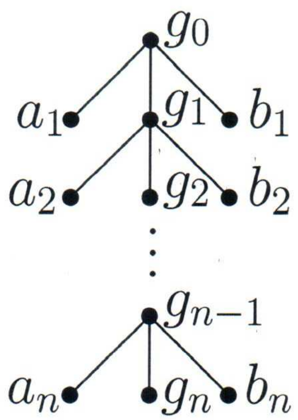
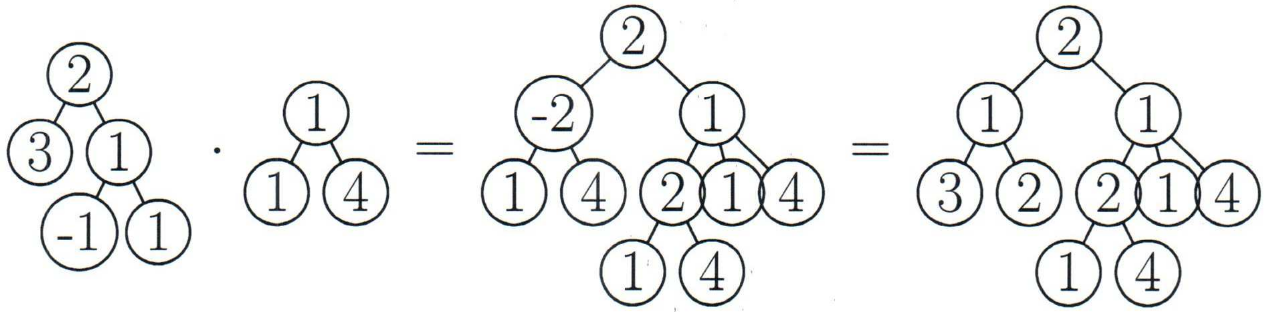
Proposition 2. *Let* $(A;+)$ *and* G *be groups with a left action of* G *on the set* A. *The unit group of* $T_A(G)$ *is the free product* $G^{\pm} * (A+1+A)$, *where* $A+1+A \cong A^{+} \times (A^{+})^{\mathrm{op}}$.

As trees, the units in $T_A(G)$ look as follows:



For the trivial group G, the near-ring $T_0(1)$ is the initial object $\mathbb{Z}$ in the category of unital near-rings. For the 2-element group $G = C_2$, the group $G^{\pm}$ is the infinite dihedral group D_∞, the adjoint group of the infinite cyclic brace. Hence $T_0(C_2)$ consists of integer-labelled trees. Example for multiplication in $T_0(C_2)$:



The rightmost tree gives the normalized form where only the leaves may be negative.

5. The initial near-ring of a skew-brace. For a skew-brace A, the adjoint group A° acts on A by $b \mapsto a \circ b$. So we can form the near-ring $T_A(A^\circ)$. The identity map 1_A induces a bijection $\varepsilon\colon A \to A^\circ$ between the submodule $A = T_A(A^\circ)_c$ and the subgroup A° of $T_A(A^\circ)^\times$. Let I be the ideal of $T_A(A^\circ)$ generated by the elements $1 - \varepsilon(a) + \varepsilon(a+b) - \varepsilon(b)$ with $a, b \in A$. We define the **associated near-ring** of A to be $N(A) := T_A(A^\circ)/I$. So there is a natural epimorphism $\pi\colon T_A(A^\circ) \twoheadrightarrow N(A)$.

Theorem 3. *Let A be a skew-brace. Then $N(A)_c = A$. The map $e\colon A \to N(A)$ with $e(a) := \pi\varepsilon(a) - 1$ is an exponential of $N(A)$, and $N(A)$ is the initial object of* **Exp**(A).

