). Now from (2)(b) and (1)(c), (1)(d), we obtain (2)(a), (2)(c). $\square$

Definition 3.8. We define the following two subsets of the generating set of $\mathbb{F}$:

(1) $A^J = \{J^+_{a,b}\}_{a \geqslant b} \cup \{J^A_{a,b}\}_{a \geqslant b}$.
(2) $A^S = \{S_{\widehat{a},0,0}\}_{a \in \mathbb{Z}}$.

We will eventually see (Theorem 4.4(1)) that $A^J \cup A^S$ is a basis for $\mathbb{G}$. At this stage we show:

Lemma 3.9. *$A^J \cup A^S$ is a spanning set for $\mathbb{G}$.*

Proof. From the relation $J^B_{a,b} = J^A_{a-1,b-1}$ it follows that A^J spans all symbols of type J. It follows from Proposition 3.6 that in $\mathbb{G}/\mathit{span}\,A^J$ the relations of Lemma 3.7 hold, and so A^S spans $\mathbb{G}/\mathit{span}\,A^J$. $\square$

4. Proof of main result

Returning to our invariant $F : \mathcal{C}_0 \to \mathbb{X} \oplus \mathbb{Y}$ we now compute $F^{(1)}$. It is easy to see that indeed the value of $F^{(1)}$ depends only on the symbol of a given curve $c \in \mathcal{C}_1$, which by Lemma 3.2 proves that indeed F is an order 1 invariant. For the sake of completeness, we find the value of $F^{(1)}$ on all symbols, though we will need it only for the generators $A^J \cup A^S$ of $\mathbb{G}$.

By Figs. 8, 9 we have

- $F^{(1)}(J^+_{a,b}) = X_{a,b} + X_{b,a} + 2Y_{a+b}$.
- $F^{(1)}(J^A_{a,b}) = X_{a,b+1} + X_{b,a+1} + Y_{a+b-1} + Y_{a+b+3}$.

It follows from this and the relation $J^B_{a,b} = J^A_{a-1,b-1}$ that

- $F^{(1)}(J^B_{a,b}) = X_{a-1,b} + X_{b-1,a} + Y_{a+b-3} + Y_{a+b+1}$.

By Fig. 10 we have:

- $F^{(1)}(S_{a,b,c}) = -X_{a,b+c+2} - X_{b,c+a+2} - X_{c,a+b+2} + X_{a,b+c} + X_{b,c+a} + X_{c,a+b} - Y_{a+b+c+4} + Y_{a+b+c-2}$.

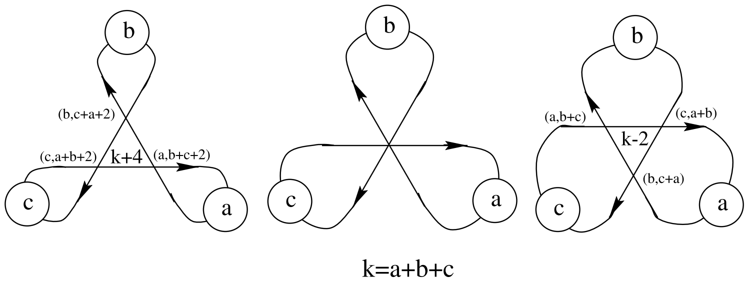

Fig. 10. $F^{(1)}(S_{a,b,c}) = -X_{a,b+c+2} - X_{b,c+a+2} - X_{c,a+b+2} + X_{a,b+c} + X_{b,c+a} + X_{c,a+b} - Y_{a+b+c+4} + Y_{a+b+c-2}$.

It follows from the above values of $F^{(1)}$ and the relations of Proposition 3.6 that

- $F^{(1)}(S_{\widehat{a},b,c}) = -X_{c,a+b+1} - X_{b+c-1,a} - X_{b,c+a+1} + X_{b+c+1,a} + X_{b,c+a-1} + X_{c,a+b-1} - Y_{a+b+c+1} + Y_{a+b+c-1}$.
- $F^{(1)}(S_{\widehat{a},\widehat{b},c}) = -X_{c,a+b+1} - X_{c+a-1,b} - X_{b+c-1,a} + X_{b+c+1,a} + X_{c,a+b-1} + X_{c+a+1,b} - Y_{a+b+c-1} + Y_{a+b+c+1}$.
- $F^{(1)}(S_{\widehat{a},\widehat{b},\widehat{c}}) = -X_{b+c-2,a} - X_{c+a-2,b} - X_{a+b-2,c} + X_{a+b,c} + X_{b+c,a} + X_{c+a,b} - Y_{a+b+c-4} + Y_{a+b+c+2}$.

