Factorized groups and solubility

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Malta, March 2018

Factorized groups

A group G is called **factorized**, if

$$G = AB = \{ab \mid a \in A,\ b \in B\}$$

is the product of two subgroups A and B of G.

More generally, consider a group $G = A_1 \ldots A_n$ which is the product of finitely many pairwise permutable subgroups A_1, ..., A_n such that $A_iA_j = A_jA_i$ for all $i, j \in \{1, \ldots, n\}$.

Problem. What can be said about the structure of the factorized group G if the structures of its subgroups A_i are known?

Factorized subgroups

Let N be a normal subgroup of a factorized group $G = AB$. Then clearly the factor group G/N inherits the factorization

$$G/N = (AN/N)(BN/N).$$

Definition (a) A subgroup S of $G = AB$ is **factorized** if $S = (A \cap S)(B \cap S)$ and $A \cap B \subseteq S$.

(b) If U is a subgroup of $G = AB$, the $X(U)$ denotes the smallest factorized subgroup of $G = AB$ which contains U, $X(U)$ is called the **factorizer** of U in G.

Groups with a triple factorization

<u>**Lemma**</u>. Let N be a normal subgroup of $G = AB$. Then the factorizer of N has the form

$$X(N) = AN \cap BN = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN).$$

$\square$

Therefore the critical situation that has to be studied is the following **triply factorized group**

$G = AB = AM = BM$ with a normal subgroup M of G.

If in particular M is abelian, then $(A \cap M)(B \cap M)$ is a normal subgroup of G, which we may factor out to have in addition

$A \cap M = B \cap M = 1$

(in this case A and B are **complements of** M in G).

Construction of triply factorized groups

Let R be a radical ring. Then the adjoint group $A = R^\circ$ operates on the additive group $M = R^+$ via

$$x^a = x(1+a) = x + xa (a \in A, x \in M)$$

Form the **associated group**

$$G(R) = A \ltimes M = \{(a, x) \mid a \in A, x \in M\}$$

Take for A the set of all $(a, 0), a \in A$, for M the set of all $(0, x), x \in M$, and for B the diagonal group of all $(x, x), x \in R$. Then we have

$G(R) = AM = BM = AB$, where M is a normal subgroup of G such that $A \cap M = B \cap M = A \cap B = 1$.

Here A and B are isomorphic to R° and M is isomorphic to R^+.

Reference

By using various radical rings many more interesting examples can be constructed.

A general reference is the monograph

Products of groups Ref. [AFG]

by B.A., Silvana Franciosi, Francesco de Giovanni

Oxford Mathematical Monographs

Clarendon Press, Oxford (1992)

Triply factorizations with three abelian subgroups

Proposition. (see [AFG], Proposition 6.1.4)

Let the group G be triply factorized by two abelian subgroups A, B and an abelian normal subgroup M of G such that

$G = M \rtimes A = M \rtimes B = AB$ and $A \cap B = 1$.

Then there exists a radical ring R and an isomorphism α from $G(R)$ onto G such that $A(R)^\alpha = A$, $B(R)^\alpha = B$ and $M(R)^\alpha = M$.

Hyperabelian groups and finiteness conditions

A group G is **hyperabelian** if every nontrivial epimorphic image of G contains a nontrivial abelian normal subgroup.
Thus in particular, every soluble group is hyperabelian.

A group-theoretical property $\mathfrak{X}$ is called a **finiteness condition** if every finite group belongs to $\mathfrak{X}$.

The following group-theoretical properties are finiteness conditions:

- the class of groups with minimum condition,
- the class of groups with maximum condition,
- the class of minimax groups,
- the class of groups with finite Prüfer rank,
- the class of groups with finite torsionfree rank
- the class of groups with finite abelian section rank.

The main theorem

Several authors have contributed to the following

Main Theorem.

Let $G = AB$ be a hyperabelian group (in particular a soluble group). If the two subgroups A and B satisfy any of the above finiteness conditions $\mathfrak{X}$, then also G is an $\mathfrak{X}$-group.

All these results are proved by a reduction to a triply factorized group as explained above and then considering G as a $\mathbb{Z}A$-module. Thus also Representation Theory and Cohomology Theory may be applied.

Semidirect products of groups and derivations

Let a group A act on a group M, i.e. there is a homomorphism from A into the automorphism group $Aut(M)$ of M, and let $G = M \rtimes A$ be the semidirect product of M by A.

