Research Article

Mikhail G. Katz* and Tahl Nowik

A systolic inequality with remainder in the real projective plane

https://doi.org/10.1515/math-2020-0050
received November 16, 2019; accepted July 1, 2020

Abstract: The first paper in systolic geometry was published by Loewner's student P. M. Pu over half a century ago. Pu proved an inequality relating the systole and the area of an arbitrary metric in the real projective plane. We prove a stronger version of Pu's systolic inequality with a remainder term.

Keywords: systole, geometric inequality, Riemannian submersion, Cauchy-Schwarz theorem, probabilistic variance

MSC 2020: 53C23, 53A30

1 Introduction

Loewner's systolic inequality for the torus and Pu's inequality [1] for the real projective plane were historically the first results in systolic geometry. Great stimulus was provided in 1983 by Gromov's paper [2] and later by his book [3].

Our goal is to prove a strengthened version with a remainder term of Pu's systolic inequality $\mathrm{sys}^2(g) \leq \frac{\pi}{2}\mathrm{area}(g)$ (for an arbitrary metric g on $\mathbb{RP}^2$), analogous to Bonnesen's inequality $L^2 - 4\pi A \geq \pi^2(R-r)^2$, where L is the length of a Jordan curve in the plane, A is the area of the region bounded by the curve, R is the circumradius and r is the inradius.

Note that both the original proof in Pu ([1], 1952) and the one given by Berger ([4], 1965, pp. 299–305) proceed by averaging the metric and showing that the averaging process decreases the area and increases the systole. Such an approach involves a five-dimensional integration (instead of a three-dimensional one given here) and makes it harder to obtain an explicit expression for a remainder term. Analogous results for the torus were obtained in ref. [5] with generalizations in ref. [6–17].

2 The results

We define a closed three-dimensional manifold $M \subseteq \mathbb{R}^3 \times \mathbb{R}^3$ by setting

$$M = \{(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 : v \cdot v = 1, w \cdot w = 1, v \cdot w = 0\},$$

where $v \cdot w$ is the scalar product on $\mathbb{R}^3$. We have a diffeomorphism $M \to SO(3, \mathbb{R})$, $(v, w) \mapsto (v, w, v \times w)$, where $v \times w$ is the vector product on $\mathbb{R}^3$. Given a point $(v, w) \in M$, the tangent space $T_{(v,w)}M$ can be identified by differentiating the three defining equations of M along a path through (v, w). Thus,

*** Corresponding author: Mikhail G. Katz,** Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel, e-mail: katzmik@macs.biu.ac.il
Tahl Nowik: Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel, e-mail: tahl@math.biu.ac.il

$$T_{(v,w)}M = \{(X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3 : X \cdot v = 0, Y \cdot w = 0, X \cdot w + Y \cdot v = 0\}.$$

We define a Riemannian metric g_M on M as follows. Given a point $(v, w) \in M$, let $n = v \times w$ and declare the basis $(0, n)$, $(n, 0)$, $(w, -v)$ of $T_{(v,w)}M$ to be orthonormal. This metric is a modification of the metric restricted to M from $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. Namely, with respect to the Euclidean metric on $\mathbb{R}^6$ the above three vectors are orthogonal and the first two have length 1. However, the third vector has Euclidean length $\sqrt{2}$, whereas we have defined its length to be 1. Thus, if $A \subseteq T_{(v,w)}M$ denotes the span of $(0,n)$ and $(n,0)$, and $B \subseteq T_{(v,w)}M$ is spanned by $(w, -v)$, then the metric g_M on M is obtained from the Euclidean metric g on $\mathbb{R}^6$ (viewed as a quadratic form) as follows:

$$g_M = g\lfloor_A + \frac{1}{2} g\lfloor_B. \tag{1}$$

Each of the natural projections $p, q : M \to S^2$ given by $p(v, w) = v$ and $q(v, w) = w$ exhibits M as a circle bundle over S^2.