A key step for dealing with the above complicated expressions for $F^{(1)}$ is the following change of basis for $\mathbb{X} \oplus \mathbb{Y}$. For $(a,b) \in \mathbb{Z}^2$ define

$$\Phi_{a,b} = \frac{a-b}{2} Y_{a+b-2} + (b-a-1) Y_{a+b} + \frac{a-b}{2} Y_{a+b+2}$$

and let $Z_{a,b} = X_{a,b} - \Phi_{a,b}$. Then we claim $\{Z_{a,b}, Y_d\}$ is a basis for $\mathbb{X} \oplus \mathbb{Y}$. Indeed the linear map from $\mathbb{X} \oplus \mathbb{Y}$ to itself taking $X_{a,b} \mapsto Z_{a,b}$, $Y_d \mapsto Y_d$ which is given in the $\{X_{a,b}, Y_d\}$ basis by $X_{a,b} \mapsto X_{a,b} - \Phi_{a,b}$, $Y_d \mapsto Y_d$, has inverse given by $X_{a,b} \mapsto X_{a,b} + \Phi_{a,b}$, $Y_d \mapsto Y_d$. (Since $\Phi_{a,b} \in \mathbb{Y}$ it is fixed by these maps.)

Substituting $X_{a,b} = Z_{a,b} + \Phi_{a,b}$ in the above expressions for $F^{(1)}$, on the generators $A^J \cup A^S$, we obtain:

- $F^{(1)}(J^+_{a,b}) = Z_{a,b} + Z_{b,a}$.
- $F^{(1)}(J^A_{a,b}) = Z_{a,b+1} + Z_{b,a+1}$.
- $F^{(1)}(S_{\widehat{a},0,0}) = -2Z_{0,a+1} - Z_{-1,a} + Z_{1,a} + 2Z_{0,a-1} - \frac{1}{2}(a-3)Y_{a-3} + \frac{3}{2}(a-1)Y_{a-1} - \frac{3}{2}(a+1)Y_{a+1} + \frac{1}{2}(a+3)Y_{a+3}$.

We see that the change of basis has simplified the formulas for $F^{(1)}(J^+_{a,b})$, $F^{(1)}(J^A_{a,b})$. It also simplifies the formulas for $\psi_1, \dots, \psi_6$:

Proposition 4.1. *The linear maps $\psi_1, \dots, \psi_6 : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}$ of Section 2 all vanish on each $Z_{a,b}$.*

Proof. This technical verification is best done by defining new linear maps $\psi'_i : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}$ $i = 1, \dots, 6$, via the $\{Z_{a,b}, Y_d\}$ basis, by $\psi'_i(Z_{a,b}) = 0$ and $\psi'_i(Y_d) = \psi_i(Y_d)$. Then substitute $X_{a,b} = Z_{a,b} + \Phi_{a,b}$ into the formulas for ψ'_i and check that the formulas for ψ_i are obtained. $\square$

The following two lemmas contain the linear algebra portion of our analysis.

Lemma 4.2. *The images of A^J under $F^{(1)} : \mathbb{G} \to \mathbb{X} \oplus \mathbb{Y}$ are independent, and span the subspace $Z = span\{Z_{a,b}\colon (a,b) \in \mathbb{Z}^2\}$.*

Proof. We need to show that the set $\{Z_{a,b} + Z_{b,a}\colon a \geqslant b\} \cup \{Z_{a,b+1} + Z_{b,a+1}\colon a \geqslant b\}$ is a basis for Z. For every $n \in \mathbb{Z}$, let $Z_n = span\{Z_{a,b}\colon a+b=n\}$, then it is enough to show that $\{Z_{a,b} + Z_{b,a}\colon a \geqslant b,\ a+b=n\} \cup \{Z_{a,b+1} + Z_{b,a+1}\colon a \geqslant b,\ a+b=n-1\}$ is a basis for Z_n. For this end we look at the following model space. Let $\mathbb{E}$ be the vector space with basis all symbols E_i, $i \in \mathbb{Z}$. Let D be the following set of element in $\mathbb{E}$: $D = \{E_{-i} + E_i\}_{i \geqslant 0} \cup \{E_{-i} + E_{1+i}\}_{i \geqslant 0}$. We claim that D is a basis for $\mathbb{E}$. Indeed, order the symbols E_i as follows: $E_0, E_1, E_{-1}, E_2, E_{-2}, E_3, E_{-3}, \dots$. Order the elements of D by alternatingly taking an element from the first and second set, that is

$$2E_0, \quad E_0 + E_1, \quad E_{-1} + E_1, \quad E_{-1} + E_2, \quad E_{-2} + E_2, \quad E_{-2} + E_3, \quad E_{-3} + E_3, \quad \dots .$$

By induction, the span of the first m elements in the first list coincides with that of the second list, which establishes our claim.

