Finiteness conditions on the injective hull of simple modules.

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jointly with

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Injective Hulls

Definition (Injective hull)

The injective hull $E(M)$ of a (left R)-module M is an injective module such that M embeds as an essential submodule in it, i.e. $M \cap U \neq 0$ for all $0 \neq U \subseteq E(M)$.

Theorem (Matlis, 1960)

The injective hull of a simple module over a commutative Noetherian ring is Artinian.

Question

What can be said if either "commutative" or "Noetherian" is dropped?

Non-commutativity

Definition (Jans, 1968)

A ring is co-Noetherian if the injective hull of any simple module is Artinian.

Proposition (Hirano, 2000)

The 1st Weyl algebra $A_1(\mathbb{Z})$ is co-Noetherian, but $A_1(\mathbb{Q})$ is not.

For any infinite set $\{x - a_1, x - a_2, \ldots\}$ in $\mathbb{Q}[x]$, the localisations

$$\mathbb{Q}[x]_{S_1} \supset \mathbb{Q}[x]_{S_2} \supset \cdots \supset \mathbb{Q}[x]_{S_n} \supset \cdots \supset \mathbb{Q}[x]$$

form a descending chain of $A_1(\mathbb{Q})$-modules, where S_n is the multiplicatively closed set generated by $x - a_i$, for $i \geq n$.
$\Rightarrow$ $E(\mathbb{Q}[x])$ is not an Artinian $A_1(\mathbb{Q})$-module.

Locally Artinian

Definition

$(\diamond)$ any injective hull of a simple R-module is locally Artinian.

Definition

A left ideal I of R is subdirectly irreducible (SDI) if R/I has an essential simple socle.

R satisfies $(\diamond)$ if and only if R/I is Artinian for all left SDI's.

Proposition (Krull intersection)

Suppose finitely generated Artinian left R-modules are Noetherian. If R satisfies $(\diamond)$ then $\bigcap(I + \mathrm{Jac}(R)^n) = I$ for any left ideal I.

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Krull dimension 1

Any semiprime Noetherian ring of Krull dimension ≤ 1 satisfies $(\diamond)$.

Example (Goodearl-Schofield, 1986)

$\exists$ Noetherian ring with Krull dimension 1 not satisfying $(\diamond)$.

Relies on a skew field extension $F \subseteq E$ with E finite dimensional over F on the right, but transcendental on the left. Then $\begin{pmatrix} E[t] & E[t] \\ 0 & F[t] \end{pmatrix}$ does not satisfy $(\diamond)$, but is Noetherian and has Krull dimension 1.

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FBN

Theorem (Jategaonkar, 1974)

Any fully bounded Noetherian ring satisfies $(\diamond)$. In particular any Noetherian semiprime PI-ring satisfies $(\diamond)$.

Theorem (Carvalho, Musson, 2011)

The q-plane $R = K_q[x,y] = K\langle x,y\rangle / \langle xy - qyx\rangle$ satisfies $(\diamond)$ if and only if q is a root of unity.

If q is not a root of unity, then

$$0 \to R/R(xy-1) \to R/R(xy-1)(x-1) \to R/R(x-1) \to 0$$

is an essential embedding of a simple into a non-Artinian module.

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Weyl algebras

$A_1(K)$ satisfies $(\diamond)$ since it is a Noetherian domain of Kdim 1

Theorem (Stafford, 1985)

Let $n > 1$ and $\lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\alpha = x_1 + \left(\sum_{i=2}^{n} \lambda_i y_i x_i \right) y_1 + \sum_{i=2}^{n} (x_i + y_i) \in A_n = A_n(\mathbb{C})$$

generates a maximal left ideal of A_n and

$$0 \to A_n/A_n\alpha \longrightarrow A_n/A_n\alpha x_1 \longrightarrow A_n/A_n x_1 \to 0$$

is an essential embedding with $Kdim(A_n/A_n x_1) = n - 1$.

Exploiting Stafford's theorem

Example

Let $\mathfrak{h}_n = \mathrm{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ with $[x_i, y_i] = z$. Then $U(\mathfrak{h}_n)$ satisfies $(\diamond)$ if and only if $n = 1$ as $U(\mathfrak{h}_n)/\langle z - 1\rangle \simeq A_n$.