Lemma 2.1. *The maps p and q on (M, g_M) are Riemannian submersions, over the unit sphere $S^2 \subseteq \mathbb{R}^3$.*

Proof. For the projection p, given $(v, w) \in M$, the vector $(0, n)$ as defined above is tangent to the fiber $p^{-1}(v)$. The other two vectors, $(n, 0)$ and $(w, -v)$, are thus an orthonormal basis for the subspace of $T_{(v,w)}M$ normal to the fiber and are mapped by dp to the orthonormal basis n, w of $T_v S^2$. □

The projection p maps the fiber $q^{-1}(w)$ onto a great circle of S^2. This map preserves length since the unit vector $(n,0)$, tangent to the fiber $q^{-1}(w)$ at (v, w), is mapped by dp to the unit vector $n \in T_v S^2$. The same comments apply when the roles of p and q are reversed.

In the following proposition, integration takes place, respectively, over great circles $C \subseteq S^2$, over the fibers in M, over S^2, and over M. The integration is always with respect to the volume element of the given Riemannian metric. Since p and q are Riemannian submersions by Lemma 2.1, we can use Fubini's theorem to integrate over M by integrating first over the fibers of either p or q, and then over S^2; cf. [18, Lemma 4]. By the remarks above, if $C = p(q^{-1}(w))$ and $f : S^2 \to \mathbb{R}$, then $\int_{q^{-1}(w)} f \circ p = \int_C f$.

Proposition 2.2. *Given a continuous function $f : S^2 \to \mathbb{R}^+$, we define $m \in \mathbb{R}$ by setting*

$$m = \min\left\{ \int_C f : C \subseteq S^2 \text{ a great circle} \right\}.$$

Then,

$$\frac{m^2}{\pi} \le \frac{1}{4\pi} \left(\int_{S^2} f \right)^2 \le \int_{S^2} f^2,$$

where equality in the second inequality occurs if and only if f is constant.

Proof. Using the fact that M is the total space of a pair of Riemannian submersions, we obtain

$$\int_{S^2} f = \int_{S^2} \left(\frac{1}{2\pi} \int_{p^{-1}(v)} f \circ p \right) = \frac{1}{2\pi} \int_M f \circ p = \frac{1}{2\pi} \int_{S^2} \left(\int_{q^{-1}(w)} f \circ p \right) \ge \frac{1}{2\pi} \int_{S^2} m = 2m,$$

proving the first inequality. By the Cauchy-Schwarz inequality, we have

$$\left(\int_{S^2} 1\cdot f\right)^2 \leq 4\pi \int_{S^2} f^2,$$

proving the second inequality. Here, equality occurs if and only if f and 1 are linearly dependent, i.e., if and only if f is constant. □

We define the quantity V_f by setting $V_f = \int_{S^2} f^2 - \frac{1}{4\pi}\left(\int_{S^2} f\right)^2$. Then, Proposition 2.2 can be restated as follows.

Corollary 2.3. *Let $f : S^2 \to \mathbb{R}^+$ be continuous. Then,*

$$\int_{S^2} f^2 - \frac{m^2}{\pi} \geq V_f \geq 0,$$

and $V_f = 0$ if and only if f is constant.

Proof. The proof is obtained from Proposition 2.2 by noting that $a \leq b \leq c$ if and only if $c - a \geq c - b \geq 0$. □

We can assign a probabilistic meaning to V_f as follows. Divide the area measure on S^2 by 4π, thus turning it into a probability measure μ. A function $f : S^2 \to \mathbb{R}^+$ is then thought of as a random variable with expectation $E_\mu(f) = \frac{1}{4\pi}\int_{S^2} f$. Its variance is thus given by

$$\mathrm{Var}_\mu(f) = E_\mu(f^2) - (E_\mu(f))^2 = \frac{1}{4\pi}\int_{S^2} f^2 - \left(\frac{1}{4\pi}\int_{S^2} f\right)^2 = \frac{1}{4\pi}V_f.$$

The variance of a random variable f is non-negative, and it vanishes if and only if f is constant. This reproves the corresponding properties of V_f established above via the Cauchy-Schwarz inequality.

