Results on potential algebras: contraction algebras and Sklyanin algebras

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We consider finiteness conditions and questions of growth of noncommutative algebras, known as A_{con}s.

They appear in M.Wemyss work on minimal model program and noncommutative resolution of singularities. Namely, they serve as noncommutative invariants attached to a birational flopping contraction:

$f : X \to Y$

which contracts rational curve $C \simeq \mathbb{P}^1 \subset X$ to a point. X is a smooth quasi-projective 3-fold.

It is known due to [Van den Bergh], that A_{con}s are potential.

Finiteness questions are essential, because algebras with geometrical origin are finite dimensional or have a linear growth.

Def *Potential algebra* (Jacobi, vacualgebra) given by cyclic invariant polynomial F is an algebra

$$A_F = k\langle x, y\rangle/\mathrm{id}(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$$

where noncommutative derivations $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} : k\langle x, y\rangle \to k\langle x, y\rangle$ are defined via action on monomials as:

$$\frac{\partial w}{\partial x} = \begin{cases} u & \text{if } w = xu, \\ 0 & \text{otherwise,} \end{cases} \qquad \frac{\partial w}{\partial y} = \begin{cases} u & \text{if } w = yu, \\ 0 & \text{otherwise.} \end{cases}$$

Polynomial F is *cyclic invariant* means $F = F \circlearrowleft$ where $u \circlearrowleft$ is a sum of all cyclic permutations of the monomial $u \in K\langle X\rangle$.

In [Iyudu, Smoktunowicz, IMRN 2017 and IHES/M/16/19] we prove the following theorems on the finiteness conditions for 2-generated potential algebras.

It was shown by Michael Wemyss that the completion of a potential algebra can have dimension 8 and he conjectured that this is the minimal possible dimension. We show that his conjecture is true.

Theorem 1. *Let A_F be a potential algebra given by a potential F having only terms of degree 3 or higher. The minimal dimension of A_F is at least 8. Moreover, the minimal dimension of the completion of A_F is 8.*

Proof We use Golod-Shafarewich theorem, Gröbner bases arguments **plus** relation, which holds in any potential algebra:

$$[x, \frac{\partial F}{\partial x}] + [y, \frac{\partial F}{\partial y}] = 0$$

Non-Homogeneous case

Using the improved version of the Golod–Shafarevich theorem and involving the fact of potentiality we derive the following fact.

Theorem 2. *Let* A_F *be a potential algebra given by a not necessarily homogeneous potential* F *having only terms of degree* 5 *or higher. Then* A_F *is infinite dimensional.*

Homogeneous case

Theorem 3. *For the case of homogeneous potential of degree* $\geqslant 3$*,* A_F *is always infinite dimensional.*

Namely, we prove the following two theorems.

First, we deal with the case of homogeneous potentials of degree 3.

We classify all of them up to isomorphism.

From this we see that the corresponding algebras are infinite dimensional. We also compute the Hilbert series for each of them.

Classification of potential algebras, with homogeneous potential of degree 3.

Theorem 4. *There are three non isomorphic potential algebras with homogeneous potential of degree* 3*.*

1. $F = x^3$, $A = \mathbb{K}\langle x, y\rangle / \mathrm{Id}(x^2)$.

2. $F = x^2y + xyx + yx^2$, $A = \mathbb{K}\langle x, y\rangle / \mathrm{Id}(xy + yx, x^2)$.

3. $F = x^2y + xyx + yx^2 + xy^2 + yxy + y^2x$, $A = \mathbb{K}\langle x, y\rangle / \mathrm{Id}(xy + yx + y^2, x^2 + xy + yx) = \mathbb{K}\langle x, y\rangle / \mathrm{Id}(xy + yx + y^2, x^2 - y^2)$.

In each case:

*These relations form a Gröbner basis (w.r.t. degLex and $x > y$).

*A_F is infinite dimensional.

