Groups, Rings, Braces and Set-theoretic Solutions of the Yang-Baxter Equation

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Abstract

set-theoretical solution: (X, r), X a set and $r : X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r).$$

Write:

$$r(x, y) = (\sigma_x(y), \gamma_y(x)),$$

where σ_x and γ_y are maps $X \to X$.

Drinfeld in 1992: study the set-theoretical solutions of the Yang-Baxter equation.

Motivation

Gateva-Ivanova and Van den Bergh (1998): a class of Noetherian finitely presented algebras

$$R = K\langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \rangle$$

with $\binom{n}{2}$ quadratic homogeneous relations and each word $x_i x_j$ appears at most in one of the defining relations. Hence, one has a bijective map (with $X = \{x_1, \ldots, x_n\}$),

$$r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x)).$$

Theorem, Gateva-Ivanova - Van den Bergh 1998

Let $S = \langle x_1, \ldots, x_n \mid xy = \sigma_x(y)\gamma_y(x)\rangle$, a monoid. If the bijective map $r : X^2 \to X^2 : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))$ is

1. involutive, i.e $r^2 = \mathrm{id}$,
2. right non-degenerate, i.e. each γ_x is bijective,
3. r is a set-theoretic solution of the Yang-Baxter equation,

then S is a monoid of (left) I-type, i.e. there exists an I-structure $v : FaM_n = \langle u_1, \ldots, u_n \mid u_i u_j = u_j u_i\rangle \to S$, that is, for all $a \in FaM_n$,

$$v(1) = 1 \text{ and } \{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\}.$$

The converse also holds.

Examples of non-dgenerate involutive solutions

1. The flip map: $r(x, y) = (y, x)$. So, S is a free abelian monoid.
2. Let σ be a permutation on X, $r(x, y) = (\sigma(y), \sigma^{-1}(x))$.
3. $X = \{x, y\}$, $r : X^2 \to X^2$ with $\sigma = (x, y)$. So, $S = \langle x, y \mid x^2 = y^2 \rangle$

Theorem, Jespers - Okninski, 2005

A monoid S is of (left) I-type if and only if $S \cong \{(a, \psi(a)) \mid a \in FaM_n\}$, a submonoid of $Fa_n \rtimes \mathrm{Sym}_n$.

In particular, S is of left I-type if and only of S is of right I-type (and thus left and right non-degenerate).

Furthermore, in this case, $S \subseteq G = \{(a, \Psi(a)) \mid a \in Fa_n\}$, a group (called a group of I-type)

Properties of the group $G(X, r)$

1. $G = G(X, r) \cong \mathrm{gr}(x_1, \ldots, x_n \mid xy = \sigma_x(y)\gamma_y(x)\rangle) \subseteq Fa_n \rtimes \mathcal{G}(X, r)$, with $\mathcal{G}(X, r) = gr(\sigma_x \mid x \in X)$, called IYB-group (involutive Yang-Baxter group). $G(X, r)$ is called the structure group.
2. G is abelian-by-finite,
3. G is solvable (Etingof, Schedler, Soloviev),
4. G is torsion free,i.e. a Bieberbach group (Gateva-Ivanova, Van den Bergh),
5. Rump 2005, G is decomposable if the defining relations are square free.

If

$$G = \{(a, \Psi(a)) \mid a \in Fa_n\}$$

is a group of left I-type then Fa_n is equipped with two operations $+$ and $\cdot$ such that $(Fa_n, +)$ is an abelian group, $(Fa_n, \cdot)$ is a group (with $a \cdot b = a + \Psi(a)(b)$) and, for all $a, b, c \in Fa_n$,

$$a \cdot (b + c) + a = a \cdot b + a \cdot c.$$

In other words, there is a bijective 1-cocycle $G \to Fa_n$ (Etingof, Schedler, Soloviev).
Such a structure $(G, +, \cdot)$ is called a (left) brace (introduced by Rump).

Bachiller, Cedo, Vendramin 2017

Let $G = \langle x_1, \dots, x_n \mid xy = \sigma_x(y)\gamma_y(x)\rangle$ be a group of I-type.

The following are equivalent

1. G is poly-$\mathbb{Z}$,
2. G is left orderable,
3. $r : X^2 \to X^2$ is a mulipermutation solution, that is, $\mathrm{Ret}^m(X, r)$ has only one element for some m, where $\mathrm{Ret}^{m+1}(X, r) = \mathrm{Ret}(\mathrm{Ret}^m(X, r))$ and $\mathrm{Ret}(X, r)$ is the solution induced by the equivalence $\sim$ on X by $x \sim y$ if and only if $\sigma_x = \sigma_y$.

Gateva-Ivanova - Van den Bergh, Jespers-Okninski

Let $S = S(X,r)$ be a monoid of I-type. The algebra $K[S] = K[M(X,r)] = K\langle x_1,\dots,x_n \mid xy = \sigma_x(y)\gamma_y(x)\rangle$ satisfies the following properties ($M(X,R)$ is called the structure monoid)

1. Noetherian,
2. satisfies a polynomial identity,
3. is a maximal order,
4. $\mathrm{clKdim}(K[S]) = \mathrm{GK}(K[S]) = n$.
5. the height one primes of $K[S]$ that do intersect S non-trivially are of the form $K[Ss] = K[sS]$.