Now let g_0 be the metric of constant Gaussian curvature $K = 1$ on $\mathbb{RP}^2$. The double covering $\rho : S^2 \to (\mathbb{RP}^2, g_0)$ is a local isometry. Each projective line $C \subseteq \mathbb{RP}^2$ is the image under ρ of a great circle of S^2.

Proposition 2.4. *Given a function $f : \mathbb{RP}^2 \to \mathbb{R}^+$, we define $\overline{m} \in \mathbb{R}$ by setting*

$$\overline{m} = \min\left\{\int_C f : C \subseteq \mathbb{RP}^2 \text{ a projective line}\right\}.$$

Then,

$$\frac{2\overline{m}^2}{\pi} \leq \frac{1}{2\pi}\left(\int_{\mathbb{RP}^2} f\right)^2 \leq \int_{\mathbb{RP}^2} f^2,$$

where equality in the second inequality occurs if and only if f is constant.

Proof. We apply Proposition 2.2 to the composition $f \circ \rho$. Note that we have $\int_{\rho^{-1}(C)} f \circ \rho = 2\int_C f$ and $\int_{S^2} f \circ \rho = 2\int_{\mathbb{RP}^2} f$. The condition for f to be constant holds since f is constant if and only if $f \circ \rho$ is constant. □

For $\mathbb{RP}^2$ we define $\overline{V}_f = \int_{\mathbb{RP}^2} f^2 - \frac{1}{2\pi}\left(\int_{\mathbb{RP}^2} f\right)^2 = \frac{1}{2}V_{f\circ\rho}$. We obtain the following restatement of Proposition 2.4.

Corollary 2.5. *Let $f : \mathbb{RP}^2 \to \mathbb{R}^+$ be a continuous function. Then,*

$$\int_{\mathbb{RP}^2} f^2 - \frac{2\overline{m}^2}{\pi} \geq \overline{V}_f \geq 0,$$

where $\overline{V}_f = 0$ if and only if f is constant.

Relative to the probability measure induced by $\frac{1}{2\pi}g_0$ on $\mathbb{RP}^2$, we have $E(f) = \frac{1}{2\pi}\int_{\mathbb{RP}^2} f$, and therefore $\mathrm{Var}(f) = \frac{1}{2\pi}\overline{V}_f$, providing a probabilistic meaning for the quantity $\overline{V}_f$, as before.

By the uniformization theorem, every metric g on $\mathbb{RP}^2$ is of the form $g = f^2 g_0$, where g_0 is of constant Gaussian curvature +1, and the function $f : \mathbb{RP}^2 \to \mathbb{R}^+$ is continuous. The area of g is $\int_{\mathbb{RP}^2} f^2$, and the g-length of a projective line C is $\int_C f$. Let L be the shortest length of a noncontractible loop. Then, $L \leq \overline{m}$ where $\overline{m}$ is defined in Proposition 2.4, since a projective line in $\mathbb{RP}^2$ is a noncontractible loop. Then, Corollary 2.5 implies $\mathrm{area}(\mathbb{RP}^2, g) - \frac{2L^2}{\pi} \geq \overline{V}_f \geq 0$. If $\mathrm{area}(\mathbb{RP}^2, g) = \frac{2L^2}{\pi}$, then $\overline{V}_f = 0$, which implies that f is constant, by Corollary 2.5. Conversely, if f is a constant c, then the only geodesics are the projective lines, and therefore, $L = c\pi$. Hence, $\frac{2L^2}{\pi} = 2\pi c^2 = \mathrm{area}(\mathbb{RP}^2)$. We have thus completed the proof of the following result strengthening Pu's inequality.