The relation $\varepsilon(a) - \varepsilon(a+b) + \varepsilon(b) = 1$ carries the additive structure of A into A°. As the unit element 1 of A° is $\varepsilon(0)$, the equation should actually read

$$\varepsilon(a) - \varepsilon(a+b) + \varepsilon(b) = 0.$$

The elements of $N(A)$ are $(A^\circ)^\pm$-labelled ordered trees whose leaves can also be labelled by elements of A, so that adjacent leaves a, b in A can be contracted to a single leaf $a + b$. – Note that the skew-brace A is determined by the bijective 1-cocycle $\varepsilon^{-1}\colon A^\circ \to A$ from the adjoint group A° to the right A°-module A.

6. Can a skew-brace A be recovered from an associated near-ring? For the terminal object $M(A)$ this is impossible: Only the additive group can be recovered. What about the initial near-ring $N(A)$?

Proposition 3. *The defining bijective 1-cocycle of a skew-brace A is given by the restriction $c|_{A^\circ}$ of the canonical map $c\colon N(A) \to N(A)_c = A$.*

Alas, how to distinguish A° within the unit group of $N(A)$? So we should remove A from $N(A)$ to reduce the unit group $(A^\circ)^\pm * (A+1+A)$ by the free factor $A+1+A$. (Then we are left with the problem to distinguish A° within the free product $A^\circ * \{\pm 1\}$.) Consider the commutative diagram

$$\begin{array}{ccc} T_A(A^\circ) & \twoheadrightarrow & N(A) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ T_0(A^\circ) & \twoheadrightarrow & N_0(A) \end{array}$$

where $N_0(A)$ is obtained from $T_0(A^\circ)$ by factoring out the same relation as in the passage from $T_A(A^\circ)$ to $N(A)$. Both problems are solved:

Proposition 4. *Let A be a skew-brace. The unit group of $N(A)$ is $N_0(A)^\times = A^\circ * \{\pm 1\}$, and*

$$A^\circ = \{x \in N_0(A)^\times \mid 1 - x + 1 \in N_0(A)^\times\}.$$

Solving the two problems, a new problem occurs: We saved the adjoint group by throwing the additive group overboard. But remember:

"The relation $\varepsilon(a) - \varepsilon(a+b) + \varepsilon(b) = 1$ carries the additive structure of A into A°."

So we should search for the additive group in A° where it has not been destroyed by the passage from $N(A)$ to $N_0(A)$. The solution is again found by the concept of **skew-ring**. Namely, every unital near-ring N can be viewed as a unital skew-ring in two ways:

1. The above mentioned skew-ring S_+N with the same additive group and multiplication $a^b := a(b+1)$.

2. The skew-ring $S_\times N$ with the adjoint monoid $a \circ b := ab$ taken from the multiplication in N and the modified additive group

$$a \oplus b := a - 1 + b.$$

The unit element in $S_\times N$ is $1+1$.

Theorem 4. *Let A be a skew-brace. Then A° is a sub-skew-ring of $S_\times N_0(A)$ which is isomorphic to the skew-brace A.*

Together with Proposition 4, this recovers A from $N_0(A)$. Since $N_0(A)$ is derived from $N(A)$, we obtain:

Corollary. *For any skew-brace A, the initial near-ring $N(A)$ in* $\mathbf{Exp}(A)$ *is a complete invariant of A.*

The ubiquity of skew-braces thus implies that near-rings are even more universal. For the various types of skew-braces, the associated near-ring could be used for obtaining new invariants, e. g., for knots and links.

7. Local near-rings and K-linear braces. A skew-brace with abelian additive group is said to be a **brace**. A **K-brace** over a field K is a brace A with a K-vector space as additive group such that the exponentiation maps are K-linear. Accordingly, we define a **K-linear near-ring** to be a near-ring N with a K-vector space as additive group such that $\lambda(rs) = (\lambda r)s = r(\lambda s)$ holds for $r, s \in N$ and $\lambda \in K$.

A near-ring N is said to be **local** (Maxson 1968) if N is unital and the elements $r \in N$ with $Nr \neq N$ form a submodule $J(N)$ of N, the **radical** of N. Then $N^{\times} = N \setminus J(N)$. In particular, $r + s = 1$ implies that r or s is a unit. If $J(N) = 0$, then N is said to be a **near-field**.

