ℓ-WEAK IDENTITIES AND CENTRAL POLYNOMIALS FOR MATRICES

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ABSTRACT. We develop the theory of ℓ-weak identities in order to provide a feasible way of studying the central polynomials of matrix algebras. We describe the weak identities of minimal degree of matrix algebras in any dimension.

1. INTRODUCTION

One basic question in PI-theory is to determine the polynomial identities (PI's) of the matrix algebra $\mathrm{M}_n(\mathbb{Q})$. Specht's celebrated problem is whether every set of polynomial identities of an algebra is **finitely based**, i.e., is a consequence of a finite number of identities, solved affirmatively by Kemer in 1988 and 1990, cf. [K]. However, his solution is difficult to implement to obtain a finite (PI) base for the identities of $\mathrm{M}_n(\mathbb{Q})$, in the sense that every PI of the algebra is a consequence of the base identities. Indeed, a base is known only for $\mathbb{Q}$ and $\mathrm{M}_2(\mathbb{Q})$. A multilinear polynomial $f(x_1, \dots, x_m)$ is an **ℓ-weak identity** of $\mathrm{M}_n(\mathbb{Q})$ if substitution of matrices for x_i sends f to zero whenever $\mathrm{tr}(x_1) = \cdots = \mathrm{tr}(x_\ell) = 0$, and an **$\ell$-weak central polynomial** if such substitution sends f to a central element. Our overriding goal here is to obtain partial information about bases, mostly in terms of weak identities and weak central polynomials.

Section 2 provides a brief overview of polynomial identities. We define and discuss ℓ-weak identities in Section 3, developing an inductive procedure to compute spaces of ℓ-weak identities (see Remark 3.4). Aided by computer computations, we obtain the following results.

(1) Explicit generators for the ℓ-weak identities of $\mathrm{M}_2(F)$ in degrees 3 and 4, for any ℓ (Section 6).
(2) When $\mathrm{char}\, F \neq 3$ there are no weak identities of degree 5 for $\mathrm{M}_3(F)$ (Subsection 7.1).
(3) However, s_4 is a weak central polynomial of $\mathrm{M}_3(F)$ over a field of characteristic 3, so $[s_4, x_5]$ is a 4-weak polynomial identity of degree 5 (Subsection 7.2).

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(4) We present dimensions and module decomposition for the ℓ-weak identity spaces in degree 6 for $\mathrm{M}_3(F)$, correcting a minor omission in [DR] (Subsection 8.1).
(5) We obtain a trace identity of degree 4 for $\mathrm{M}_3(F)$ from the Okubo composition algebra, and deduce Halpin's 4-weak identity of degree 6 from it (Subsection 8.3).
(6) For $n \geq 4$, there are no weak identities of $\mathrm{M}_n(F)$ in degree $2n$ other than the standard identity (Section 9).

2. Preliminaries

Let F be a field. The free (associative) F-algebra generated by noncommuting variables $x_1, \dots, x_m$ is denoted $F\{x_1, \dots, x_m\}$; we refer to the elements of $F\{x_1, \dots, x_m\}$ as **polynomials**.

Definition 2.1. *A polynomial $p \in F\{x_1, \dots, x_m\}$ is called a* ***polynomial identity*** *(PI) of the F-algebra A if $p(a_1, \dots, a_m) = 0$ for all $a_1, \dots, a_m \in A$. We write* $\mathrm{id}(A)$ *for the set of identities of A.*

2.1. Identities, central polynomials and examples.

The free algebra has no nonzero identities, almost by definition. An algebra A is PI if $\mathrm{id}(A) \neq 0$. The most basic examples of PI-algebras are the matrix algebra $\mathrm{M}_n(F)$ for arbitrary n, f.d. algebras over a field, and the Grassmann algebra G, cf. [BR, Definition 1.35].

Here is a notion closely related to PI.

Definition 2.2 (Central polynomials)**.** *A polynomial $f(x_1, \dots, x_n)$ is A-***central** *if $f(A) \subseteq \mathrm{Cent}(A)$. A central polynomial $f(x_1, \dots, x_n)$ is* ***strictly A-central*** *if $f \notin \mathrm{id}(A)$; in other words, $0 \neq f(A) \subseteq \mathrm{Cent}(A)$.*

A polynomial $p(x_1, \dots, x_n)$ is k-**multilinear** if each of the variables $x_1, \dots, x_k$ appears exactly once in each of the monomials of p. We omit the preamble if p is multilinear in all of its variables. Let P_m be the subspace of multilinear polynomials in $F\{x_1, \dots, x_m\}$, for $m \geq 1$. Any PI f can be transformed into a multilinear PI through the multilinearization process (see [BR]), and the process is reversible in characteristic 0; likewise any central polynomial f can be transformed into a multilinear central polynomial through the multilinearization process, which is reversible in characteristic 0. Thus in what follows we consider polynomials in P_m.