Back to showing that $\{Z_{a,b} + Z_{b,a}\colon a \geqslant b,\ a+b=n\} \cup \{Z_{a,b+1} + Z_{b,a+1}\colon a \geqslant b,\ a+b=n-1\}$ is a basis for Z_n, assume first that n is even and let $n = 2m$. Define an isomorphism $\varphi : \mathbb{E} \to Z_n$ by $\varphi(E_i) = Z_{m-i,m+i}$ for all $i \in \mathbb{Z}$. Then for all $i \geqslant 0$:

- $\varphi(E_{-i}+E_i)=Z_{m+i,m-i}+Z_{m-i,m+i}=Z_{a,b}+Z_{b,a}$, where $a=m+i$, $b=m-i$.
- $\varphi(E_{-i}+E_{1+i})=Z_{m+i,m-i}+Z_{m-1-i,m+1+i}=Z_{a,b+1}+Z_{b,a+1}$, where $a=m+i$, $b=m-1-i$.

We note that the pairs $a=m+i$, $b=m-i$ for all $i\geqslant 0$, are precisely all $a\geqslant b$ with $a+b=n$, and the pairs $a=m+i$, $b=m-1-i$ for all $i\geqslant 0$, are precisely all $a\geqslant b$ with $a+b=n-1$.

For n odd, let $n=2m+1$. This time we define the isomorphism $\varphi:\mathbb{E}\to Z_n$ by $\varphi(E_i)=Z_{m+i,m+1-i}$ for all $i\in\mathbb{Z}$. We have for all $i\geqslant 0$:

- $\varphi(E_{-i}+E_i)=Z_{m-i,m+1+i}+Z_{m+i,m+1-i}=Z_{b,a+1}+Z_{a,b+1}$, where $a=m+i$, $b=m-i$.
- $\varphi(E_{-i}+E_{1+i})=Z_{m-i,m+1+i}+Z_{m+1+i,m-i}=Z_{b,a}+Z_{a,b}$, where $a=m+1+i$, $b=m-i$.

We note that the pairs $a=m+i$, $b=m-i$ for all $i\geqslant 0$, are precisely all $a\geqslant b$ with $a+b=n-1$, and the pairs $a=m+1+i$, $b=m-i$ for all $i\geqslant 0$, are precisely all $a\geqslant b$ with $a+b=n$. $\square$

Lemma 4.3. *The images of $A^J\cup A^S$ under $F^{(1)}:\mathbb{G}\to\mathbb{X}\oplus\mathbb{Y}$ are independent, and span the subspace $(\mathbb{X}\oplus\mathbb{Y})^{\psi_1,\dots,\psi_6}$.*

Proof. By Lemma 4.2 it is enough to show that the images of A^S are independent in the quotient space $(\mathbb{X}\oplus\mathbb{Y})/Z$, and span $((\mathbb{X}\oplus\mathbb{Y})/Z)^{\psi_1,\dots,\psi_6}$ ($\psi_1,\dots,\psi_6$ are indeed well defined on $(\mathbb{X}\oplus\mathbb{Y})/Z$ since they vanish on Z). Let $P_Z:\mathbb{X}\oplus\mathbb{Y}\to\mathbb{Y}$ be the projection with respect to the direct summand Z. Then the above is the same as proving that the images of A^S under $P_Z\circ F^{(1)}:\mathbb{G}\to\mathbb{Y}$ are independent, and span $\mathbb{Y}^{\psi_1,\dots,\psi_6}$.

For each $a\in\mathbb{Z}$, let $\gamma_a=P_Z\circ F^{(1)}(S_{\widehat{a},0,0})$, then

$$\gamma_a=-\frac{1}{2}(a-3)Y_{a-3}+\frac{3}{2}(a-1)Y_{a-1}-\frac{3}{2}(a+1)Y_{a+1}+\frac{1}{2}(a+3)Y_{a+3}.$$

One checks directly that indeed each γ_a satisfies the six equation $\psi_1=0,\dots,\psi_6=0$. Since these six equations are clearly independent on $\mathbb{Y}$, it is enough to show that $\{\gamma_a\}_{a\in\mathbb{Z}}$ are independent and span a subspace of $\mathbb{Y}$ of codimension 6. For this we show that $\{Y_d\}_{-2\leqslant d\leqslant 2}\cup\{Y_3-Y_{-3}\}\cup\{\gamma_a\}_{a\in\mathbb{Z}}$ are a basis for $\mathbb{Y}$. For this it is enough to prove that for every $n\geqslant 3$, the $(2n+1)$-dimensional subspace of $\mathbb{Y}$ *span*$\{Y_d\}_{-n\leqslant d\leqslant n}$, is spanned by the $2n+1$ elements $\{Y_d\}_{-2\leqslant d\leqslant 2}\cup\{Y_3-Y_{-3}\}\cup\{\gamma_a\}_{-n+3\leqslant a\leqslant n-3}$. This we show by induction on $n\geqslant 3$. For $n=3$ indeed *span*$\{Y_{-3},\dots,Y_3\}$ is spanned by $\{Y_{-2},\dots,Y_2,Y_3-Y_{-3},\gamma_0=\frac{3}{2}(Y_{-3}-Y_{-1}-Y_1+Y_3)\}$. For the induction step one needs to notice that the coefficient of Y_{-n} in γ_{-n+3} is $\frac{n}{2}\neq 0$ and the coefficient of Y_n in γ_{n-3} is $\frac{n}{2}\neq 0$. $\square$