A mapping $\delta : A \to M$ is a **derivation** (or a 1-cocycle) from A into M if $(ab)^\delta = (a^\delta)^b b^\delta$ for all elements $a, b \in A$.

For instance, for each $m \in M$ the mapping $\delta : a \to [a, m] = a^{-1}m^{-1}am$ with $a \in A$ is a derivation from A into M, because $[ab, m] = [a, m]^b[b, m]$ for all $a, b \in A$. Such a derivation is called **inner**.

If A acts trivially on M, then every non-trivial homomorphism $\delta : A \to M$ is a non-inner derivation from A into M and conversely.

If N is an A-invariant subgroup of M, then the full preimage B of A in N (i.e. the set of all $a \in A$ such that $a^\delta \in N$) is a subgroup of A, because $1^\delta = 1$ and $(a^{-1})^\delta = a(a^\delta)^{-1}a^{-1}$.

On the other hand, the image A^δ of A in M under δ is not necessarily a subgroup of M. If for some subgroup N of M there exists a subgroup C of A such that N is the set of all c^δ with $c \in C$, then we will say that N is a **derivation image** of C.

The following result describes some properties of derivations in terms of the complements of M in the semidirect product $G = M \rtimes A$.

Triply factorized groups and derivations

Theorem.

Let A be a group acting on a group M and let $G = M \rtimes A$ be the semidirect product of M and A. If $\delta : A \to M$ is a derivation and $B = \{aa^{\delta} \mid a \in A\}$, then B is a complement to M, and the following holds:

1. The derivation δ is inner if and only if B is conjugate to A in G,
2. $ker\delta = A \cap B$ and in particular δ is injective if and only if $A \cap B = 1$,
3. The derivation δ is surjective if and only if $G = AB$. In other words, M is a derivation image of A if and only if $G = M \rtimes A = M \rtimes B = AB$.

Bijective Derivations and triply factorized groups

As a particular case of this theorem we have the following characterization of bijective derivations in terms of triply factorized groups.

Corollary.

A derivation δ from A to M is bijective if and only if in the semidirect product $G = M \rtimes A$ there exists a complement B of M in G such that

$G = M \rtimes A = M \rtimes B = AB$ and $A \cap B = 1$.

Braces

Definition. An additive abelian group V with a multiplication $VxV \to V$ is called a **(right) brace** if for all $u, v, w \in V$ the following holds

1. (u+v)w=uw+vw,
2. u(vw+v+w) = (uv)w + uv + uw,
3. the map $v \to uv + v$ is bijective.

Every radical ring R is a brace under the addition and multiplication in R. Every brace whose multiplication is either associative or two-sided distributive is a radical ring.

As in a radical ring, the set of all elements of any brace V forms a group with neutral element 0 under the adjoint multiplication $u \circ v = u + v + uv$, which is also called the **adjoint group** V° of V.

Braces and triply factorized groups

Theorem.

Let A be a group and V be an A-module. Then the following statements are equivalent:

(1) V is a brace whose adjoint group is isomorphic to A,

(2) there exists a bijective derivation $d : A \to M$ such that $u.v = ud^{-1}(v) - u$ for all $u, v \in V$,

(3) the integer group ring $\mathbb{Z}A$ contains a right ideal $\mathfrak{a}$ such that V is the brace determined by $\mathfrak{a}$,

(4) in the semidirect product $G = M \rtimes A$ there exists a subgroup B such that $G = M \rtimes A = M \rtimes B = AB$ and $A \cap B = 1$.

Some solubility criteria for factorized groups

When is a factorized group soluble or at least generalized soluble in some sense?

The most important criterion is the following

Theorem (N. Itô 1955). If the group $G = AB$ is the product of two abelian subgroups A and B, then G is metabelian.

Question. Let the group $G = AB$ be the product of two abelian-by-finite subgroups A and B, (i.e. A and B have abelian subgroups of finite index, perhaps even with index at most 2)

Does then G have a soluble (or even metabelian) subgroup of finite index?

Some previous results

This seemingly simple question has a positive answer for **linear goups** (Ya. Sysak 1986) and for **residually finite groups** (J. Wilson 1990).

Theorem (N.S. Chernikov 1981). If the group $G = AB$ is the product of two central-by-finite subgroups A and B, then G is soluble-by-finite.
(It is unknown whether G is metabelian-by-finite in this case.)