The radical $J(N)$ of a local near-ring N is a sub-skew-ring on which the exponentiation is bijective. Hence $J(N)$ is a skew-brace. So it is natural to ask:

Question: Which skew-braces arise as $J(N)$?

For a near-field, the additive group is always abelian (B. H. Neumann 1940). For a local near-ring, this is almost true: It holds unless the centre of the additive group is contained in the radical (easy exercise!).

We say that a local near-ring N **splits** if there is a sub-near-field K with $N = K \oplus J(N)$. If K can be chosen in the centre

$$Z(N) := \{r \in N \mid \forall\, s \in N \colon rs = sr\},$$

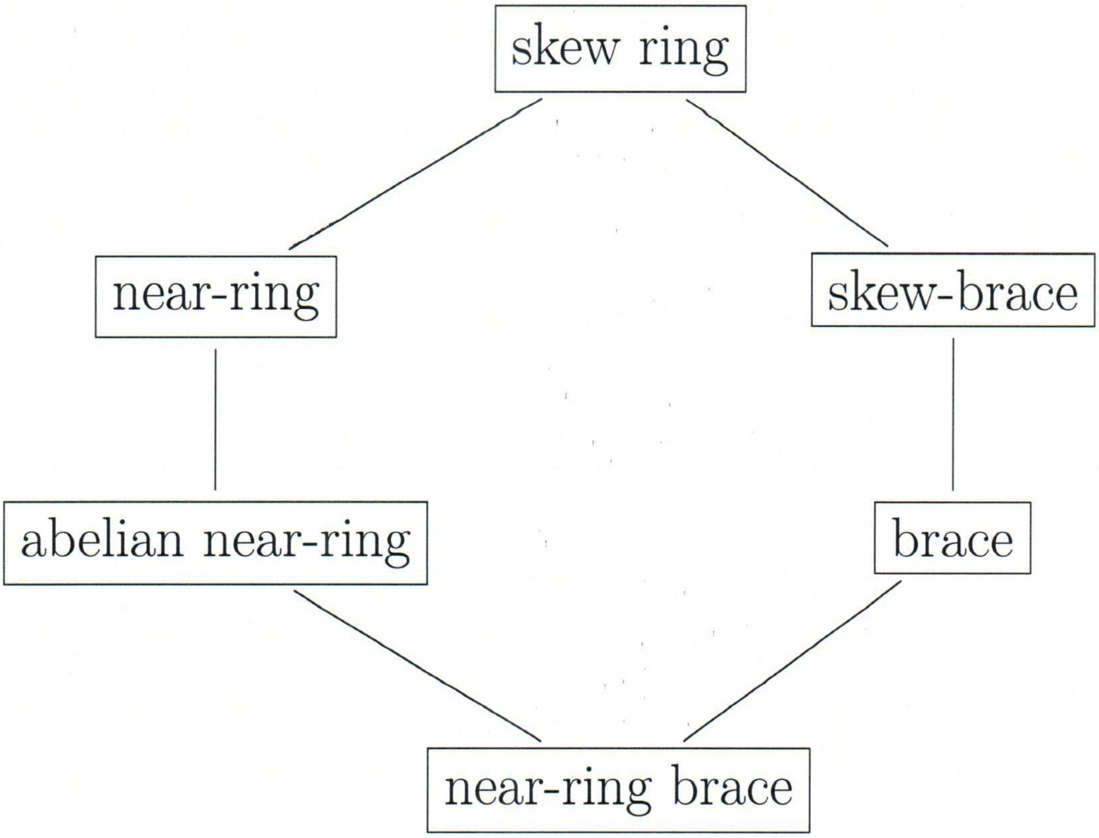
we say that N **totally splits**. In this case, $J(N)$ is an ideal. (In general, this is still open!) If N totally splits, K is commutative, hence a field.