Example 2.3. (i) *The polynomial x_1 is central for any commutative algebra.*
(ii) *The polynomial $[x_1, x_2]$ is central for the Grassmann algebra.*
(iii) *Let $\mathrm{UT}(n)$ denote the algebra of upper triangular matrices over a given commutative base ring C. Any product of n strictly upper triangular $n \times n$ matrices is 0. Since $[a, b]$ is strictly upper triangular, for any upper triangular matrices a, b, we conclude that the algebra $\mathrm{UT}(n)$ satisfies the identity*

$$[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}].$$

(iv) *(Wagner's identity) The matrix algebra* $\mathrm{M}_2(F)$ *satisfies the identity* $g_2 := [[x,y]^2, z]$ *or, equivalently, the central polynomial* $[x,y]^2$ *and its multilinearization. (This is because the square of a trace-zero* 2×2 *matrix is scalar.)*

(v) *Fermat's Little Theorem translates to the fact that any field* F *of* q *elements satisfies the identity* $x^q - x$*. Its multilinearization is the symmetric polynomial, but in going back we only get* qx^q *which is identically zero.*

(vi) *The* ***standard polynomial***

$$s_m := \sum_{\pi \in S_m} \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(m)}$$

is a PI of $\mathrm{M}_n(\mathbb{Q})$ *precisely when* $m \geq 2n$*.*

(vii) *By Razmyslov* [Ra2] *and Drensky* [D1] $\{s_4, g_2\}$ *is a PI base for* $\mathrm{M}_2(F)$*. A base for* $\mathrm{M}_3(\mathbb{Q})$ *remains unknown.*

The **PI degree** of an algebra A, denoted $\mathrm{PIdeg}\, A$, is the minimal degree of an identity of this algebra. Thus $\mathrm{PIdeg}\, \mathrm{M}_n(F) = 2n$, and $\mathrm{PIdeg}\, G = 3$.

2.2. **Spechtian polynomials.** A multilinear polynomial is i-**Spechtian** if it vanishes when 1 is substituted for x_i. We write Sp_m^i for the subset of i-Spechtian polynomials in P_m, and Sp_m^I for the subset $\bigcap_{i\in I} \mathrm{Sp}_m^i$ of polynomials that vanishes when 1 is substituted for x_i, for any $i \in I$. In particular $\mathrm{Sp}_m^{\emptyset} = P_m$. We write $\mathrm{Sp}_m = \mathrm{Sp}_m^{\{1,\dots,m\}}$ for the set of **Spechtian** polynomials (also called **proper** in the literature). The polynomial s_{2k} is Spechtian.

Definition 2.4. *Define* ***higher commutator*** *inductively, as a commutator* $[f, g]$ *of either letters or higher commutators.*

In the proof of [BR, Proposition 6.2.1]*, by specializing* x_i *to 1, we see that a polynomial* f *can be written as* $f_1 + f_2$ *where* x_i *does not appear in* f_1 *and* f_2 *is* i*-Spechtian. It follows that* f *is Spechtian if and only if it is a sum of products of higher commutators.*

We write $\mathrm{id}_{\mathrm{Sp}}(A)$ *for the subset of Spechtian identities of* A *and* $\mathrm{id}_{m,\mathrm{Sp}}(A)$ *for* $\mathrm{Sp}_m \cap \mathrm{id}(A)$*.*

In [BR, Corollary 6.2.2] it is shown that any base of identities can be comprised of Spechtian identities.

3. Weak identities

3.1. **Weak and strong variables.** We refine Definition 2.1 with respect to the matrix algebra $A = \mathrm{M}_n(F)$.

Definition 3.1. *Let* $p(x_1, \dots, x_m)$ *be an* ℓ*-multilinear polynomial. We say that* p *is an* ℓ***-weak identity*** *of* A *if it vanishes under every substitution of matrices of trace* 0 *in* $x_1, \dots, x_\ell$ *and arbitrary matrices in the other variables.*

More generally, for $I \subseteq \{1, \dots, m\}$*, we say that* p *is an* I***-weak identity*** *of* A *if it vanishes under every substitution of matrices of trace* 0 *in* $\{x_i : i \in I\}$ *and* *arbitrary matrices in the other variables (in this context we say that x_i, $i \in I$ are* **weak** *variables in p, while x_i, $i \notin I$ are* **strong***).*

We write $\mathrm{id}_m^I = \mathrm{id}_m^I(A)$ for the set of I-weak multilinear identities of degree m. In particular, a 0-weak identity is simply an identity, namely $\mathrm{id}_m^\emptyset = \mathrm{id}_m$. On the other extreme, if p is m-weak we omit the prefix and say that p is a weak identity. For $I \subseteq J$ we have that $\mathrm{id}_m^I \subseteq \mathrm{id}_m^J$ and $\mathrm{Sp}_m^I \supseteq \mathrm{Sp}_m^J$.