Theorem (Hatipoglu-L. 2012)

Let $\mathfrak{g}$ be a finite dimensional nilpotent complex Lie (super)algebra. Then $U(\mathfrak{g})$ satisfies $(\diamond)$ if and only if

1. *g has an Abelian ideal of codimension 1 or*
2. *$\mathfrak{g} \simeq \mathfrak{h} \times \mathfrak{a}$ with $\mathfrak{a}$ Abelian and $\mathfrak{h} = \mathrm{span}(e_1, \ldots, e_m)$ with either*
   (i) *$m = 5$ and $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$ or*
   (ii) *$m = 6$ and $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, $[e_1, e_2] = e_6$.*

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Ore extensions

Theorem (Carvalho,Hatipoglu, L. 2015)

Let σ be an automorphism of K and d a σ-derivation. Then $K[x][y;\sigma,d]$ satisfies $(\diamond)$ if and only if

(i) *$\sigma = id$ and d is locally nilpotent or*

(ii) *$\sigma \neq id$ has finite order.*

Theorem (Vinciguerra, 2017)

Let $R = \mathbb{C}[x,y]$ and d a non-zero derivation of it. Then $S = R[\theta,d]$ satisfies $(\diamond)$ if and only if

(i) *every maximal ideal of R contains an Darboux element*

(ii) *$d(R) \subseteq Rp$, for any Darboux element p contained in a d-stable maximal ideal.*

An element is Darboux if it generates a d-stable ideal.

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Skew-polynomial rings

Theorem (Brown,Carvalho,Matczuk 2017)

Let K be an uncountable field and R a commutative affine K-algebra, and let α be a K-algebra automorphism of R. Then $S = R[\theta;\alpha]$ satisfies $(\diamond)$ if and only if all simple S-modules are finite dimensional over K.

Many more interesting results and open question can be found in the paper "Simple modules and their essential extensions for skew polynomial rings" by Brown, Carvalho and Matczuk (arXiv:1705.06596).

Commutative, but not Noetherian?

From now on R will be commutative.

Theorem (Vamos, 1968)

The following statements are equivalent for a commutative ring R.

(a) *The injective hull of a simple module is Artinian.*

(b) *The localisation of R by a maximal ideal is Noetherian.*

Theorem

The following statements are equivalent for a commutative ring R.

(a) *R satisfies $(\diamond)$*

(b) *$R_{\mathfrak{m}}$ satisfies $(\diamond)$ for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$.*

$E(R/\mathfrak{m})$ is an injective hull of $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-module.

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Comparison

Theorem

For a local ring $(R, \mathfrak{m})$ the following are equivalent:

(a) R *is (co-)Noetherian.*

(b) R *satisfies* $(\diamond)$ *and* $\mathfrak{m}/\mathfrak{m}^2$ *is finitely generated.*

(c) $\bigcap(I + \mathfrak{m}^n) = I$ *for any ideal* I *and* $\mathfrak{m}/\mathfrak{m}^2$ *is finitely generated.*

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Local rings with nilpotent radical

Proposition

If R has nilpotent radical, then the following are equivalent

(a) *R satisfies $(\diamond)$*

(b) *For any module M: $\mathrm{Soc}(M)$ f.g implies $\mathrm{Soc}(M/\mathrm{Soc}(M))$ f.g.*

Example

Any local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^2 = 0$ satisfies $(\diamond)$, because if I is SDI, then $\mathfrak{m}/I \cap \mathfrak{m}$ has dimension ≤ 1 as vector space over $R/\mathfrak{m}$, i.e. R/I has length at most 2.

For example the trivial extension

$$R = \left\{ \begin{pmatrix} a & v \\ 0 & a \end{pmatrix} \mid a \in K, v \in V \right\}.$$

Radical cube zero

Theorem (Local rings with radical cube zero)

Let $(R, \mathfrak{m})$ be local with $\mathfrak{m}^3 = 0$. Then there exists a bijective correspondence between SDI's I not containing $\mathrm{Soc}(R)$ and non-zero $f \in \mathrm{Hom}(\mathrm{Soc}(R), F)$. Corresponding pairs (I, f) satisfy:

$$\mathrm{Soc}(R) + I = V_f := \{a \in \mathfrak{m} \mid f(\mathfrak{m}a) = 0\}.$$

Then R satisfies $(\diamond)$ iff $\dim(\mathfrak{m}/V_f) < \infty$ for all $f \in \mathrm{Soc}(R)^*$.

