28. In dealing a poker hand of five cards, what is the probability of (a) a flush (five cards of the same suit)? (b) A royal flush (A K Q J 10 in a suit)? (c) A straight (any five cards in sequence)?

29. Faces 1 and 6 of a die are painted green while the rest are white. If the die is rolled seven times, what is the probability of at least five green?

30. Raspberry- and strawberry-flavored Jello is on special sale at a local Food Mart, one package to a customer. What is the probability that the seventh package sold in the special will be the third package of strawberry?

31. A purse contains four \$5 bills and three \$1 bills. Bob removes a bill at random, then Paul does the same, and so on alternately, without replacements. What is the probability that Bob will be the first to obtain a \$5 bill?

32. A sack contains three red and four green apples. Two are removed at random and are replaced by one of each color. Then two are removed again. What is the probability that they are of the same color?

33. If it is known that a bridge hand is void of spades, what is the probability that it is also void of hearts?

34. A postal clerk slips 12 duplicate advertisements at random into 8 boxes. What is the probability that (a) exactly one box is empty? (b) At least one is empty? (*Hint:* See Example 1.30.)

35. A dart is thrown five times at a target which has a horizontal and vertical line through its bull's eye. What is the probability that either a majority will strike in the first quadrant or that the strikes will alternate in this regard?

36. In nine births at a hospital, what is the probability that exactly three of the first five will be boys and exactly three of the last five will be girls? (Assume boy and girl babies to be equally likely.)

37. A nickel, dime, and quarter are tossed as a group twice. What is the probability that (a) each coin will give the same result on the second toss as on the first? (b) The same number of heads will appear? (c) Extend (a) to the case of n coins instead of three.

38. What is the probability that either North or South will be dealt exactly two aces in a single bridge deal?

39. What is the probability that a bridge hand will contain some spades or some aces?

40. What is the probability that a five-card hand will contain (a) a flush with three face cards? (b) A straight flush (five cards in sequence in the same suit)?

41. The eighth, ninth, and tenth floors of an office building serve the medical–dental professions with 12 offices, 4 on each floor. What is the probability (a) that 9 patients will go 3 to each floor? (b) That all 9 will go to offices in the same relative position on their floors?

42. What is the probability that a five-card hand will contain (a) a flush or four aces? (b) Two hearts or two spades or one diamond, exactly?

43. Marion deals five ESP cards faceup on a table. Martha shuffles a duplicate set and places these below the first five. What is the probability that there will be (a) at least one match? (b) No match? (*Hint:* Use (1.13).)

the table. Are the events *blue shirt* and *Doug* independent? What about *Mark* and *green shirt*?

	Y	B	G
D	4	3	2
M	5	7	9

14. The first drawer of a cabinet contains one gold and one silver coin; the second contains two gold coins; and the third contains two silver coins. A drawer is chosen at random and a coin is removed at random. If it is gold, what is the probability that it came from the second drawer?

15. The freshmen class at a college contains 300 boys and 200 girls, while the sophomore class is composed of 250 boys and 120 girls. A class is chosen by the flip of a coin and a student is chosen from the class by random choice of ID numbers. It turns out to be a girl. What is the probability that she is a freshman?

16. A player in a crap game throws a sum of 5 on the two dice. (a) What is the probability of throwing another 5 before throwing a total of 7? (b) The same if his "point" (original throw) is a 10.

17. A "point" in craps is a sum of 4, 5, 6, 8, 9, or 10 on the pair of dice. What is the probability that a player will roll a point and then make his point; that is, roll the same sum before rolling a 7?

18. A player wins in a crap game if he rolls a 7, or 11, or if he rolls a point and then makes it. What is his probability of winning?

19. A bridge deck is shuffled and spread out in a line. (a) What is the probability that the first face card precedes the first 2? (b) That the aces appear in the order S, H, D, C?

20. A pair of dice is rolled repeatedly. What is the probability of obtaining a 5 three times before the first 7?

21. A custodian has two keyrings on his belt, the first having seven room keys and one to his supply cabinet, and the other having three room keys and two masters. He selects a ring at random, tries a key and finds that it opens his cabinet. What is the probability that it was not a master?

22. A box contains two red and three white chips. One is removed, and if it is white it is replaced in the box by two white chips, otherwise the chip is set aside. Then a second chip is removed. What is the probability that the first chip is red if the second is red?

23. One hundred children, 250 women, and 150 men are patients at a hospital. Eighty percent of the children, 40% of the women, and 60% of the men are seriously ill. What is the probability that a seriously ill patient is a women?

24. In a certain community there are twice as many Republicans as Democrats. Half of the Republicans and one-fifth of the Democrats are opposed to a certain proposal. What proportion of those opposed are Democrats?

and all the rest lose the denomination in dollars, what is the expecte value of the gain?

9. A half million one-dollar tickets are sold in a state lottery. There is or $50,000 prize, five $10,000 prizes, and one hundred $1,000 prizes. What the expected gain on a single ticket?