Theorem (O. Kegel 1961, H. Wielandt 1958, L. Kazarin 1981). Let the finite group $G = AB$ be the product of two subgroups A and B, which both have nilpotent subgroup of index at most 2. Then G is soluble.
(It is unknown whether this holds for infinite groups in general)

Generalized dihedral groups

A group is **dihedral** if it is generated by two involutions.

Definition. A group G is **generalized dihedral** if it is **of dihedral type**, i.e. G contains an abelian subgroup X of index at most 2 and an involution τ which inverts every element in X.

Then $A = X \rtimes < a >$ is the semi-direct product of an abelian subgroup X and an involution a, so that $x^a = x^{-1}$ for each $x \in X$.

Clearly every (finite or infinite) dihedral group is also generalized dihedral. A periodic generalized dihedral group is locally finite and every finite subgroup is contained in a finite dihedral subgroup.

Products of generalized dihedral subgroups

The following solubility criterion widely generates Itô's theorem.

Theorem 1. (B.A., Ya. Sysak, J. Group Theory 16 (2013), 299-318).

(a) Let the group $G = AB$ be the product of two subgroups A and B, each of which is either abelian or generalized dihedral. Then G is soluble.

(b) If, in addition, one of the two subgroups, B say, is abelian, then the derived length of G does not exceed 5.

Products of two (locally cyclic)-by-(index at most 2) subgroups

A group A is **locally cyclic**, if every finitely generated subgroup is cyclic.

Corollary.

Let the group $G = AB$ be the product of two subgroups A and B.

(a) If both A and B contain torsionfree locally cyclic subgroups of index at most 2, then G is soluble and metabelian-by-finite.

(b) If A and B are cyclic-by-(index at most 2), then G is metacyclic-by-finite.

Some special cases of Theorem 1 that were proved previously

Let the group $G = AB$ be the product of two generalized dihedral subgroups A and B.

1. The second case of the corollary was first proved in B.A., Ya. Sysak, Arch. Math. 90 (2008), 101-111.
2. The special case of the theorem when A and B are periodic generalized dihedral was already treated in B.A., A. Fransman, L. Kazarin, J. Alg. 350 (2012), 308-317.
3. If A and B are Chernikov groups and (abelian)-by-(index at most 2), and one of the two is generalized dihedral, then G is a soluble Chernikov group. This was shown in B.A., L. Kazarin, Israel J. Math. 175 (2010), 363-389.

Remarks on the proof of Theorem 1

The proof of Theorem 1 is elementary and almost only uses computations with involutions. Extensive use is made by the fact that every two involutions of a group generate a dihedral subgroup.

A main idea of the proof is to show that

the normalizer in G of a non-trivial normal subgroup of one of the factors A or B has a non-trivial intersection with the other factor.

If this is not the case we may find commuting involutions in A and B and produce a nontrivial abelian normal subgroup by other computations.

We may assume that $|A \cap B| \leq 2$.

Properties of generalized dihedral subgroups

Lemma. Let A be generalized dihedral. Then the following holds

1) every subgroup of X is normal in A;
2) if A is non-abelian, then every non-abelian normal subgroup of A contains the derived subgroup A' of A;
3) $A' = X^2$ and so the commutator factor group A/A' is an elementary abelian 2-group;
4) the center of A coincides with the set of all involutions of X;
5) the coset aX coincides with the set of all non-central involutions of A;
6) two involutions a and b in A are conjugate if and only if $ab^{-1} \in X^2$;
7) if A is non-abelian, then X is characteristic in A.

Products of pairwise permutable abelian subgroups

Consider groups of the form $G = A_1 \dots A_n$ with pairwise commuting abelian subgroups A_i

Theorem (H. Heineken and J. Lennox 1983).

(a) A product of finitely many pairwise permutable finitely generated abelian groups is polycyclic.

(b) A product of finitely many pairwise permutable cyclic subgroups is supersoluble.

Theorem (M. Tomkinson 1986).

A product of finitely many pairwise permutable abelian minimax subgroups is a soluble minimax group.

(Here a **minimax group** is a group with a finite series such that its factors satisfy the minimum or the maximum condition).

Products of pairwise permutable abelian subgroups of finite Prüfer rank

A group G has **finite Prüfer rank** r if every finitely generated subgroup of G can be gererated by r elements, and r is minimal with this property.

Theorem 2. (B. A. and Ya. Sysak, Advances in Group Theory and Applications 2 (2016), 13-24).

Let the group $G = A_1 A_2 .... A_n$ be the product of finitely many pairwise permutable abelian subgroups $A_1, A_2, ..., A_n$, each of which has finite Prüfer rank. Then G is hyperabelian with finite Prüfer rank.

