Garside structures on the structure group of finite solutions of the Yang–Baxter equation

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Solutions of the YBE

Definition 1.1

Let X be a non-empty set. A set-theoretic solution of the Yang-Baxter equation on X is a bijective map $r\colon X \times X \to X \times X$ such that

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where $r_1 = r \times \mathrm{id}_X$ and $r_2 = \mathrm{id}_X \times r$ are maps from $X \times X \times X$ to itself.

We write $r(x, y) = (\sigma_x(y), \gamma_y(x))$, for all $x, y \in X$.
The map r is involutive if $r^2 = \mathrm{id}_{X^2}$.
We say that r is non-degenerate if the maps $\sigma_x, \gamma_x\colon X \to X$ are bijective, for all $x \in X$.

Solutions of the YBE

Convention 1.2

By a solution of the YBE we mean an involutive non-degenerate set-theoretic solution of the Yang-Baxter equation.

Let (X, r) be a solution of the YBE. Etingof, Schedler and Soloviev [3] introduced the structure group associated to (X, r),

$$G(X, r) = \langle X \mid xy = \sigma_x(y)\gamma_y(x),\ \forall x, y \in X \rangle,$$

where $r(x, y) = (\sigma_x(y), \gamma_y(x))$.

Braces and the Yang-Baxter equation

In 2007 Rump [4] introduced braces as a generalization of radical rings to study solutions of the YBE. The following definition is equivalent to the original definition of Rump.

Definition 1.3

A left brace is a set B with two binary operations, $+$ and $\cdot$, such that $(B,+)$ is an abelian group, $(B,\cdot)$ is a group, and for every $a,b,c \in B$,

$$a\cdot(b+c)+a = a\cdot b + a\cdot c.$$

Note that in a left brace B, $1=0$ (taking $a=1$ and $b=c=0$ in the above formula).
In any left brace B there is an action $\lambda\colon (B,\cdot)\to \mathrm{Aut}(B,+)$ defined by $\lambda(a)=\lambda_a$ and $\lambda_a(b)=ab-a$, for $a,b\in B$.

Braces and the Yang-Baxter equation

Rump proved that each left brace B produces a solution of the YBE: $r_B\colon B\times B\to B\times B,\quad r_B(a,b)=(\lambda_a(b),\lambda^{-1}_{\lambda_a(b)}(a))$.

Definition 1.4

An ideal I of a left brace B is a normal subgroup I of the multiplicative group of B such that $\lambda_a(y)\in I$ for all $a\in B$ and $y\in I$.

It is easy to check that every ideal I of a left brace B also is a subgroup of the additive group of B. Note that

$$a-b=bb^{-1}a-b=\lambda_b(b^{-1}a),$$

thus $a-b\in I$ if and only if $b^{-1}a\in I$. Therefore the natural sum and multiplication on B/I define a natural structure of left brace, the quotient left brace of B modulo I.

Braces and the Yang-Baxter equation

The socle of a left brace B is defined as the set

$$\mathrm{Soc}(B) = \{a \in B : \lambda_a = \mathrm{id}\} = \{a \in B : a + b = ab \text{ for all } b \in B\}.$$

The socle of B is an ideal of B.
Let (X, r) be a solution of the YBE.
It is known that there exists a unique left brace structure over the structure group $G(X, r)$ such that the additive group of $G(X, r)$ is isomorphic to $\mathbb{Z}^{(X)}$, and $\lambda_x(y) = \sigma_x(y)$, for all $x, y \in X$.

Garside monoid

Definition 2.1

A Garside monoid is a pair (M, Δ) where M is a cancellative monoid such that:

1. *there exists $d : M \longrightarrow \mathbb{N}$ satisfying $d(ab) \geq d(a) + d(b)$ and $a \neq 1$ implies $d(a) \neq 0$,*
2. *any two elements of M have a left- and a right-lcm and a left- and a right-gcd,*
3. *Δ is a Garside element of M, this meaning that the left- and right-divisors of Δ coincide and generate M,*
4. *the family of all divisors of Δ in M is finite.*

Garside group

It is known that if (M, Δ) is a Garside monoid, then M satisfies the left and right Ore conditions. Thus it has left and right group of fractions $G = M^{-1}M = MM^{-1}$.

Definition 2.2

A group G is said to be a Garside group if it is the (left) group of fractions of a submonoid M and there exists $\Delta \in M$ such that (M, Δ) is a Garside monoid.

In [1, Theoremm 3.3], Chouraqui proved that the structure group of a finite solution of the YBE is a Garside group. We shall prove this result using the natural structure of left brace of the structure group of a solution of the YBE.

Garside group

Lemma 2.3

Let (X, r) be a finite solution of the YBE. Let $n = [G(X, r) : \mathrm{Soc}(G(X, r))]$. Let z be an integer. Let $\Delta_z = \sum_{x \in X} zx$. Then $g\Delta_z = g + \Delta_z$, for all $g \in G(X, r)$. In particular, $\Delta_z^m = m\Delta_z$, for all integer m, and $n\Delta_z$ is a central element of the structure group $G(X, r)$.