10. If $f(x) = 12/(25x)$, $x = 1, 2, 3, 4$, find (a) $E[X]$, (b) $E[2X+3$ (c) $E[\sqrt{X}]$.

11. Find μ and σ^2 for $f(x) = (2x)/[n(n+1)]$, $x = 1, 2, \ldots, n$.

For each of the distributions in Exercises 12–17, obtain (a) the mean, (b) th variance, and (c) the rth moment about the origin.

12. $f(x) = 1$, $0 \le x \le 1$.

13. $f(x) = ae^{-ax}$, $a > 0$, $x > 0$

14. $f(x) = kxe^{-x}$, $x > 0$. What is k?

15. $f(x) = (1/2)e^{-|x|}$

16. $f(x) = ax^{a-1}$, $0 \le x \le 1$, $a > 0$

17. $f(x) = 2(1-x)$, $0 < x < 1$

18. Find μ and σ^2 for $f(x) = a|x|e^{-x^2}$. What is the value of a?

19. Prove (2.15).

20. Find $E[X^2+1]$ for $f(x) = 1/2$, $-1 < x < 1$.

21. Prove that the distribution $f(x) = 1/x^2$, $x \ge 1$, has a median but no mea

22. Prove (2.11).

23. Prove (2.13).

24. If a density curve $y = f(x)$ is symmetric with respect to a line $x = a$, th if $E[X]$ exists, $E[X] = a$. Prove this theorem analytically.

25. Consider the density function $f(x) = xe^{-x}$, $x > 0$. Show by computing t mean and the area shaded in Figure 2.21 that these are equal in this cas

26. Use the coding to obtain $E[X]$ and $V[X]$ for the distribution $f(20) = 0.$ $f(50) = 0.1$, $f(80) = 0.3$, $f(110) = 0.2$, and $f(140) = 0.1$.

2.7 Moment Generating Functions

There are obviously an unlimited number of probability distrib tions having the same mean and variance. But under certain c cumstances, a specification of moments of *all* orders will determine distribution uniquely. We begin with the definition of an addition special expectation (2.9) in which $u(X) = e^{tX}$.

Let

$$M(t) = E[e^{tX}] \qquad 2.$$

exist for every t in some open interval containing $t = 0$. Then $M(t)$ is called *the moment generating function* of the distribution of X.

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has a limiting distribution that is standard normal, and the theorem is proved by application of the uniqueness theorem extension, which states that if the limit of a sequence of moment generating functions is that of the standard normal,then the distributions represented by the sequence converge to the standard normal distribution.

The importance of the central limit theorem can hardly be overstated. It is absolutely essential for much of statistical theory and practice. Though very little may be known about a distribution other than its mean and variance, the theorem nevertheless supplies substantial information concerning the distribution of its sample average. Actually n need not be very large for good approximations—generally $n \geq 25$ suffices, and often much smaller values give satisfactory results.

EXAMPLE 5.2

(a) Obtain the approximate probability that 27 numbers taken randomly from the interval [0, 1] will average less than 0.4. (b) Find the approximate probability that the average of 25 numbers taken at random from an interval of length 6 on the x-axis will differ from the mean by more than 0.5.

(a) Here $\mu_{\bar{x}} = \mu_x = 1/2$, $\sigma_x^2 = 1/12$, so that $\sigma_{\bar{x}}^2 = \sigma_x^2/27 = 1/324$. Therefore$P(\bar{X} < 0.4) \cong P(Z < (0.4 - 0.5)/(1/18)) = P(Z < -1.8) = 0.04$.

(b) Here $\sigma_x^2 = 3$ and $\sigma_{\bar{x}}^2 = 3/25$ so that $P(|\bar{X} - \mu| > 0.5) \cong P(|Z| > 0.5/\sqrt{3/25}) = P(|Z| > 1.44) = 0.15$. Note that it was unnecessary to know the mean.

EXERCISES 5.1

1. Compare $P(6 < X < 8)$ with $P(6 < \bar{X} < 8)$ if X is normally distributed with mean 7 and variance 25. The size of the random sample is 100.
2. A random sample of size 64 is taken from a population with mean 3.5 and variance 0.25. What is the probability that the sample average will not differ from the population mean by more than two-hundredths of a unit?
3. A random sample of size 50 is taken from a population whose mean and variance are -3 and 8, respectively. Find the value of c that satisfies $P(|\bar{X} + 3| < c) = 0.95$.
4. A population has variance 4. The average of a sample of size n is used to approximate the mean of the population. How large should n be so that 90% of the time the error will be less than 0.1 in absolute value?
5. A fair die is rolled 105 times. Find the approximate probability of obtaining a sum of spots larger than 350.
6. A sample of size 75 is taken from a normal population with mean 2 and variance 12. What is the probability that the average will lie between 1.92 and 2.28?
7. What is the probability that the mean of a population will exceed the

EXAMPLE 3.13

If there is no sex discrimination in the selection of high school teachers, and if there continue to be equal numbers of qualified applicants of each sex, what is the probability that a faculty of one hundred will have *between* 47 and 55 men?