From Lemma 3.9 and Lemma 4.3 we conclude:

Theorem 4.4.

(1) *$A^J\cup A^S$ is a basis for $\mathbb{G}$.*
(2) *$F^{(1)}:\mathbb{G}\to\mathbb{X}\oplus\mathbb{Y}$ is injective, with image $(\mathbb{X}\oplus\mathbb{Y})^{\psi_1,\dots,\psi_6}$.*

We may now prove our main result:

Theorem 4.5. *$F(\mathcal{C}_0)\subseteq(\mathbb{X}\oplus\mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$ and $F:\mathcal{C}_0\to(\mathbb{X}\oplus\mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$ is a universal order 1 invariant.*

Proof. By Lemma 3.5 it is enough to show that $\phi_F:\widehat{\mathbb{G}}\to\mathbb{X}\oplus\mathbb{Y}$ is injective with image $(\mathbb{X}\oplus\mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$. Indeed, by Theorem 4.4, $\phi_F|_{\mathbb{G}}$ is injective, and onto $(\mathbb{X}\oplus\mathbb{Y})^{\psi_1,\dots,\psi_6}$. It remains to show that $\phi_F(a_0)=F(\Gamma_0)$ and $\phi_F(a_1)=F(\Gamma_1)$ are in $(\mathbb{X}\oplus\mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$, and independent mod $(\mathbb{X}\oplus\mathbb{Y})^{\psi_1,\dots,\psi_6}$. This follows by observing the values of $\psi_1,\dots,\psi_6$ on $F(\Gamma_0)$, $F(\Gamma_1)$. We have $F(\Gamma_0)=Y_{-1}+Y_1$ and $F(\Gamma_1)=X_{0,0}+Y_{-2}+Y_0+Y_2=Z_{0,0}+Y_{-2}+Y_2$, so:

- $\psi_1(F(\Gamma_0))=2$ and $\psi_i(F(\Gamma_0))=0$ for $i\neq 1$.
- $\psi_2(F(\Gamma_1))=2$ and $\psi_i(F(\Gamma_1))=0$ for $i\neq 2$. $\square$

From now on we will always present any order 1 invariant in the form $\phi\circ F$. Whenever this is done in practice, we will define ϕ over all $\mathbb{X}\oplus\mathbb{Y}$, which will allow us to define ϕ via the $\{X_{a,b},Y_d\}$ or the $\{Z_{a,b},Y_d\}$ basis. One can then think of ϕ as restricted to $(\mathbb{X}\oplus\mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6}$.

We also remark, that from Theorem 4.4(1) it follows that the six families of elements of N that we have presented and used (Figs. 4, 5, 6, 7 and the mirror images of Figs. 6, 7) in fact span all N. Indeed let $N'\subseteq N$ be the subspace of $\mathbb{F}$ spanned by these elements, and let $K=span(A^J\cup A^S)$. Since the proof of Lemma 3.9 used only the relations coming from these six families, it implies that $K+N'=\mathbb{F}$. On the other hand Theorem 4.4(1) implies $K\cap N=\{0\}$. Together this shows $N'=N$.

5. *S*-invariants and *J*-invariants

An order 1 invariant f is called an S-invariant if $f^{(1)}$ vanishes on all symbols of type J. Similarly, f is called a J-invariant if $f^{(1)}$ vanishes on all symbols of type S. We will denote the spaces of (order 1) S- and J-invariants by V_1^S, V_1^J respectively. Clearly $V_1^J \cap V_1^S = V_0$, the space of all invariants which are constant on $\mathcal{C}^{od}$ and on $\mathcal{C}^{ev}$. A universal invariant for such subclass of invariants, say V_1^S, is an S-invariant $f^S : \mathcal{C}_0 \to W^S$ such that for any W, the natural map $Hom_{\mathbb{Q}}(W^S, W) \to V_1^S(W)$ given by $\phi \mapsto \phi \circ f^S$ is an isomorphism. For example, from the proof of Theorem 4.5 we see that $(\psi_1, \psi_2) \circ F : \mathcal{C}_0 \to \mathbb{Q}^2$ is a universal invariant for V_0. (This is of course nothing other than the invariant assigning $(2,0), (0,2)$ to all curves in $\mathcal{C}_0^{od}, \mathcal{C}_0^{ev}$, respectively.)

