Abelian, nilpotent and solvable quandles

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jointly with
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whose students are on strike today starting 12:00

Malta, March 2018

Goal

... tell you about the Gumm-Smith commutator theory

... describe abelian / solvable / nilpotent quandles

... Corollary: topologically slice knots cannot be colored by latin quandles

Abelian groups and modules

abelian $\neq$ commutative

Observation: Abelian groups $= \mathbb{Z}$-modules

... and the only groups that can be considered as modules are abelian groups

$\Rightarrow$ Idea: Jonathan D. H. Smith (1970s): abelian = "module-like"

In what sense, "module-like" ?

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$\Rightarrow$ Idea: Jonathan D. H. Smith (1970s): abelian = "module-like"

In what sense, "module-like" ?

... module up to a selection of operations ... *too strong*

... embeds a module ... *no good abstract description*

... the term condition

Abelian algebras

algebra = a set + a collection of basic operations
term operation = composition of basic operations
polynomial operation = composition of basic operations and constants

An algebra A is called abelian if

$$t(a, u_1, \dots, u_n) = t(a, v_1, \dots, v_n) \;\Rightarrow\; t(b, u_1, \dots, u_n) = t(b, v_1, \dots, v_n)$$

for every term operation $t(x, y_1, \dots, y_n)$ and every a, b, u_i, v_i in A.
Equivalently, if the diagonal is a congruence block on A^2.

Observation

Modules are abelian.

Proof: $t(x, y_1, \dots, y_n) = rx + \sum r_i y_i$, cancel ra, add rb.

Abelian groups, quandles

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for every term operation $t(x, y_1, \dots, y_n)$ and every $a, b,\ u_i, v_i$.

Observation

An abelian monoid is commutative and cancellative.

Proof: $t(x, y, z) = yxz, \quad a11 = 11a \;\Rightarrow\; ab1 = 1ba$
$t(x, y) = xy, \quad ab = ac \;\Rightarrow\; 1b = 1c$

Observation

An abelian quandle is medial.

Proof: $t(x, y, u, v) = (xy)(uv)$,
$(yy)(uv) = (yu)(yv) \;\Rightarrow\; (xy)(uv) = (xu)(yv)$

Abelian algebras = modules, sometimes

A is a *polynomial reduct* of B = basic operations of A are polynomial operations of B

A, B are *polynomially equivalent* = both ways

Observation

Polynomial reducts of modules are abelian.

Mal'tsev operation: $m(x, y, y) = m(y, y, x) = x$

Theorem (Gumm-Smith 1970s)

TFAE for algebras with a Mal'tsev polynomial operation:

1. *abelian*
2. *polynomially equivalent to a module*

Examples: groups, loops, quasigroups

Non-examples: quandles, monoids, semigroups

Abelian algebras = submodules, usually

polynomial subreduct = subalgebra of a reduct

Observation

Polynomial subreducts of modules are abelian.

The converse implication is

- false in general [Quackenbush 1980s]
- but known counterexamples are rare and unnatural
- true for algebras in a variety with no "algebraically trivial" algebras (e.g. when operations are essentially unary) [Kearnes, Szendrei 1990s]
- true for finite simple algebras [Hobby, McKenzie 1980s]
- true for quandles [JPSZ]

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- true for quandles [JPSZ]

Remember:

abelian = abstract term condition = (almost always) submodule

Coming next: abelianness for congruences

Solvability and nilpotence

A group G is solvable, resp. nilpotent, if there are $N_i \trianglelefteq G$ such that

$$1 = N_0 \leq N_1 \leq \ldots \leq N_k = G$$

and N_{i+1}/N_i is an abelian, resp. central subgroup of G/N_i, for all i.

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An arbitrary algebraic structure A is solvable, resp. nilpotent, if there are congruences α_i such that

$$0_A = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k = 1_A$$

and α_{i+1}/α_i is an abelian, resp. central congruence of A/α_i, for all i.

Need a good notion of *abelianness* and *centrality* for congruences.

Solvability and nilpotence, via commutator

$$G^{(0)} = G_{(0)} = G, \qquad G_{(i+1)} = [G_{(i)}, G_{(i)}], \qquad G^{(i+1)} = [G^{(i)}, G]$$

A group G is

- solvable iff $G_{(n)} = 1$ for some n
- nilpotent iff $G^{(n)} = 1$ for some n

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$$\alpha^{(0)} = \alpha_{(0)} = 1_A, \qquad \alpha_{(i+1)} = [\alpha_{(i)}, \alpha_{(i)}], \qquad \alpha^{(i+1)} = [\alpha^{(i)}, 1_A]$$

An arbitrary algebraic structure A is

- solvable iff $\alpha_{(n)} = 0_A$ for some n
- nilpotent iff $\alpha^{(n)} = 0_A$ for some n

Need a good notion of *commutator of congruences*.