Now comes the counterpart to skew-rings which have passed the test of their usefulness: We define a **near-ring brace** to be a brace A with a near-ring structure satisfying

$$\boxed{a(b^c) = (ab)^c} \qquad \boxed{(a^b)c = a(bc + c)}$$

for all $a, b, c \in A$. If A is K-linear as a brace and as a near-ring, we speak of a **near-ring K-brace**. The concept gives a **common specialization** of near-rings and braces: Every brace is a near-ring brace with a trivial near-ring structure ($ab = 0$), while every near-ring with abelian additive group is a near-ring brace with trivial brace structure ($a^b = a$).

We depict this as a diagram, where an **abelian near-ring** means a near-ring with abelian additive group.

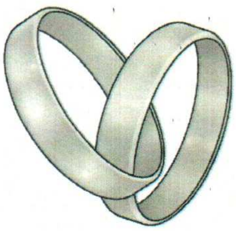


Here is our motivating example: The radical of an abelian local near-ring N is a near-ring brace!

Theorem 5. *Let K be a field. Up to isomorphism, $N \mapsto J(N)$ gives a bijection between totally split local near-rings N with $N/J(N) \cong K$ and near-ring K-braces.*

Corollary. *Under the bijection of Theorem 5, K-braces correspond to totally split near-rings N with $N/J(N) \cong K$ and $J(N)^2 = 0$.*

Skew-braces and Near-rings



Wolfgang Rump

In this talk, I will show that the connection between **skew-braces** and **near-rings** is much closer than expected: Skew-braces can be described as near-rings.

1. Skew-braces. Skew-braces arise in connection with the Yang-Baxter equation:

$$(S \times 1)(1 \times S)(S \times 1) = (1 \times S)(S \times 1)(1 \times S)$$

The structure group of any solution S is a skew-brace, and every skew-brace gives rise to a solution. Every tame knot is characterized by a special type of skew-brace. Hopf-Galois field extensions are described by skew-braces, and every skew-brace is equivalent to a triply factorized group, a group G with subgroups A, B and a normal subgroup N satisfying

$$G = AB = AN = BN, \ A \cap B = A \cap N = B \cap N = 1.$$

2. Near-rings are additive groups with a right distributive associative multiplication. **Near-fields** are important in geometry. They determine a class of translation planes. Finite sharply 2-transitive groups can be viewed as affine transformations of a near-field (Carmichael 1931). Finite near-fields (Dickson 1905) are obtained by changing the multiplication in a finite field. Zassenhaus (1936) found seven exceptional finite near-fields. Some aspects carry over to **Near-rings**.

Examples: 1. A polynomial ring $K[x]$ has two multiplications. With composition, it is a near-ring.
2. The self-maps $A \to A$ of a group $(A; +)$ form a near-ring $M(A)$.

In a near-ring N, the equation $a0 = 0$ need not hold. Elements of $N_c := \{a0 \mid a \in N\}$ are called **constants**. The constant maps in $M(A)$ identify A with $M(A)_c$.

3. A broader term. Recall that a **skew-brace** (Guarnieri-Vendramin 2017) is a group $(A; +)$ with a second group structure $(A; \circ)$ satisfying

$$(a + b) \circ c = a \circ c - c + b \circ c. \qquad (1)$$

Defining a^b by the equation $a \circ b = a^b + b$, Eq. (1) can be replaced by the exponential rules

$$\boxed{(a + b)^c = a^c + b^c} \qquad \boxed{a^{b \circ c} = (a^b)^c}$$

As a common generalisation, we define a **skew-ring** to be a group $A^+ := (A; +)$ such that $a^0 = a$ and the exponential rules hold with $a \circ b := a^b + b$. Thus, a skew-brace is just a skew-ring for which $(A; \circ)$ is a group. So a skew-ring is a bit nearer to a ring than to a skew-brace. But they are even nearer to near-rings:

Proposition 1. *A unital near-ring is the same as a unital skew-ring.*

We have to add what a unital skew-ring is. A **unit** of a skew-ring A is an element $1 \in A$ which satisfies $1^a = a + 1$ for all $a \in A$. A unit is unique modulo **constants**, that is, elements of the subgroup

$$\mathrm{Fix}(A) := \{a \in A \mid \forall\, b \in A\colon a^b = a\},$$

the *fixator* of A. If a unit 1 exists, the set of units is $1 + \mathrm{Fix}(A)$. The corresponding near-ring A has the same unit element 1 and multiplication $ab := a^{b-1}$. Its subgroup of constants is $\mathrm{Fix}(A)$.