Lemma 3.2. *Assume* $\mathrm{char}\, F$ *does not divide n.*

1. $\mathrm{id}_m^I \cap \mathrm{Sp}_m^J \subseteq \mathrm{id}_m^{I \setminus J}$ *for every $I, J \subseteq X$.*
2. $\mathrm{id}_m^I(A) \cap \mathrm{Sp}_m \subseteq \mathrm{id}_m$ *for every I.*
3. *A weak identity which is a Specht polynomial is in fact an identity.*

Proof.
1. Let $\mathrm{M}_n(F)_0 = \{a \in \mathrm{M}_n(F) \mid \mathrm{tr}(a) = 0\}$. Since $\mathrm{M}_n(F) = F \cdot 1 \oplus \mathrm{M}_n(F)_0$, the condition for an I-weak identity $f \in \mathrm{id}_m^I$ to be in $\mathrm{id}_m^{I \setminus J}$ is that for every $j \in I \cap J$, substitution $x_j \mapsto 1$ sends f to an identity.
2. Take $J = \{1, \dots, m\}$ in (1).
3. Take $I = \{1, \dots, m\}$ in (2) to obtain $\mathrm{id}_m^m(A) \cap \mathrm{Sp}_m = \mathrm{id}_m$.

□

3.2. **Modules of weak identities.** Write id_m^ℓ for $\mathrm{id}_m^{\{1,\dots,\ell\}}$, the set of ℓ-weak identities. We clearly have

$$\mathrm{id}_m(A) = \mathrm{id}_m^0(A) \subseteq \mathrm{id}_m^1(A) \subseteq \cdots \subseteq \mathrm{id}_m^m(A). \tag{1}$$

Following the Amitsur-Levitzki theorem [AmL], it is known that the minimal identities appear in $\mathrm{id}_m(\mathrm{M}_n(F))$ for $m = 2n$, where this space is 1-dimensional. As a refinement, it is desirable to describe the chain (1), at least for the minimal m for which it is nonzero.

Note that $\mathrm{id}_m^\ell(A)$ is not a submodule of P_m, since a permutation could send a weak indeterminate to a strong indeterminate.

Remark 3.3. *The space of ℓ-weak identities is a module through the natural action on weak and strong variables over the ring $F[S_\ell \times S_{m-\ell}] \cong F[S_\ell] \otimes F[S_{m-\ell}]$, which is semisimple when* $\mathrm{char}\, F = 0$*, being a direct sum of matrix rings over F.*

In particular $\mathrm{id}_m^m(A)$ and $\mathrm{id}_m^0(A)$ are S_m-modules, which can be described through their irreducible decompositions.

The level of details in a description of $\mathrm{id}_m^\ell(A)$ is a matter of taste. In increasing level of details, such a description might include:

1. An indication that the space is nonempty (for m minimal).
2. The dimension of the space, possibly given by a computer program.
3. Better still would be explicit identities, preferably ones that can be understood and demonstrated to be identities (and not just computer verified).

(4) Computations in the module $\mathrm{id}_m^\ell(A)$ can be facilitated by generators and relations. Or, more generally, the module can be endowed with a resolution of permutation modules (defined through the action on indices in a generating set).
(5) A decomposition into irreducible submodules is not hard to obtain for small m, although our experience ([V1] and [V2]) show that by itself it is not very illuminating.
(6) Finally, it is desirable to explicitly exhibit the embedding $\pi_m^{\ell-1}(A) \hookrightarrow \pi_m^\ell(A)$.

In order to study the chain of weak identity spaces (1), we compare two consecutive chains.

Remark 3.4. *The substitution map $x_\ell \mapsto 1$ defines a projection $\pi_\ell : P_m \to P_{m-1}$ (reducing the indices $\ell' > \ell$ by one), which induces the maps*

$$\begin{array}{ccccccccccccc}
\mathrm{id}_m^0(A) & \subseteq & \mathrm{id}_m^1(A) & \subseteq \cdots \subseteq & \mathrm{id}_m^{\ell-1}(A) & \subseteq & \mathrm{id}_m^\ell(A) & \subseteq \cdots \subseteq & \mathrm{id}_m^m(A) & \subseteq & P_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_\ell} \\
\mathrm{id}_{m-1}^0(A) & \subseteq & \mathrm{id}_{m-1}^1(A) & \subseteq \cdots \subseteq & \mathrm{id}_{m-1}^{\ell-1}(A) & \subseteq & P_{m-1} & = \cdots = & P_{m-1} & = & P_{m-1}
\end{array}$$

Indeed, for every $k < \ell$, if $p \in \mathrm{id}_m^k(A)$ then $p(x_1, \dots, x_k, \dots, 1, \dots, x_m)$ is a k-weak identity of degree $m-1$, so the downwards arrows are defined.

