Noncommutative and non-associative structures, braces and applications

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On speciality of Malcev algebras

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Malcev algebras:
(1955, A. I. Malcev)

$$
\begin{aligned}
x^2 &= 0 \ \textit{(anticommutativity)}, \\
J(x, xy, z) &= J(x, y, z)x \quad \textit{(Malcev identity)}
\end{aligned}
$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$.

Examples:

- tangent space $T(L)$ of an analytic Moufang loop L,
- commutator algebra $A^{(-)} = \langle A, +, [,]\rangle$ for an alternative algebra A, where $[x, y] = xy - yx$.

The most important example: the algebra of traceless octonions

$$sl\,(\mathbb{O}) = \{x \in \mathbb{O} \,|\, tr(x) = 0\} \subset \mathbb{O}^{(-)},$$

a simple non-Lie Malcev algebra.

All the algebras in the talk are assumed to be over a field F of characteristic $\neq 2, 3$.

A Malcev algebra M is called *special* if M is isomorphic to a subalgebra of the algebra $A^{(-)}$ for a certain alternative algebra A.

The Malcev Problem: Is it true that every Malcev algebra is special?

First Positive Results:

E.N.Kuzmin (1968): Simple finite-dimensional algebras are special,
V.T.Filippov (1982): Semiprime algebras are special,
S.R.Sverchkov (1986): The variety $Var\,(sl\,(\mathbb{O}))$ is special.

Consider the Filippov functions h and g:

$$\begin{aligned} h(x,y,z,t) &= \{yz,t,x\}x - \{xy,z,x\}t, \\ g(x,y,z,t,v) &= J(\{yz,t,x\},x,v) - J(\{xy,z,x\},t,v), \end{aligned}$$

where $\{u,v,w\} = J(u,v,w) + 3u(vw)$.

Denote by $\mathcal{H}$ and $\mathcal{G}$ the varieties of Malcev algebras defined by the identities $h = 0$ and $g = 0$, respectively. Observe that $\mathcal{H} \subset \mathcal{G}$.

V.T.Filippov (1983):

- For every algebra $M \in \mathcal{H}$, the subalgebra M^2 is a special algebra,
- For every algebra $M \in \mathcal{G}$, the ideal $J(M)$ is a special algebra.

It is easy to check that $h = 0$ in $sl(\mathbb{O})$, therefore $Var\,(sl\,(\mathbb{O})) \subseteq \mathcal{H}$.

The Problem: Is it true that $Var\,(sl\,(\mathbb{O})) = \mathcal{H}$?

V.T.Filippov: $Var\,(sl(2,F))) = \mathcal{H} \cap Lie.$

Results for superalgebras:

I.Shestakov (1991): Simple and prime superalgebras are special,

A.Elduque, I.Shestakov (1995): Irreducible and prime supermodules are special,

I.Shestakov, N.Zhukavets (2006): Every superalgebra generated by an odd element is special.

Free algebras and s-identities

$Malc[X]$, the free Malcev algebra,
$Alt[X]$, the free alternative algebra,
$SMalc[X] \subset (Alt[X])^{(-)}$, the free special Malcev algebra on a set of generators X.

Consider the natural epimorphism

$$\pi : Malc[X] \to SMalc[X],\ \pi|_X = id_X.$$

The algebra $Malc[X]$ is special $\Leftrightarrow \ker \pi = 0$. The elements from $\ker \pi$ are called *s-identities*.

I.Shestakov, N.Zhukavets (2006): There are no skew-symmetric s-identities,
A.I.Kornev (2010): $Malc\,[x,y,z] \cong SMalc\,[x,y,z]$.
In particular, there are no s-identities on three variables.

The Problem: Is it true that $Var\,(Malc\,[x,y,z]) = Var\,(Malc\,[x,y,z,t])$?

Cohn's elements and commutator-eaters

$SMalc$, the class of special Malcev algebras,
$\overline{SMalc} = Var(SMalc)$,
$Malc$, the variety of Malcev algebras.

$$SMalc \subseteq \overline{SMalc} \subseteq Malc$$

In case of Jordan algebras, the both corresponding inclusions are strong:

$$SJord \subsetneqq \overline{SJord} \subsetneqq Jord$$

While the strictness of the second inclusion depends on existence of s-identities, the first inclusion is related with the existence in free special Jordan algebra of Cohn's elements and tetrad-eaters.

We introduce certain analogues of these elements for Malcev algebras.

An element $f \in SMalc[X]$ is called a Cohn's element if

$$id_{Alt[X]}\langle f\rangle \cap SMalc[x] \neq id_{SMalc[X]}\langle f\rangle.$$

If $f \in SMalc[X]$ is a Cohn element then the quotient algebra $SMalc[X]/id_{SMalc[X]}\langle f\rangle$ is not special.