Proof. Let $g \in G(X, r)$. We have

$$g\Delta_z = \lambda_g(\Delta_z) + g = \lambda_g\left(\sum_{x \in X} zx\right) + g = \sum_{x \in X} z\lambda_g(x) + g = \Delta_z + g.$$

The second part is a consequence of the first and the fact that $n\Delta_z \in \mathrm{Soc}(G(X, r))$. □

Garside group

Let (X, r) be a finite solution of the YBE. Let $M(X, r)$ be the submonoid of $G(X, r)$ generated by X.

Lemma 2.4

$M(X,r) = \{\sum_{x\in X} z_x x \mid z_x \in \mathbb{N}\}$ and $M(X,r)^{-1} = -M(X,r)$.

Proof. Let $M = \{\sum_{x\in X} z_x x \mid z_x \in \mathbb{N}\}$. Note that $\lambda_g(a) \in M$, for all $g \in G(X,r)$ and all $a \in M$.
Every element of $M(X,r)$ is of the form $x_1 \cdots x_m$, with $x_1, \dots, x_m \in X$. We shall prove that $x_1 \cdots x_m \in M$ by induction on m. For $m = 1$ it is clear. Suppose that $m > 1$ and that $y_1 \cdots y_{m-1} \in M$ for all $y_1, \dots, y_{m-1} \in X$. By the induction hypothesis, $x_2 \cdots x_m \in M$, and thus

$$x_1 \cdots x_m = x_1 + \lambda_{x_1}(x_2 \cdots x_m) \in M.$$

Hence $M(X,r) \subseteq M$.

Garside group

Let $\sum_{x\in X} z_x x \in M$. We shall prove that $\sum_{x\in X} z_x x \in M(X,r)$ by induction on $t=\sum_{x\in X} z_x$. For $t=1$ it is clear. Suppose that $t>1$ and that $\sum_{x\in X} a_x x \in M(X,r)$ for all $\sum_{x\in X} a_x x \in M$ with $\sum_{x\in X} a_x < t$. Since $t>1$ there exists $x_0\in X$ such that $z_{x_0}>0$. Let $a=-\lambda_{x_0}^{-1}(x_0)+\sum_{x\in X} z_x\lambda_{x_0}^{-1}(x)$. Now we have that $a\in M$ and, by the induction hypothesis, $a\in M(X,r)$. Thus

$$\sum_{x\in X} z_x x = x_0 - x_0 + \sum_{x\in X} z_x x = x_0\lambda_{x_0}^{-1}(-x_0+\sum_{x\in X} z_x x) = x_0 a \in M(X,r).$$

Hence $M=M(X,r)$.
Note that $g^{-1}=-\lambda_{g^{-1}}(g)$ and $-g=\lambda_{(-g)^{-1}}(g)^{-1}$ for all $g\in G(X,r)$. Therefore $M^{-1}=-M$, and the result follows. $\square$

Garside group

Note that $M(X,r)$ has a degree function

$$\deg\colon M(X,r) \longrightarrow \mathbb{N}$$

defined by $\deg(x_1 \cdots x_m) = m$, for all $x_1, \dots, x_m \in X$.

Lemma 2.5

Let $a \in M(X,r)$. By Lemma 2.4, $a = \sum_{x \in X} a_x x$, for some $a_x \in \mathbb{N}$. Then $\deg(a) = \sum_{x \in X} a_x$.

Proof. We will prove the result by induction on $\deg(a)$. If $\deg(a) = 1$, then $a = x$ for some $x \in X$, and the result is clear. Suppose that $\deg(a) = m > 1$ and that the result is true for all $b \in M(X,r)$ with $\deg(b) < m$. We have that $a = x_1 \cdots x_m$, for some $x_1, \dots, x_m \in X$.

Garside group

By the induction hypothesis,

$$x_2 \cdots x_m = \sum_{x \in X} b_x x,$$

for some $b_x \in \mathbb{N}$ such that $\sum_{x \in X} b_x = m - 1$. We have

$$\begin{aligned} a &= x_1 + \lambda_{x_1}(x_2 \cdots x_m) \\ &= x_1 + \lambda_{x_1}(\sum_{x \in X} b_x x) \\ &= x_1 + \sum_{x \in X} b_x \lambda_{x_1}(x). \end{aligned}$$

Since $\lambda_{x_1}(x) \in X$, for all $x_1, x \in X$, the result follows. $\square$

Garside group

Lemma 2.6

Let $a, b \in M(X, r)$ and $c, d \in M(X, r)^{-1}$. By Lemma 2.4,

$$a = \sum_{x \in X} a_x x,\ \ b = \sum_{x \in X} b_x x,\ \ c = \sum_{x \in X} c_x x \quad \text{and} \quad d = \sum_{x \in X} d_x x,$$

for some $a_x, b_x, -c_x, -d_x \in \mathbb{N}$. Then

(i) $b^{-1}a \in M(X, r)$ *if and only if* $b_x \leq a_x$ *for all* $x \in X$.