Theorem 2.6. *Let g be a Riemannian metric on $\mathbb{RP}^2$. Let L be the shortest length of a noncontractible loop in $(\mathbb{RP}^2, g)$. Let $f : \mathbb{RP}^2 \to \mathbb{R}^+$ be such that $g = f^2 g_0$, where g_0 is of constant Gaussian curvature +1. Then,*

$$\mathrm{area}(g) - \frac{2L^2}{\pi} \geq 2\pi \mathrm{Var}(f),$$

where the variance is with respect to the probability measure induced by $\frac{1}{2\pi}g_0$. Furthermore, equality $\mathrm{area}(g) = \frac{2L^2}{\pi}$ holds if and only if f is constant.

References

[1] Pao Ming Pu, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific J. Math. **2** (1952), 55–71.

[2] Mikhael Gromov, *Filling Riemannian manifolds*, J. Differ. Geom. **18** (1983), no. 1, 1–147.

[3] Mikhael Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Based on the 1981 French original with appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by S. M. Bates, *Progress in Mathematics*, 152, Birkhäuser Boston, Boston, MA, 1999.

[4] Marcel Berger, *Lectures on Geodesics in Riemannian Geometry*, Tata Institute of Fundamental Research Lectures on Mathematics, No. 33, Tata Institute of Fundamental Research, Bombay, 1965.

[5] Charles Horowitz, Karin Usadi Katz, and Mikhail Katz, *Loewner's torus inequality with isosystolic defect*, J. Geom. Anal. **19** (2009), no. 4, 796–808.

[6] Ivan Babenko, Florent Balacheff, and Guillaume Bulteau, *Systolic geometry and simplicial complexity for groups*, J. Reine Angew. Math. **757** (2019), 247–277.

[7] Florent Balacheff, *A local optimal diastolic inequality on the two-sphere*, J. Topol. Anal. **2** (2010), no. 1, 109–121.

[8] Viktor Bangert, Christopher Croke, Sergei Ivanov, and Mikhail Katz, *Boundary case of equality in optimal Loewner-type inequalities*, Trans. Am. Math. Soc. **359** (2007), no. 1, 1–17.

[9] Paul Creutz, *Rigidity of the Pu inequality and quadratic isoperimetric constants of normed spaces*, preprint (2020), https://arxiv.org/pdf/2004.01076.pdf.

[10] Chady El Mir and Zeina Yassine, *Conformal geometric inequalities on the Klein bottle*, Conform. Geom. Dyn. **19** (2015), 240–257.

[11] Larry Guth, *Systolic inequalities and minimal hypersurfaces*, Geom. Funct. Anal. **19** (2010), no. 6, 1688–1692.

[12] James Hebda, *The primitive length spectrum of 2-D tori and generalized Loewner inequalities*, Trans. Amer. Math. Soc. **372** (2019), no. 9, 6371–6401.

[13] Sergei Ivanov and Mikhail Katz, *Generalized degree and optimal Loewner-type inequalities*, Israel J. Math. **141** (2004), 221–233.
[14] Mikhail Katz, *Systolic Geometry and Topology. With an Appendix by Jake P. Solomon*, Mathematical Surveys and Monographs, vol. 137, American Mathematical Society, Providence, RI, 2007.
[15] Mikhail Katz and Stephane Sabourau, *Hyperelliptic surfaces are Loewner*, Proc. Am. Math. Soc. **134** (2006), no. 4, 1189–1195.
[16] Yevgeny Liokumovich, Alexander Nabutovsky, and Regina Rotman, *Lengths of three simple periodic geodesics on a Riemannian 2-sphere*, Math. Ann. **367** (2017), no. 1–2, 831–855.
[17] Stephane Sabourau, *Isosystolic genus three surfaces critical for slow metric variations*, Geom. Topol. **15** (2011), no. 3, 1477–1508.
[18] Balázs Csikós and Márton Horváth, *Harmonic manifolds and tubes*, J. Geom. Anal. **28** (2018), no. 4, 3458–3476.