We begin with S-invariants. By Theorem 4.5 any W valued order 1 invariant is of the form $f = \phi \circ F$ for a unique $\phi : (\mathbb{X} \oplus \mathbb{Y})^{\psi_3,\psi_4,\psi_5,\psi_6} \to W$. By Lemma 4.2, $f^{(1)} = (\phi \circ F)^{(1)} = \phi \circ F^{(1)}$ vanishes on all symbols of type J iff ϕ vanishes on Z. So we obtain that $f^S = P_Z \circ F : \mathcal{C}_0 \to \mathbb{Y}^{\psi_3,\psi_4,\psi_5,\psi_6}$ is a universal S-invariant, where P_Z, as in the proof of Lemma 4.3, is the projection to $\mathbb{Y}$ with respect to the direct summand Z. The presentation of P_Z via the $\{X_{a,b}, Y_d\}$ basis is: $P_Z(X_{a,b}) = \Phi_{a,b}$, $P_Z(Y_d) = Y_d$.

Example 5.1. Let $\varphi : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}$ be defined via the $\{Z_{a,b}, Y_d\}$ basis as follows $\varphi(y_d) = d^2$, $\varphi(Z_{a,b}) = 0$. By the above analysis $\varphi \circ F$ is an S-invariant. In the $\{X_{a,b}, Y_d\}$ basis φ is given by $\varphi(y_d) = d^2$, $\varphi(X_{a,b}) = 4(a-b) - (a+b)^2$. By direct substitution (preferably in the $\{Z_{a,b}, Y_d\}$ basis) we get $(\varphi \circ F)^{(1)}(S_{\widehat{a},0,0}) = 24$ for all a. By Proposition 3.6 and Lemma 3.7 we conclude that $(\varphi \circ F)^{(1)} = 24$ for all symbols of type S. That is, $\frac{1}{24}\varphi \circ F$ coincides with Arnold's Strangeness invariant of spherical curves, up to choice of constants on $\mathcal{C}^{od}, \mathcal{C}^{ev}$. As noticed above, such constants may be obtained via $\psi_1 \circ F, \psi_2 \circ F$, so, $(\frac{1}{24}\varphi + k_1\psi_1 + k_2\psi_2) \circ F$ gives Arnold's strangeness invariant for spherical curves with all possible normalizations.

We now look at J-invariants.

Lemma 5.2. *An order* 1 *invariant f is a J-invariant iff $f^{(1)}$ vanishes on A^S and the value of $f^{(1)}$ on the symbols $J^+_{a,b}$ and $J^A_{a,b}$ depends only on the parity of a and b.*

Proof. If the value of $f^{(1)}$ on $J^+_{a,b}$, $J^A_{a,b}$ depends only on the parity of a, b, then by Proposition 3.6 the relations of Lemma 3.7 hold, and so if $f^{(1)}$ vanishes on A^S it vanishes on all symbols of type S. The converse is proved similarly, if f is a J-invariant then it vanishes on A^S, and by Proposition 3.6 the value of $f^{(1)}$ on $J^+_{a,b}$, $J^A_{a,b}$ depends only on the parity of a and b (recall that the indices a, b are unordered). □

It follows from Lemma 5.2 that a J-invariant $f : \mathcal{C}_0 \to W$ is determined, up to constants, by the value of $f^{(1)}$ on the following six classes of symbols, where e, o stand for even and odd respectively: $J^+_{e,e}$, $J^+_{o,o}$, $J^+_{e,o}$, $J^A_{e,e}$, $J^A_{o,o}$, $J^A_{e,o}$. In addition to the linear functions ψ_1, ψ_2 which provide the constants, we define the following 6 linear maps $\eta_i : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}$ using the $\{X_{a,b}, Y_d\}$ basis. As before, a basis element that is not mentioned is mapped to 0.

(1) $\eta_1(X_{a,b}) = (a+b)^2$ for $a+b$ odd, $\eta_1(y_d) = -d^2$ for d odd.
(2) $\eta_2(X_{a,b}) = (a+b)^2$ for $a+b$ even, $\eta_2(y_d) = -d^2$ for d even.
(3) $\eta_3(X_{a,b}) = 1$ for a even b odd.
(4) $\eta_4(X_{a,b}) = 1$ for a even b even.
(5) $\eta_5(X_{a,b}) = 1$ for a odd b even.
(6) $\eta_6(X_{a,b}) = 1$ for a odd b odd.