(i) $c^{-1}d \in M(X, r)^{-1}$ *if and only if* $c_x \geq d_x$ *for all* $x \in X$.

Garside group

Proof. (i) Note that

$$b^{-1}a = b^{-1} + \lambda_{b^{-1}}(a) = -\lambda_{b^{-1}}(b) + \lambda_{b^{-1}}(a) = \lambda_{b^{-1}}(a-b).$$

Therefore $b^{-1}a \in M(X,r)$ if and only if $a-b \in M(X,r)$ and the result follows easily by Lemma 2.4.
(ii) As above $c^{-1}d = \lambda_{c^{-1}}(d-c)$. Therefore $c^{-1}d \in M(X,r)^{-1}$ if and only if $d-c \in M(X,r)^{-1}$ and the result follows easily by Lemma 2.4. $\square$

Garside group

Lemma 2.7

Any two elements of $M(X, r)$ have a left- and a right-lcm and a left- and a right-gcd.

Proof. Let $a, b \in M(X, r)$. By Lemma 2.4

$$a = \sum_{x \in X} a_x x \quad \text{and} \quad b = \sum_{x \in X} b_x x,$$

for some $a_x, b_x \in \mathbb{N}$. By Lemma 2.6 it is clear that the left-gcd of a and b is

$$\text{l-gcd}(a, b) = \sum_{x \in X} \min(a_x, b_x) x,$$

and that its right-lcm is

$$\text{r-lcm}(a, b) = \sum_{x \in X} \max(a_x, b_x) x.$$

Garside group

Now consider the monoid $M(X,r)^{-1}$. Since $a^{-1}, b^{-1} \in M(X,r)^{-1}$, by Lemma 2.4

$$a^{-1} = \sum_{x \in X} a'_x x \quad \text{and} \quad b^{-1} = \sum_{x \in X} b'_x x,$$

for some non-positive integers a'_x, b'_x. By Lemma 2.6 it is clear that the left-gcd of a^{-1} and b^{-1} in $M(X,r)^{-1}$ is

$$\text{l-gcd}(a^{-1}, b^{-1}) = \sum_{x \in X} \max(a'_x, b'_x)x,$$

and that its right-lcm is

$$\text{r-lcm}(a^{-1}, b^{-1}) = \sum_{x \in X} \min(a'_x, b'_x)x.$$

Garside group

Note that if $d^{-1} = \text{l-gcd}(a^{-1}, b^{-1})$, then d is the right-gcd of a and b in $M(X,r)$, and if $m^{-1} = \text{r-lcm}(a^{-1}, b^{-1})$, then m is the left-lcm of a and b in $M(X,r)$. Thus the result follows. $\square$

Theorem 2.8 (Chouraqui [1, Theorem 3.3])

Let (X,r) be a finite solution of the YBE. Let m be a positive integer. Then $(M(X,r), \Delta_m)$ is a Garside monoid and thus $G(X,r)$ is a Garside group.

Proof. It is clear that the submonoid $M(X,r)$ of $G(X,r)$ is cancellative. We have seen that $M(X,r)$ has a degree function deg. In particular condition (1) in the definition of Garside monoid is satisfied.

Garside group

By Lemma 2.7, condition (2) in the definition of Garside monoid is satisfied by $M(X, r)$.
By Lemma 2.4, $\Delta_m \in M(X, r)$ and, by Lemma 2.3, $\Delta_m^{-1} = -\Delta_m$.
By Lemma 2.6, the set of left-divisors of Δ_m in $M(X, r)$ is

$$D_1 = \{\sum_{x\in X} z_x x \mid z_x \in \mathbb{Z},\ 0 \leq z_x \leq m,\ \text{for all } x \in X\},$$

and the set of left-divisors of Δ_m^{-1} in the monoid $M(X, r)^{-1}$ is

$$D_2 = \{-\sum_{x\in X} z_x x \mid z_x \in \mathbb{Z},\ 0 \leq z_x \leq m,\ \text{for all } x \in X\}.$$

Note that

$$(\sum_{x\in X} z_x x)^{-1} = -\lambda_{(\sum_{x\in X} z_x x)^{-1}}(\sum_{x\in X} z_x x) = -\sum_{x\in X} z_x \lambda_{(\sum_{x\in X} z_x x)^{-1}}(x),$$

for all $z_x \in \mathbb{Z}$.

Garside group

Hence $D_2^{-1} = D_1$. Thus the set of right-divisors of Δ_m in $M(X,r)$ also is D_1, which is finite and contains X. Therefore Δ_m is a Garside element of $M(X,r)$. Hence $(M(X,r), \Delta_m)$ is a Garside monoid, and clearly $G(X,r)$ is its group of left (right) group of fractions. $\square$

In [2] Dehornoy describes finite quotients of $G(X,r)$ that play a role similar to the role that Coxeter groups play for Artin-Tits groups. This can be also done using the left brace structure of $G(X,r)$.

References

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Thank you for your attention!