It has exponential growth for $F = x^3$ and the Hilbert series is $H_A = 1 + 2t + 2t^2 + 2t^3 + \ldots$ in the other two cases (the normal words are y^n and $y^n x$).

Next, we consider the main case, when F is of degree $\geqslant 4$.

Theorem 5. *If $F \in \mathbb{K}\langle x, y\rangle$ is a homogeneous potential of degree $\geqslant 4$, then the potential algebra $= \mathbb{K}\langle x, y\rangle/\mathrm{Id}(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ is infinite dimensional.*

Moreover, the minimal Hilbert series in the class $\mathcal{P}_n$ of potential algebras with homogeneous potential of degree $n+1 \geqslant 4$ is $H_n = \frac{1}{1-2t+2t^n-t^{n+1}}$.

Corollary 6. *Growth of a potential algebra with homogeneous potential of degree 4 can be polynomial (non-linear), but starting from degree 5 it is always exponential.*

Conjecture formulated in [Wemyss and Donovan, Duke 2015]

The conjecture says that the difference between the dimension of a potential algebra and its abelianization is a linear combination of squares of natural numbers starting from 2, with non-negative integer coefficients.

In [Toda, 2014] it is shown, that these integer coefficients do coincide with Gopakumar - Vafa invariants.

We give an example of solution of the conjecture using Gröbner bases arguments, for one particular type of potential, namely for the potential $F = x^2y+xyx+yx^2+xy^2+yxy+y^2+a(y)$, where $a = \sum_{j=3}^{n} a_j y^j \in \mathbb{K}[y]$ is of degree $n > 3$ and has only terms of degree $\geqslant 3$.

Results on another class of potential algebras - Sklyanin algebras: we prove Koszulity via the calculation of Hilbert series.

(Obtained using the same potential complex and Gröbner basis theory):

[Iyudu, Shkarin, J.Algebra 2017],

[Iyudu, Shkarin, MPIM preprint 49.17]

For $(p,q,r) \in \mathbb{K}^3$, the *Sklyanin algebra* $S^{p,q,r}$ is the quadratic algebra over a field $\mathbb{K}$ with generators x, y, z given by 3 relations

$$\begin{aligned} pyz + qzy + rxx &= 0, \\ pzx + qxz + ryy &= 0, \\ pxy + qyx + rzz &= 0. \end{aligned}$$

One of the main methods in the investigation of exactly solvable models in quantum mechanics and statistical physics is the 'inverse problem method'. The method leads to study the meromorphic matrix functions $L(u)$ satisfying the equation

- $R(u-v)L^1(u)L^2(v) = L^2(v)L^1(u)R(u-v)$

Here $L^1 = L \otimes 1, L^2 = 1 \otimes L$ and $R(u)$ is a chosen solution of the Yang-Baxter equation (depending on parameter):

- $R^{12}(u-v)R^{13}(u)R^{23}(v) = R^{23}(v)R^{13}(u)R^{12}(u-v)$

In the seminal paper Sklyanin [1983] *On some algebraic structures related to Yang-Baxter equation. II Representations of quantum algebras.*

Sklyanin considered a specific series of elliptic solutions to YBE expressed via Pauli matrices: $R(u) = 1 + \sum_{\alpha=1}^{3} W_\alpha \sigma_\alpha \otimes \sigma_\alpha$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and discovered that for that solution one can obtain a series of solutions to the first equation of the form

$L(u) = S_0 + \sum_{\alpha=1}^{3} W_\alpha(u) S_\alpha$

for any matrices S_0, S_1, S_2, S_3 (not depending on parameter any more) satisfying the following relations

$[S_\alpha, S_0] = -iJ_{\beta,\gamma}[S_\beta, S_\gamma]_+, \ [S_\alpha, S_\beta] = i[S_0, S_\gamma]_+.$

So any information on this algebra and its representations becomes important, since it gives a family of solutions, and in these cases model is integrable.

Then it was notices that the analogous thing exists for any n, and especially extensive study begins for 3-dimensional Sklyanin algebras.