By direct substitution we see that $\eta_i \circ F^{(1)}$ all satisfy the conditions of Lemma 5.2, so $\eta_i \circ F$ are all J-invariants. The value of each η_i on each of the six classes of symbols of type J mentioned above is as follows, where blank spaces are 0.

	$J^+_{e,o}$	$J^A_{e,e}$	$J^A_{o,o}$	$J^+_{e,e}$	$J^+_{o,o}$	$J^A_{e,o}$
η_1	0	−8	−8			
η_3	1	2	0			
η_5	1	0	2			
η_2				0	0	−8
η_4				2	0	1
η_6				0	2	1

This is a nonsingular matrix, and so, if we define $\phi^J : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}^8$ via the eight linear maps $\psi_1, \psi_2, \eta_1, \ldots, \eta_6$, then $f^J = \phi^J \circ F : \mathcal{C}_0 \to \mathbb{Q}^8$ is a universal J-invariant.

Example 5.3. Let $\phi^+, \phi^- : \mathbb{X} \oplus \mathbb{Y} \to \mathbb{Q}$ be defined by:

- $\phi^+(X_{a,b}) = 4 + (a+b)^2$ for all a, b, $\phi^+(Y_d) = -d^2$ for all d.
- $\phi^-(X_{a,b}) = (a+b)^2$ for all a, b, $\phi^-(Y_d) = -d^2$ for all d.

Then $\phi^+ = \eta_1 + \eta_2 + 4(\eta_3 + \eta_4 + \eta_5 + \eta_6)$ and $\phi^- = \eta_1 + \eta_2$, so $\phi^+ \circ F$ and $\phi^- \circ F$ are J-invariants. By direct substitution or by applying the matrix above, we obtain that:

- $(\phi^+ \circ F)^{(1)}(J^+_{a,b}) = 8$, $(\phi^+ \circ F)^{(1)}(J^A_{a,b}) = 0$, for all a, b.
- $(\phi^- \circ F)^{(1)}(J^A_{a,b}) = -8$, $(\phi^- \circ F)^{(1)}(J^+_{a,b}) = 0$, for all a, b.

That is, $\frac{1}{4}\phi^+ \circ F, \frac{1}{4}\phi^- \circ F$ coincide with Arnold's J^+, J^- invariants of spherical curves, respectively, up to choice of constants on $\mathcal{C}^{\mathrm{od}}, \mathcal{C}^{\mathrm{ev}}$. So, $(\frac{1}{4}\phi^+ + k_1\psi_1 + k_2\psi_2) \circ F$ and $(\frac{1}{4}\phi^- + k_1\psi_1 + k_2\psi_2) \circ F$ give Arnold's J^+ and J^- invariants for spherical curves with all possible normalizations.

An order 1 invariant will be called an *SJ*-invariant if it is the sum of an S-invariant and a J-invariant. The space of *SJ*-invariants is thus $V_1^S + V_1^J$. We recall that $V_1^S \cap V_1^J = V_0$. It follows that $f^{SJ} = (P_Z \oplus (\eta_1, \ldots, \eta_6)) \circ F : \mathcal{C}_0 \to \mathbb{Y}^{\psi_3,\psi_4,\psi_5,\psi_6} \oplus \mathbb{Q}^6$ is a universal *SJ*-invariant. We also see that $\phi \circ F$ is an *SJ*-invariant iff ϕ vanishes on $Z^{\eta_1,\ldots,\eta_6}$. (This explains our remark in the introduction, that the space of *SJ*-invariants is much smaller than the full space of order 1 invariants.)

We have considered several linear maps H on $\mathbb{X} \oplus \mathbb{Y}$ of the special form $Y_d \mapsto h(d)$, $X_{a,b} \mapsto -h(a+b)$, for function h on $\mathbb{Z}$, namely, $\psi_1, \psi_2, \psi_3, \psi_4, \eta_1, \eta_2$ and $\phi^- = \eta_1 + \eta_2$. For H of this form we can interpret $H \circ F(c)$ in terms of the smoothing of c and its complementary regions in S^2. Indeed, from Fig. 1b it is clear that for H of this form, $H \circ F(c) = \sum_E \chi(E)h(d(E))$, where the sum is over all complementary regions E of the smoothing of c, and $d(E)$ is minus the sum of Whitney numbers of the components of the smoothing of c, considered in $S^2 - \{p\} \cong \mathbb{R}^2$ for some $p \in E$. (Note that the total Whitney number is preserved under smoothing.) Conversely, this equality shows that for any h the invariant $\sum_E \chi(E)h(d(E))$ is of order 1. Via this interpretation, there is a clear similarity between our formulas above for the two invariants J^+, J^- for spherical curves, and Viro's formulas for J^+, J^- for plane curves, appearing in [27]. The notion of *index* with respect to a point in $\mathbb{R}^2$, appearing in Viro's formulas, is replaced in our formulas by the notion of *Whitney number* with respect to a point in S^2.