An elementl $f \in SMalc[X]$ we call a *commutator eater* if $f \circ [g,h]^2 \in SMalc[X]$ for any $f, g \in SMalc[X]$, where $x \circ y = xy + yx$.

Lemma. If Malcev polynomial $f = f(x, \dots, y)$ satisfies the identities

$$f(x^2, \dots, y) = x \circ f(x, \dots,),$$
$$f([x, y]^2, \dots, z) = 0,$$

then $f(x, \dots, y)$ is a commutator eater.

The Filippov g-function $g_a(x, y, z, t)$ satisfies the identities of Lemma, hence it is a commutator eater, and the element

$$s(a, b, c, x, y, z, t) = g_a(x, y, z, t) \circ [b, c]^2$$

is a Malcev polynomial in $Alt[X]$, that is, belongs to the free special Malcev algebra $SMalc[X]$.

Let $S = S(a, b, c, x, y, z, t)$ be the preimage of s in the free Malcev algebra $Malc[a, b, c, x, y, z, t]$ obtained by replacing the commutators with Malcev multiplication, and let G be the ideal of this algebra generated by $g_a(x, y, z, t)$.

Theorem (A.Buchnev, V.Filippov, I.Sh., S.Sverchkov):
The quotient algebra $M = Malc[a,b,c,x,y,z,t]/G$ is not special.

For the proof, we had first to verify whether the element S is non-zero in $Malc[a,b,c,x,y,z,t]$. We observe that

$$S(a,b,c,x,y,z,t) = 0 \iff \tilde{S}(a,b,c,x,x,x,x) = 0$$

in the free Malcev superalgebra $Malc[a,b,c;x]$ on even generators a,b,c and odd generator x, where $\tilde{S}$ is a "superization" of the element S. We used the MALCEV computer algebra system developed by authors and verified that $\tilde{S} \neq 0$.

Observe that $s = \pi(S)$. Thus if $s = 0$, we have a non-trivial s-identity $S(a,b,c,x,y,z,t)$. Unfortunately, the calculations in $SMalc[X]$ is much harder than in $Malc[X]$, and we do not know whether $s = 0$. But we prove that

1) If $s \neq 0$ then $g_a(x,y,z,t)$ is a Cohn's element in $SMalc[a,b,c,x,y,z,t]$ and $M \in \overline{SMalc} \setminus SMalc$,

2) If $s = 0$ then the s-identity S is non-zero in M and $M \in Malc \setminus \overline{SMalc}$

In both cases, $M \in Malc \setminus SMalc$.

Non-alterative enveloping algebras

Given an algebra A, the alternative nucleus of A is defined as

$$N_{alt}(A) = \{a \in A \mid (a,y,z) = -(y,a,z) = (y,z,a)\}$$

for any $y, z \in A$, where $(x,y,z) = (xy)z - x(yz)$. This subspace of A is closed under the commutator product $[x,y] = xy - yx$ and $N_{alt}(A)^{(-)} = (N_{alt}(A), [\,,\,])$ is a Malcev algebra. In case that A is alternative then $A = N_{alt}(A)$.

J.M.Pérez–Izquierdo, I.P.Shestakov (2004):
For every Malcev algebra M over F, there are an algebra $U(M)$ and an injective homomorphism $i\colon M \to N_{alt}(U(M))^{(-)}$; moreover, the algebra $U(M)$ is a universal object under such homomorphisms.

If M is a Lie algebra then $U(M)$ agrees with the universal enveloping algebra of M as a Lie algebra. In general, $U(M)$ shares many features with the universal enveloping algebras of Lie algebras.

- $U(M)$ has no zero divisors,
- $gr\, U(M) \cong S(M)$,
- $U(M)$ has a PBW-basis over M,
- $U(M)$ has a structure of a bialgebra with comultiplication defined by

$$\Delta(m) = 1 \otimes m + m \otimes 1, \; m \in M,$$

with $Prim\,(U(M), \Delta) = M$ if char $F = 0$.

The bialgebra $U(M)$ is co-associative and co-commutative and satisfies the Hopf-Moufand identity

$$\sum_{(y)} ((xy_{(1)})z)y_{(2)} = \sum_{(y)} x(y_{(1)}(zy_{(2)})).$$

Another example of a Hopf-Moufang bialgebra is provided by a loop algebra FL for a Moufang loop L, with the coproduct $\Delta(l) = l \otimes l$.

In a Hopf-Moufang bialgebra H the space of primitive elements forms a Malcev algebra, and the set of groop-like elements forms a Moufang loop.

G.Benkart, S.Madariaga,J .M.Pérez–Izquierdo: The category of Hopf-Moufang bialgebras is equivalent to the category of co-commutative Hopf algebras with triality.