Here $p = 1/2$, $\mu = np = 50$, and $\sigma^2 = npq = 25$. We wish to obtain $P(47 < X < 55)$ which is the area of the binomial histogram from 47.5 to 54.5. Standardizing, we obtain the corresponding values $(47.5 - 50)/5 = -0.5$ and $(54.5 - 50)/5 = 0.9$. Thus the desired probability is approximately $P(-0.5 \leq Z \leq 0.9)$ where Z is the standard normal variable. Table VII gives $P = 0.5074$.

We now have *three* methods for evaluating binomial probabilities. When n is small, tables or direct computation can be used. When n is large and p or q is near zero, the Poisson approximation is available. In the remaining case, when n is large and p is not near zero or one, we can use the standard normal approximation.

EXERCISES 3.7

1. Compare the exact and the approximate values of $b(4; 12, 1/4)$.
2. Find approximate values of the following:
   (a) $b(23; 54, 0.4)$ (b) $b(65; 100, 0.7)$
3. Find approximate values of the following:
   (a) $\Sigma_{x=69}^{75} b(x; 144, 0.5)$ (b) $\Sigma_{x=36}^{39} b(x; 80, 0.45)$ (c) $\Sigma_{x=88}^{91} b(x; 150, 0.6)$
4. A fair coin is tossed 256 times. What is the approximate probability of obtaining (a) more than 140 heads? (b) Between 120 and 138 heads, noninclusive? (c) Exactly 128 heads?
5. A fair die is rolled 180 times. What is the approximate probability of obtaining (a) between 32 and 36 fours? (b) More than 39? (c) Exactly 29?
6. The probability of success in a certain experiment is 1/5. The experiment is repeated independently for a total of n times. What must n be so that the probability of at least 29 successes will be no less than 0.0166?
7. In a certain chemistry course 80% of the students receive a C or better. If this is so, what is the approximate probability that more than 85 of 100 students in the course will achieve this distinction?
8. It is desired to test the assumption that the proportion of voters who will cast their ballots for a certain candidate is 55%. If this is true, what is the probability that fewer than 90 of 176 sampled favor him?
9. An asymmetrical coin for which $P(\text{H}) = 0.64$ is tossed 100 times. The probability that the total number of heads obtained is between $64 - c$ and $64 + c$, inclusive, is 95%. Find c to the nearest integer.
10. The probability that a fruit fly will die before completion of a certain laboratory experiment is 2/3. Find c so that the probability of c or more survivors in 1800 is approximately 95%.
11. Prove that the moment generating function for the standardized Poisson variable has $e^{t^2/2}$ as limit as μ becomes infinite. What are the implications?

We approximate the mean of the Poisson distribution by the average number of cells per square, $60/25 = 2.4$. Then the desired probability is $p(0; 2.4) + p(1; 2.4) + p(2; 2.4) = 0.57$.

EXAMPLE 3.8

A certain book of 300 pages contains 60 typographical errors spread independently throughout the book. (We need to question this hypothesis since fatigue and other conditions may tend to produce dependent or linked errors.) What is the probability that a page contains at least 2 errors?

Here the role of "interval of length w" is taken by a page. The average number of errors per page is 0.2, which is therefore our estimate of μ. Hence the required probability is $P = 1 - p(0; 0.2) - p(1; 0.2) = 0.018$.

EXERCISES 3.3

1. Obtain the variance of the Poisson distribution without the use of the moment generating function. First obtain $E[X(X-1)]$.
2. Prove (3.17).
3. Obtain μ and σ^2 using the moment generating function.
4. For what values of μ will the Poisson histograms form a decreasing set only? Is an entirely increasing set possible?
5. Construct and compare tables for the Poisson distribution with mean 2 and the binomial distributions with the same mean using $n = 4$, 8, and 20.
6. The probability of a 5 in a set of 10 random digits is 0.1. (a) Using the Poisson distribution, find the approximate probability of x 5's ($x = 0, 1, \ldots, 25$) in a set of 25 random digits. (b) Obtain the relative frequency of 5's in 40 sets of 25 random digits from Table XVI and compare with the results in part (a).
7. The probability that a certain city will have a severe storm during a 365-day period is approximated by a Poisson distribution with mean 2.7. What is the probability that four such storms will strike during the twelve months beginning today? Fewer than three?
8. A Geiger counter registers radioactive emissions in a Poisson pattern with an average of 114 per minute. (a) Find the probability of more than four emissions in one second. (b) At most one in two seconds.
9. Fifty Toll House cookies are to be made using 75 chocolate chips. What is the probability that a cookie will contain no chocolate chips? More than two chips? What is the mode?
10. A cake contains approximately 75 raisins. It is cut into 10 equal parts. What is the probability that one piece will contain no raisins? At least 4?
11. A card is drawn from an ordinary deck 130 times with replacement and shuffling. What is the approximate probability of obtaining the ace of spades more than twice?