Commutator theory

[mid 1970s by Smith, Gumm, Herrmann, ..., the Freese-McKenzie 1987 book]

Centralizing relation for congruences α, β, δ of A:

$C(\alpha, \beta; \delta)$ iff for every term $t(x, y_1, \dots, y_n)$ and every $a \overset{\alpha}{\equiv} b$, $u_i \overset{\beta}{\equiv} v_i$

$$t(a, u_1, \dots, u_n) \overset{\delta}{\equiv} t(a, v_1, \dots, v_n) \;\Rightarrow\; t(b, u_1, \dots, u_n) \overset{\delta}{\equiv} t(b, v_1, \dots, v_n)$$

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The *commutator* $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$.

A congruence α is called

- abelian if $C(\alpha, \alpha; 0_A)$, i.e., if $[\alpha, \alpha] = 0_A$.
- central if $C(\alpha, 1_A; 0_A)$, i.e., if $[\alpha, 1_A] = 0_A$.

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Fact (not difficult, certainly not obvious)

In groups, this gives the usual commutator, abelianness, centrality.

Deep theory: works well in varieties with modular congruence lattices.

Quandles

An algebraic structure $(Q, *, \backslash)$ is called a *quandle* if

- $x * x = x$
- all left translations $L_x(y) = x * y$ are automorphisms, with $L_x^{-1}(y) = x \backslash y$.

Multiplication group, displacement group:

$$\mathrm{LMlt}(Q) = \langle L_x : x \in Q \rangle \leq \mathrm{Aut}(Q)$$
$$\mathrm{Dis}(Q) = \langle L_x L_y^{-1} : x, y \in Q \rangle \leq \mathrm{LMlt}(Q)$$

A quandle is called *connected* if $\mathrm{LMlt}(Q)$ is transitive on Q.

Affine quandles (aka Alexander) $Aff(A, f)$:
$x * y = (1 - f)(x) + f(y)$ on an abelian group A, $f \in Aut(A)$

... i.e., a reduct of a $\mathbb{Z}[t, t^{-1}]$-module $(A, +, f)$

Big picture

quandle		$\mathrm{Dis}(Q)$
affine	$\Leftrightarrow$	abelian, semiregular, "balanced"
$\Downarrow$		$\Downarrow$
abelian	$\Leftrightarrow$	abelian, semiregular
$\Downarrow$		$\Downarrow$
nilpotent	$\Rightarrow$	nilpotent
	$\Leftarrow$ if Mal'tsev	
$\Downarrow$		$\Downarrow$
solvable	$\Rightarrow$	solvable
	$\Leftarrow$ if Mal'tsev	

[JPSZ, BonS]

Fact

A quandle has a Mal'tsev operation iff all subquandles are connected.

Quandles and abelianness

Theorem (JPSZ)

TFAE for a quandle Q:

1. *abelian*
2. *subquandle of an affine quandle*
3. $\mathrm{Dis}(Q)$ *abelian, semiregular*
4. $Q \simeq Ext(A, f, \bar{d})$, *a certain kind of extension of $Aff(A, f)$*

Theorem (JPSZ)

TFAE for a quandle Q:

1. *abelian and "balanced orbits"*
2. *affine*
3. $\mathrm{Dis}(Q)$ *abelian, semiregular and "balanced occurences of generators"*
4. $Q \simeq Ext(A, f, \bar{d})$ *and $\bar{d}$ is a multi-transversal of $A/\mathrm{Im}(1 - f)$*

Congruences of quandles

Let $N(Q) = \{N \leq \mathrm{Dis}(Q) : N \text{ is normal in } \mathrm{LMlt}(Q)\}$

There is a Galois correspondence

$$Con(Q) \longleftrightarrow N(Q)$$

$$\alpha \to \mathrm{Dis}_\alpha(Q) = \langle L_x L_y^{-1} : x \,\alpha\, y \rangle$$

$$\alpha_N = \{(x, y) : L_x L_y^{-1} \in N\} \leftarrow N$$

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Proposition

TFAE for $\alpha, \beta \in Con(Q)$, Q a quandle:

1. *α centralizes β over 0_Q, i.e., $C(\alpha,\beta;0_Q)$*
2. *$\mathrm{Dis}_\beta(Q)$ centralizes $\mathrm{Dis}_\alpha(Q)$ and acts α-semiregularly on Q*

α-semiregularly means $g(a) = a \;\Rightarrow\; g(b) = b$ for every $b \overset{\alpha}{\equiv} a$

Abelian congruences and solvable quandles

Theorem

TFAE for a congruence α of a quandle Q:

1. α *is abelian*
2. $\mathrm{Dis}_\alpha(Q)$ *is abelian and acts α-semiregularly*
3. Q *is an abelian extension of* $F = Q/\alpha$, *i.e.,* $(F \times A, *)$ *with*
$$(x, a) * (y, b) = (xy,\ \varphi_{x,y}(a) + \psi_{x,y}(b) + \theta_{x,y})$$
where A *is an abelian group,* $\varphi : Q^2 \to End(A)$, $\psi : Q^2 \to Aut(A)$, $\theta : Q^2 \to A$ *satisfying the cocycle condition.*

The last item only assuming that α has connected blocks.

