The question of Collins on the word and conjugacy problems.

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The word and conjugacy problems in groups

The word and conjugacy problems (as formal languages)

Given $G = \langle S \rangle$, S is finite and symmetric (i.e. $S = S^{-1}$),

$$\mathrm{WP}(S) = \{w \in S^* \mid w =_G 1\} \subseteq S^*$$

$$\mathrm{CP}(S) = \{(u, v) \in S^* \times S^* \mid u \sim_{conj} v \text{ in } G\} \subseteq S^* \times S^*$$

The word and conjugacy problems

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Britton-Higman, 1973

A f.g. group G has a decidable word problem iff it embeds into a simple subgroup of a finitely presented group. (Later, Thompson showed that the simple subgroup can be requested to be finitely generated.)
Later, Sacerdote obtained a similar characterization for the decidability of the conjugacy problem.

Birget-Ol'shanskii-Rips-Sapir, 2001

The word problem of a f.g. group is decidable in NP time iff it embeds into a finitely presented group with polynomially bounded Dehn function.

(Un)decidability of the word and conjugacy problems

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There exists a finitely presented group with undecidable word problem.

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Miller III, 1971 and Collins, 1972

There exist groups with decidable word problem but undecidable conjugacy problem.

(Un)decidability of the word and conjugacy problems

The Novikov-Boone theorem, 1955, 1958

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Miller III, 1971 and Collins, 1972

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The example of Miller III was finitely generated. In contrast, the example of Collins was finitely presented.

Embedding theorems

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Olshanskii-Sapir, 2003

Every f.g. group with decidable CP embeds into a finitely presented group with decidable CP.

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*For the case of groups with torsions a negative answer was obtained by Macintyre. The main specifics of non torsion-free case is that if two elements of a group are conjugate, then they have the same order. For the torsion free f.g. groups we have the following.

Osin, 2000's

Every f.g. torsion-free group embeds into a f.g. group with exactly two conjugacy classes.

For the groups with decidable power problem, Olshanskii and Sapir obtained the following positive answer to the question of Collins.

Olshanskii-Sapir, 2004

Every countable group with decidable power problem is embeddable into a 2-generated finitely presented group with decidable conjugacy and power problems.

However, in general, the answer to Collins' question is negative:

D., 2017

There exists a 2-generated torsion-free group $\mathcal{G}$ with decidable word problem that does not embed into a group with decidable conjugacy problem. Moreover, the group $\mathcal{G}$ can be chosen to be solvable of solvability length 4 or be finitely presented.

However, in general, the answer to Collins' question is negative:

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There exists a 2-generated torsion-free group $\mathcal{G}$ with decidable word problem that does not embed into a group with decidable conjugacy problem. Moreover, the group $\mathcal{G}$ can be chosen to be solvable of solvability length 4 or be finitely presented.

In connection with Osin's theorem we obtain the following

Corollary

There exists a 2-generated group G with decidable word problem that does not embed into a recursively presented group with finitely many conjugacy classes.

Computable presentations for groups, Rabin, 1960, Mal'cev, 1961

The presentation $G = \langle x_1, x_2, \ldots \mid r_1, r_2, \ldots \rangle$ is called computable if for any word w from $\{x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}^*$, the lexicographically smallest word from the same alphabet that is equal to w in G can be computably found.

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Example

The presentation

$$G_1 = \langle x_1, x_2, \ldots \mid x_i^{-1} x_j^{-1} x_i x_j = 1, \forall i, j \in \mathbb{N} \rangle$$

is a computable presentation for the free abelian group of countable rank $\bigoplus \mathbb{Z}$.

Now let us fix a non-recursive set $\mathcal{N} \subset 2\mathbb{N}$ and consider the following presentation

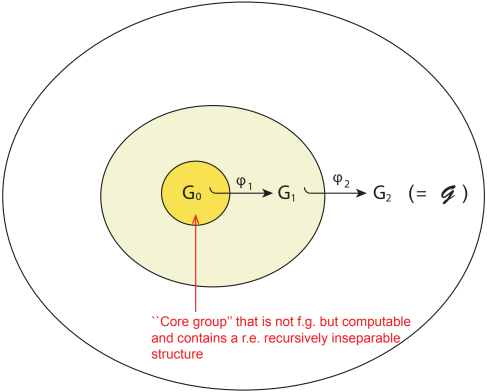
$$G_2 = \langle x_1, x_2, \ldots \mid [x_i, x_j] = 1, \forall i, j \in \mathbb{N} \ \& \ x_{2i} = x_{2i+1} \text{ iff } i \in \mathcal{N} \rangle.$$

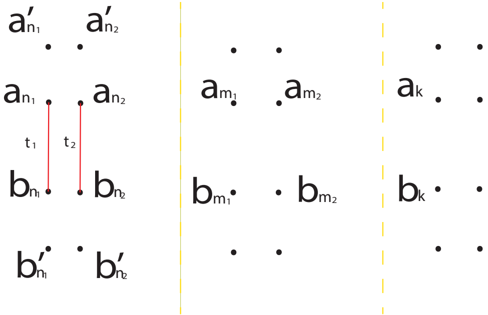
The last presentation is not computable but still it is a presentation of the group $\bigoplus \mathbb{Z}$.

Embedding theorem for groups, D., 2015

Let $G = \langle X \rangle$ be a group with countable generating set $X = \{x_1, x_2, \ldots\}$. Then there exists an embedding $\Phi_X : G \hookrightarrow K$ into a two-generated group $K = \langle f, c \rangle$ such that:

1. K has a recursive presentation if and only if G has a recursive presentation with respect to the generating set X;
2. K has decidable word problem if and only if G is computable with respect to the generating set X;
3. If $X = \{x_1, x_2, \ldots\}$ is recursively enumerated, then there exists a computable map $\phi_X : i \mapsto \{f^{\pm 1}, s^{\pm 1}\}^*$ such that ϕ_X represents the element $\Phi_X(x_i)$ in K;
4. There exists $N \lhd K$ such that $\Phi_X(G) \lhd N$, and K/N, $N/\Phi_X(G)$ are abelian groups;
5. The membership problem for the subgroup $\Phi_X(G) \leq K$ is decidable.


G_0
φ_1
G_1
φ_2
G_2 $(= \mathcal{G})$
``Core group'' that is not f.g. but computable and contains a r.e. recursively inseparable structure



{n_1, n_2, ...} and {m_1, m_2, ...}
are recursively enumerable and recursively inseparable subsets of N

Thank you!