Remarks on the proof of Theorem 2

Every hyperabelian product of two subgroups with finite Prüfer rank has likewise finite Prüfer rank by the "Main Theorem" above; see ([AFG], 4.6.11 and 4.6.12).

To show that G is hyperabelian it suffices to show that the group $G \neq 1$ contains a nontrivial abelian normal subgroup.

If n=2, then G is metabelian by Ito's theorem and if $G \neq 1$, then there exists an abelian normal subgroup $N \neq 1$ of G contained in A_1 or A_2 by a theorem of D.I. Zaicev (see [AFG], Theorem 7.1.2).

If $n \geq 3$, and assume that $A_1A_2 \neq 1$. Then there exists a normal subgroup $N \neq 1$ of A_1A_2 contained in A_2 without loss by Zaicev.

Hence, by induction on n, the normal subgroup N^G of G is hyperabelian with finite Prüfer rank. This implies that there exists a normal subgroup $M \neq 1$ of N^G which is a finite abelian p-group of torsionfree abelian.

Products of pairwise permutable abelian subgroups of finite abelian section rank

A group G has **finite abelian section rank** if it has no infinite elementary abelian p-section for any prime p.

Theorem.

Let the group $G = A_1 A_2 .... A_n$ be the product of finitely many pairwise permutable abelian subgroups $A_1, A_2 ... A_n$, each of which has finite abelian section rank. Then G is hyperabelian with finite abelian section rank.

Remark. Use that every hyperabelian product of two subgroups with finite abelian section rank has likewise finite abelian section rank by the "Main Theorem" above (see [AFG], 4.6.11).

Periodic products $G = ABC$ of three pairwise permutable locally cyclic groups A, B, C

A group A is **locally cyclic**, if every finitely generated subgroup is cyclic, i.e. if only if, A is abelian group of rank 1.

By the above theorem **every group $G = ABC$ with pairwise permutable locally cyclic subgroups A, B and C is hyperabelian**.

On the other hand, there exist **periodic products $G = ABC$ of three pairwise permutable locally cyclic subgroups A, B, C which are not soluble** (see [AFG], Proposition 7.6.3).

Question. Is every periodic product of pairwise permutable locally cyclic subgroups A, B and C locally supersoluble?

An example

Groups of the form $G = ABC$ with pairwise permutable abelian subgroups A, B and C may be very complicated and for instance contain free nonabelian subgroups of infinite rank.

Let $\mathbb{Q}_2$ be the ring of rational numbers with odd denominator and $2\mathbb{Q}_2$ its ideal with even numerators. If U is the multiplicative group of $\mathbb{Q}_2$, then

$$U = \langle -1 \rangle \times \prod_p \langle \frac{1}{p} \rangle$$

with free abelian subgroup $\prod_p \langle \frac{1}{p} \rangle$ of infinite Prüfer rank. Consider the ring R of all (2×2)-matrices over $\mathbb{Q}_2$ and put

$$G = \begin{pmatrix} 1 + 2\mathbb{Q}_2 & 2\mathbb{Q}_2 \\ 2\mathbb{Q}_2 & 1 + 2\mathbb{Q}_2 \end{pmatrix}.$$

Then G is a group under matrix multiplication. The Jacobson radical J of R consists of all (2×2)-matrices over $2\mathbb{Q}_2$ and G is isomorphic to the adjoint group of J.

Put

$$A = \left\{ \begin{pmatrix} 1+a & 0 \\ 0 & 1+b \end{pmatrix} \mid a, b \in 2\mathbb{Q}_2 \right\}$$

$$B = \left\{ \begin{pmatrix} 1+c & c-d \\ 0 & 1+d \end{pmatrix} \mid c, d \in 2\mathbb{Q}_2 \right\},$$

$$C = \left\{ \begin{pmatrix} 1+e & 0 \\ -e+f & 1+f \end{pmatrix} \mid e, f \in 2\mathbb{Q}_2 \right\}.$$

Then A, B, C are abelian subgroups of G, each of which is isomorphic to $U \times U$. We also have that $B = x^{-1}Ax$ and $C = y^{-1}Ay$ with $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

It can easily be verified that $G = ABC$ and the subgroups A, B and C are pairwise permutable. The matrices $g = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ are contained in G, and it is well-known that the subgroup $\langle g, h \rangle$ is free non-abelian. In particular, G is non-soluble.