Every skew-ring A has an **exponential** embedding into a near-ring

$$e\colon A \hookrightarrow M(A),$$

given by $e(a)(b) := a^b$. The map e is a morphism of skew-rings. In particular, any skew-brace A embeds into a near-ring and is determined by $M(A)$ and e.

The near-ring $M(A)$ is just an extremal one among a class of near-rings associated to a skew-brace A. For any near-ring N, there is a split short exact sequence

$$N_0 \underset{q}{\overset{i}{\rightleftarrows}} N \underset{j}{\overset{c}{\rightleftarrows}} N_c \tag{2}$$

with $N_0 := \{r \in N \mid r0 = 0\}$, the **zero-symmetric** part. If $M(A)$ is replaced by an arbitrary near-ring N, then $A = N_c$, and the **exponential map** becomes $e\colon N_c \hookrightarrow N$ which extends (2) to a **recollement**

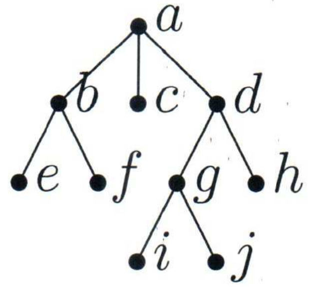
$$N_0 \begin{array}{c}\xleftarrow{\quad p \quad}\\ \xrightarrow{\quad i \quad}\\ \xleftarrow[\quad q \quad]{}\end{array} N \begin{array}{c}\xleftarrow{\quad e \quad}\\ \xrightarrow{\quad c \quad}\\ \xleftarrow[\quad j \quad]{}\end{array} N_c \tag{3}$$

Note that recollements occur in several branches of mathematics. The standard example is a ringed space $(X, \mathscr{O})$ with an open subset $j\colon U \hookrightarrow X$. Then N in (3) corresponds to the category of $\mathscr{O}$-modules on X, and the maps in (3) to Grothendieck's "six functors".

Theorem 1. *Let N be a unital near-ring. Every exponential map $e\colon N_c \to N$ makes N_c into a skew-brace. Every skew-brace arises in this way.*

Thus any skew-brace A is given in several ways by a near-ring N with an exponential map e, hence a recollement (3). The exponentals $e\colon A \to N$ form a category $\mathbf{Exp}(A)$. Terminal object is $A \hookrightarrow M(A)$.

4. Near-ring of a group action. For a deeper analysis of the near-rings associated with a skew-brace, we have to introduce a special type of near-ring. Let $(A;+)$ be a group, and let G be a (multiplicative) group acting on A as a set. With $(-1)a := -a$, the action extends to an action of the free product $G^{\pm} := G * \{\pm 1\}$ of G and the two-element group $\{\pm 1\}$. We construct a near-ring $T_A(G)$ of ordered trees, labelled by elements of $G^{\pm}$ or non-zero elements of A (for leaves only). For example, the tree



stands for the element $a(b(e+f)+c+d(g(i+j)+h))$.

Theorem 2. *$T_A(G)$ is a unital near-ring with $G^{\pm}$ as a multiplicative subgroup and $A = T_A(G)_c$.*

The construction of $T_A(G)$ is functorial in A. So the map $A \to 0$ gives a morphism $T_A(G) \twoheadrightarrow T_0(G)$ onto a near-ring $T_0(G)$ which merely depends on the group G. The morphism is obtained by cancelling the leaves in A. So the complexity of $T_A(G)$ comes mainly from the group.