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Theorem

Let $(R, \mathfrak{m})$ be a local ring with residue field F and $\mathfrak{m}^3 = 0$. Then

$$R \text{ satisfies } (\diamond) \text{ if and only if } \mathrm{gr}(R) = F \oplus (\mathfrak{m}/\mathfrak{m}^2) \oplus \mathfrak{m}^2 \text{ does.}$$

Definition

For a field F, vector spaces V and W and a symmetric bilinear form $\beta : V \times V \to W$ we can consider the generalised matrix ring

$$\left\{ \begin{pmatrix} a & v & w \\ 0 & a & v \\ 0 & 0 & a \end{pmatrix} \mid a \in F, v \in V, w \in W \right\}$$

which we identify by $S = F \times V \times W$ with multiplication

$$(a_1, v_1, w_1)(a_2, v_2, w_2) = (a_1 a_2, a_1 v_2 + v_1 a_2, a_1 w_2 + \beta(v_1, v_2) + w_1 a_2).$$

Then $\mathrm{Soc}(S) = 0 \times V_\beta^\perp \times W$ where

$$V_\beta^\perp = \{v \in V \mid \beta(V, v) = 0\}.$$

Clearly $\mathfrak{m} = 0 \times V \times W$ and $\mathfrak{m}^2 = 0 \times 0 \times \mathrm{Im}(\beta)$.

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Examples

Examples

Let $F = \mathbb{R}$ and $V = C([0,1])$, space of continuous real valued functions on $[0,1]$. Define $\beta : V \times V \to \mathbb{R}$ by

$$\beta(f,g) = \int_0^1 f(x)g(x)dx,$$

then $S = \mathbb{R} \times V \times \mathbb{R}$ has an 1-dimensional **essential** socle, but S is not Artinian, i.e. S does not satisfy $(\diamond)$.

Example

Example

Let F be any field and V be any vector space with basis $\{v_i : i \geq 0\}$. Define

$$\beta(v_i, v_j) = \begin{cases} 1 & (i,j) = (0,0) \\ 0 & \text{else} \end{cases}$$

Then $S = F \times V \times F$ satisfies $(\diamond)$, because

$$\mathfrak{m}/\mathrm{Soc}(S) = (0 \times V \times F)/(0 \times V_\beta^\perp \times F) \simeq V/V_\beta^\perp \simeq F$$

Note that $S = \mathrm{gr}\left(F[x_0, x_1, x_2 \ldots]/\langle x_0^3, x_i x_j : (i,j) \neq (0,0)\rangle\right)$.

Here: β not non-degenerated $\Rightarrow$ pass to $F \times V/V_\beta^\perp \times F$.

Local rings with radical cube zero

Theorem

Let $(R, \mathfrak{m})$ be a local ring with residue field F and $\mathfrak{m}^3 = 0$. Then the following are equivalent:

(a) *R **does not** satisfies $(\diamond)$*

(b) *R has a factor R/I such that $\mathrm{gr}(R/I)$ has the form $F \times V \times F$ for a non-degenerated form $\beta : V \times V \to F$ and $\dim(V) = \infty$.*

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Constructions coming from Algebras

Let A be an F-algebra. Then $S = F \times A \times A$ becomes a ring using the multiplication μ as bilinear form. Since μ is non-degenerated, $\mathrm{Soc}(S) = 0 \times 0 \times A$. Hence $\mathrm{Soc}(S)^* = A^*$. For any $f \in A^*$:

$V_f = \{a \in A : f(Aa) = 0\}$ is the largest ideal contained in $\ker(f)$.

Hence A/V_f is finite dimensional if and only if $f \in A^0$.

Proposition

$S = F \times A \times A$ satisfies $(\diamond)$ if and only if for any $A^* = A^0$.

Example: $A = F \times V$ the trivial extension satisfies $A^* = A^0$.

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Example

Let $\mathrm{char}(F) = 0$ and $A = F[x]$. Set $f(x^n) = \frac{1}{n+1}$ for any $n \geq 0$. Then the only ideal contained in $\ker(f)$ is the zero ideal, i.e.

$$V_f = \{0\}.$$

Therefore, $\beta = f \circ \mu : F[x] \times F[x] \to F$ is a non-degenerated symmetric bilinear form and $S = F \times F[x] \times F$ does not satisfy $(\diamond)$.

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Thank you for your attention!