6. Odds, evens, and ends

In this section we will clarify how all our above results and constructions split between $\mathcal{C}^{\mathrm{od}}$ and $\mathcal{C}^{\mathrm{ev}}$. This should give a better understanding of various details of our work. We will also obtain two equalities satisfied by any spherical curve, using our linear maps ψ_5, ψ_6.

Let $\mathbb{X}^{\mathrm{od}} \subseteq \mathbb{X}$ (respectively $\mathbb{X}^{\mathrm{ev}} \subseteq \mathbb{X}$) be the subspace spanned by $X_{a,b}$ with $a+b$ odd (respectively even). Let $\mathbb{Y}^{\mathrm{od}} \subseteq \mathbb{Y}$ (respectively $\mathbb{Y}^{\mathrm{ev}} \subseteq \mathbb{Y}$) be the subspace spanned by Y_d with d odd (respectively even).

Proposition 6.1. *$F(\mathcal{C}_0^{\mathrm{od}}) \subseteq \mathbb{X}^{\mathrm{od}} + \mathbb{Y}^{\mathrm{od}}$ and $F(\mathcal{C}_0^{\mathrm{ev}}) \subseteq \mathbb{X}^{\mathrm{ev}} + \mathbb{Y}^{\mathrm{ev}}$.*

Proof. Let $c \in \mathcal{C}_0^{\mathrm{od}}$. Given a double point v of c, let U be a small neighborhood of v, $D = S^2 - U$ and c_1, c_2 the two exterior arcs in D. Since the endpoints of c_1, c_2 are not intertwined along ∂D, there exists a regular homotopy of c_1, c_2 in D at the end of which they are disjoint. Such regular homotopy preserves $i(c_1), i(c_2)$, so we may assume c_1, c_2 are disjoint. Since $c \in \mathcal{C}_0^{\mathrm{od}}$, the number of double points of c is even and so the total number of double points of c_1 and c_2 is odd. Since the parity of $i(c_1), i(c_2)$ coincides with the parity of the number of double points of c_1, c_2, the contribution of v to $f^X(c)$ is in $\mathbb{X}^{\mathrm{od}}$. This is true for each double point v, and so $f^X(c) \in \mathbb{X}^{\mathrm{od}}$. The fact that $f^Y(c) \in \mathbb{Y}^{\mathrm{od}}$ is by definition of $\mathcal{C}^{\mathrm{od}}$. The same argument shows that for $c \in \mathcal{C}_0^{\mathrm{ev}}$, $F(c) \in \mathbb{X}^{\mathrm{ev}} + \mathbb{Y}^{\mathrm{ev}}$. $\square$

In fact, from the proof of Theorem 4.5 we obtain the following more precise statement:

Theorem 6.2.

(1) *The affine span of $F(\mathcal{C}_0^{\mathrm{od}})$ is the affine subspace of $(\mathbb{X}^{\mathrm{od}} + \mathbb{Y}^{\mathrm{od}})$ determined by the following three equations on $(\mathbb{X}^{\mathrm{od}} + \mathbb{Y}^{\mathrm{od}})$: $\psi_1 = 2$, $\psi_3 = 0$, $\psi_5 = 0$.*
(2) *The affine span of $F(\mathcal{C}_0^{\mathrm{ev}})$ is the affine subspace of $(\mathbb{X}^{\mathrm{ev}} + \mathbb{Y}^{\mathrm{ev}})$ determined by the following three equations on $(\mathbb{X}^{\mathrm{ev}} + \mathbb{Y}^{\mathrm{ev}})$: $\psi_2 = 2$, $\psi_4 = 0$, $\psi_6 = 0$.*

Let $\mathbb{F}^{\mathrm{od}} \subseteq \mathbb{F}$ be the subspace spanned by all symbols corresponding to curves in $\mathcal{C}_1^{\mathrm{od}}$. Let $N^{\mathrm{od}} \subseteq N$ be the subspace spanned by $v(c)$ for all generic loops in $\mathcal{C}^{\mathrm{od}}$, then $N^{\mathrm{od}} \subseteq F^{\mathrm{od}}$, and let $\mathbb{G}^{\mathrm{od}} = \mathbb{F}^{\mathrm{od}}/N^{\mathrm{od}}$. Let $\mathbb{F}^{\mathrm{ev}}, N^{\mathrm{ev}}, \mathbb{G}^{\mathrm{ev}}$ be defined in the same way. Then $\mathbb{F} = \mathbb{F}^{\mathrm{od}} \oplus \mathbb{F}^{\mathrm{ev}}$ and $\mathbb{G} = \mathbb{G}^{\mathrm{od}} \oplus \mathbb{G}^{\mathrm{ev}}$.