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The last item only assuming that α has connected blocks.

Corollary

- Q *solvable (of rank n)* $\Rightarrow$ $\mathrm{Dis}(Q)$ *solvable (of rank $\leq 2n-1$)*
- $\mathrm{Dis}(Q)$ *solvable, Q has Mal'tsev operation* $\Rightarrow$ Q *solvable*

Central congruences and nilpotent quandles

Theorem

TFAE for a congruence α of a quandle Q:

1. α *is central*
2. $\mathrm{Dis}_\alpha(Q)$ *is central and* $\mathrm{Dis}(Q)$ *acts α-semiregularly*
3. Q *is a central extension of $F = Q/A$, i.e., $(F \times A, *)$ with*
   $$(x, a) * (y, b) = (xy,\ (1 - f)(a) + f(b) + \theta_{x,y})$$
   where A is an abelian group, $\theta : Q^2 \to A$ satisfying the cocycle condition.

The last item only assuming that Q has Mal'tsev operation.

Corollary

- Q *nilpotent (of rank n)* $\Rightarrow$ $\mathrm{Dis}(Q)$ *nilpotent (of rank $\leq 2n - 1$)*
- $\mathrm{Dis}(Q)$ *nilpotent, Q has Mal'tsev operation* $\Rightarrow$ Q *nilpotent*

Extensions by constant cocycles (aka coverings)

Theorem

TFAE for a congruence α of a quandle Q:

1. α *is strongly abelian*
2. $\mathrm{Dis}_\alpha(Q) = 1$
3. Q *is an extension by constant cocycle of* $F = Q/\alpha$*, i.e.,* $(F \times A, *)$ *with*
$$(x,a) * (y,b) = (xy,\ \rho_{x,y}(b))$$
where A *is a set,* $\rho : Q^2 \to Sym(A)$ *satisfying the cocycle condition.*

... coverings are a special case of our abelian extensions ($\varphi_{x,y} = 0$)

... coverings have a natural universal algebraic meaning (*strongly abelian congruences*)

An application to quandles

Classification of connected quandles of order p^3 [Bianco, Bonatto]

Classification of latin quandles of order pq [Bonatto]

An application to quandles

Classification of connected quandles of order p^3 [Bianco, Bonatto]

Classification of latin quandles of order pq [Bonatto]

Theorem (Stein 2001)

If Q is a finite latin quandle, then $\mathrm{LMlt}(Q)$ is solvable.

Since latin quandles have Mal'tsev operation, we obtain

Corollary

Finite latin quandles are solvable.

An application to knot theory

Coloring by affine quandles $\leftrightsquigarrow$ *Alexander invariant*

Theorem (Bae, 2011)

Let K be a link and f its Alexander polynomial.

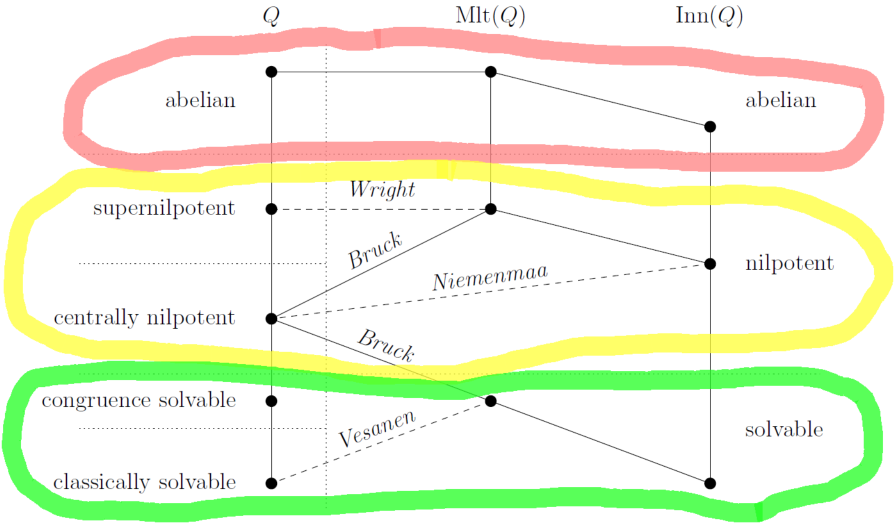
- $f = 0 \Rightarrow$ *colorable by every affine quandle*
- $f = 1 \Rightarrow$ *not colorable by any affine quandle*
- *else, colorable by* $\mathrm{Aff}(\mathbb{Z}[t, t^{-1}]/(f), f)$.

Corollary

- $f = 1 \Rightarrow$ *not colorable by any solvable quandle (in particular, latin)*

Solvability and nilpotence for loops

[S., Vojěchovský]



Solvability and nilpotence for ???

What about other interesting classes of algebras,

in particular other types of solutions to the Yang-Baxter equation?