It follows that $K[G(X,r)]$ is a Noetherian PI-algebra that is a maximal order and its height one prime ideals are principally generated by a normal element.

Jespers-Okninski-Van Campenhout 2015, Jespers-Van Campenhout 2017

Let $R = K\langle x_1, \dots, x_n \mid xy = \sigma_x(y)\gamma_y(x)\rangle$ with $r^2 = \text{id}$ and non-degenerate. Let $S = \langle x_1, \dots, x_n \mid xy = \sigma_x(y)\gamma_y(x)\rangle$.

Then $K[S]$ is a left and right Noetherian algebra that satisfies a polynomial identity and $\text{GKdim}(K[S]) \leq n = |X|$.

Furthermore,

S is of I-type if and only if S is cancellative and satisfies the cyclic condition,

i.e. $x_1, y \in X$, there exist elements $x_2, y_1, y_2, z_1, z_2 \in X$ such that $x_1 y = y_1 z_1$ and $x_2 y_1 = y_2 z_2$,
with $x_2 x_1$ and $z_2 z_1$ non-rewritable.

If S satisfies the cyclic condition then the following conditions are equivalent.

1. $\mathrm{GKdim}(K[S]) = n = |X|$,
2. S is cancellative,
3. S is of I-type,
4. $K[S]$ is prime,
5. $K[S]$ is a domain,

Braces

In 2007 Rump introduced braces as a generalization of radical rings to study solutions of the YBE. The following definition is equivalent to the original definition of Rump.

Definition

A left brace is a set B with two binary operations, $+$ and $\cdot$, such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group, and for all $a, b, c \in B$,

$$a \cdot (b + c) + a = a \cdot b + a \cdot c.$$

Note that in a left brace B, $1 = 0$ (taking $a = 1$ and $b = c = 0$ in the above formula).
In any left brace B there is an action $\lambda : (B, \cdot) \to \mathrm{Aut}(B, +)$ defined by $\lambda(a) = \lambda_a$ and $\lambda_a(b) = a \cdot b - a$, for $a, b \in B$.

Two-sided braces correspond to radicals rings.

Let G be a left brace. The following properties hold.

1. $a\lambda_a^{-1}(b) = b\lambda_b^{-1}(a)$.
2. $\lambda_a\lambda_{\lambda_a^{-1}(b)} = \lambda_b\lambda_{\lambda_b^{-1}(a)}$,
3. The map $r : G \times G \to G \times G$ defined by $r(x, y) = (\lambda_x(y), \lambda_{\lambda_x^{-1}(y)}(x))$ is a (non-degen. inv.) solution of the YBE; called the associated solution of the brace.

If the brace is two-sided then the associated solution is a multipermuation solution.

skew braces

Guarnieri and Vendramin (2017) : skew left brace $(B,\cdot,\circ)$, $(B,\cdot)$ (not necessarily abelian) and $(B,\circ)$ and

$$a\circ(b\cdot c)=(a\circ b)\cdot a^{-1}\cdot(a\circ c) \qquad (*),$$

where a^{-1} is the inverse of a in $(B,\cdot)$.
For a non-degenerate solution r there exists a 1-cocycle

$$G(X,r)\to A(X,r),$$

where

$$\begin{aligned} A(X,r) \;&=\; \langle x\in X \mid y_1y=y_2y_1 \text{ if } r(x,y)=(x_1,y_1) \\ &\qquad\qquad \text{and } r(x_1,y_1)=(x_2,y_2)\rangle \end{aligned}$$

Then, $G(X,r)$ is skew left brace, and skew left braces are induced from the regular subgroups of the holomorph of these groups.

semi-brace

Catino, Colazzo and Stefanelli: a left cancellative left semi-brace $(B,\cdot,\circ)$, $(B,\cdot)$ is a left cancellative semigroup, $(B,\circ)$ is a group and

$$a\circ(b\cdot c)=(a\circ b)\cdot(a\circ(\overline{a}\cdot c)) \qquad (**),$$

where $\overline{a}$ denotes the inverse of a in $(B,\circ)$.

Theorem: a left cancellative left semi-brace B is decomposable as a product of a skew left brace G and a left semi-brace E such that $(E,\cdot)$ consists of idempotents (actually it is a right semigroup and B is a matched product of G and E).

With a left cancellative left semi-brace B there is a left non-degenerate solution associated,

$$r(x, y) = (\lambda_x(y), \rho_y(x)),$$

where $\lambda_x(y) = x \circ (\overline{x} \cdot y)$ and $\rho_y(x) = \overline{\overline{x} \cdot y} \circ y$.
Solutions are not necessarily involutive, also idempotent solutions show up.

Jespers - Van Antwerpen

A description of finite left semi-braces B.

Corollary

Let $(B, \cdot, \circ)$ be a finite left semi-brace such that ρ is an anti-homomorphism. Then, $KM(B)$ is a Noetherian, PI-algebra of finite Gelfand-Kirillov dimension equal to that of $KM(1_\circ B 1_\circ)$. In particular, this dimension is at most $|1_\circ B 1_\circ|$ and it is precisely equal to $|1_\circ B 1_\circ|$ if B is a left brace.