Proposition 6.3. *$F^{(1)}(\mathbb{G}^{\mathrm{od}}) \subseteq \mathbb{X}^{\mathrm{od}} + \mathbb{Y}^{\mathrm{od}}$ and $F^{(1)}(\mathbb{G}^{\mathrm{ev}}) \subseteq \mathbb{X}^{\mathrm{ev}} + \mathbb{Y}^{\mathrm{ev}}$.*

Proof. For each generating symbol T of $\mathbb{G}^{\mathrm{od}}$, $F^{(1)}(T)$ is the difference $F(c) - F(c')$ for two curves $c, c' \in \mathcal{C}_0^{\mathrm{od}}$ (the two resolutions of a curve in $\mathcal{C}_1^{\mathrm{od}}$) and so by Proposition 6.1, $F^{(1)}(T) \in \mathbb{X}^{\mathrm{od}} + \mathbb{Y}^{\mathrm{od}}$. The same argument holds for a generating symbol of $\mathbb{G}^{\mathrm{ev}}$. $\square$

Again our work implies a more precise statement:

Proposition 6.4. *$F^{(1)}(\mathbb{G}^{\mathrm{od}}) = (\mathbb{X}^{\mathrm{od}} + \mathbb{Y}^{\mathrm{od}})^{\psi_1,\psi_3,\psi_5}$ and $F^{(1)}(\mathbb{G}^{\mathrm{ev}}) = (\mathbb{X}^{\mathrm{ev}} + \mathbb{Y}^{\mathrm{ev}})^{\psi_2,\psi_4,\psi_6}$.*

Finally, we would like to determine which are the symbols corresponding to curves in $\mathcal{C}_1^{\mathrm{od}}$ and which to curves in $\mathcal{C}_1^{\mathrm{ev}}$. We can do this by observing the curves themselves, in which case we must be cautious regarding the way the ends of the different exterior arcs are intertwined along ∂D. Alternatively, we can use Proposition 6.3 and our explicit formulas for $F^{(1)}$. We obtain: The symbols corresponding to curves in $\mathcal{C}_1^{\mathrm{od}}$ are: $J^+_{a,b}$ for $a+b$ odd, $J^A_{a,b}, J^B_{a,b}$ for $a+b$ even, $S_{a,b,c}, S_{\widehat{a},\widehat{b},\widehat{c}}$ for $a+b+c$ odd, and $S_{\widehat{a},b,c}, S_{\widehat{a},\widehat{b},c}$ for $a+b+c$ even. The symbols corresponding to curves in $\mathcal{C}_1^{\mathrm{ev}}$ are the complementary set, that is: $J^+_{a,b}$ for $a+b$ even, $J^A_{a,b}, J^B_{a,b}$ for $a+b$ odd, $S_{a,b,c}, S_{\widehat{a},\widehat{b},\widehat{c}}$ for $a+b+c$ even, and $S_{\widehat{a},b,c}, S_{\widehat{a},\widehat{b},c}$ for $a+b+c$ odd.

We conclude this work with the presentation of two equalities satisfied by any spherical curve. Define $x_{a,b} : \mathcal{C}_0 \to \mathbb{Z}$ and $y_d : \mathcal{C}_0 \to \mathbb{Z}$ via

$$F(c) = \sum_{a,b} x_{a,b}(c) X_{a,b} + \sum_d y_d(c) Y_d.$$

That is, $x_{a,b}(c)$ is the number of double points of c of type (a,b) and $y_d(c)$ is the number of complementary regions of c of type d. Theorem 6.2 gives six equalities satisfied by the curves in $\mathcal{C}_0$, which can be interpreted as six relations between the invariants $x_{a,b}(c), y_d(c)$. We exclude the equalities coming from $\psi_1, \psi_2, \psi_3, \psi_4$ since they can be proved directly, by Euler Characteristic arguments, via the observation in the concluding paragraph of Section 5. The equalities coming from ψ_5 and ψ_6 are:

Theorem 6.5. *The following two equalities hold for any spherical curve $c \in \mathcal{C}_0$:*

$$(1) \qquad \sum_{d \text{ odd}} \frac{1}{d} y_d(c) + \sum_{a+b \text{ odd}} \frac{4(a-b+1) - (a+b)^2}{(a+b)((a+b)^2 - 4)} x_{a,b}(c) = 0.$$

$$(2) \qquad y_0(c) + \sum_{a+b=0} (b-a-1) x_{a,b}(c) + \sum_{a+b=\pm 2} \frac{a-b}{2} x_{a,b}(c) = 0.$$

Note that by Proposition 6.1, equality (1) is trivially satisfied on $\mathcal{C}^{\mathrm{ev}}$ and equality (2) is trivially satisfied on $\mathcal{C}^{\mathrm{od}}$.

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