Even more is true:

Remark 3.5. *Assume* $\mathrm{char}\, F$ *is prime to m. For $\ell \le m$,*

$$\mathrm{id}_m^{\ell-1}(A) = \mathrm{id}_m^\ell(A) \cap \pi_\ell^{-1}(\mathrm{id}_{m-1}^{\ell-1}(A)).$$

Indeed, if $p \in \mathrm{id}_m^\ell(A)$ and $\pi_\ell(p) \in \mathrm{id}_{m-1}^{\ell-1}(A)$, then as long as $x_1, \dots, x_{\ell-1}$ are weak variables in p, x_ℓ is weak by the former assumption, and becomes strong by the latter.

We thus have an inductive procedure to compute the chain (1): once the chain was computed in degree $m-1$, the chain in degree m can be computed from $\mathrm{id}_m^m(A)$ by reverse induction on ℓ. In order to apply the condition $\pi_\ell(p) \in \mathrm{id}_{m-1}^{\ell-1}(A)$, we will need a hold on $\pi_\ell(\mathrm{id}_m^\ell(A)) \subseteq P_{m-1}$, whose elements in general are not even weak identities. For example, π_ℓ induces an embedding $\pi_\ell : \mathrm{id}_m^\ell(A)/\mathrm{id}_m^{\ell-1}(A) \hookrightarrow P_{m-1}/\mathrm{id}_{m-1}^{\ell-1}(A)$ which bounds the dimension of $\mathrm{id}_m^{\ell-1}(A)$ from below in terms of previously known quantities:

$$\dim(\mathrm{id}_m^{\ell-1}(A)) \ge \dim(\mathrm{id}_m^\ell(A)) - [(m-1)! - \dim(\mathrm{id}_{m-1}^{\ell-1}(A))].$$

For the minimal degree we can state this procedure more explicitly:

Remark 3.6. *Assume* $\mathrm{char}\, F$ *is prime to n. Assume m is the minimal degree of a weak identity for A. Then for every $\ell < m$,*

$$\mathrm{id}_m^\ell(A) = \{f \in \mathrm{id}_m^m(A) \mid \pi_{\ell+1}(f) = \cdots = \pi_m(f) = 0\}.$$

4. Central polynomials for matrices

The polynomials comprising a base of the T-ideal are hard to ascertain, unknown even for $\mathrm{M}_3(\mathbb{Q})$. So we look for minimal identities (e.g., s_{2n} for $\mathrm{M}_n(\mathbb{Q})$) and central polynomials. Surprisingly, even the minimal possible degree of a nonidentity which is a 1-weak identity (and thus provides a strict central polynomial, see Theorem 4.3 below) for $\mathrm{M}_n(F)$ is not known in general.

Halpin found an example of a central polynomial:

Lemma 4.1 ([BR, Lemma 1.4.14])**.** *The multilinearization of*

$$s_{n-1}([x,y],[x^2,y],\dots,[x^{n-2},y],[x^n,y])$$

is an $\frac{n^2-n+2}{2}$*-weak identity of* $\mathrm{M}_n(F)$*, of degree*

$$\frac{n^2-n+2}{2}+n-1=\frac{n^2+n}{2}=\frac{n(n+1)}{2}.$$

As explained in [BR, p. 37], this yields a 1-weak identity of total degree n^2:

Remark 4.2. *For* $0 \le \ell' < \ell$*, every* ℓ*-weak identity of degree* m *can be viewed as an* ℓ'*-weak identity of degree* $m+(\ell-\ell')$*, by substituting* $x_i \mapsto [x_i', x_i'']$ *for* $i=\ell'+1,\dots,\ell$*. In particular every* ℓ*-weak identity of degree* m *can be viewed as an identity of degree* $m+\ell$*.*

However, the 1-weak identity resulting from Halpin's polynomial is not an identity of $\mathrm{M}_n(\mathbb{Q})$. We thus have the existence of strict central polynomials. Formanek's polynomial [For1] also has degree n^2, and for some time this was thought the lowest possible, but in 1983, 1985, Drensky and Kasparian [DK2] discovered by a computer search a strict central polynomial for $M_3(\mathbb{Q})$ of degree 8, further explained in terms of weak identities by Drensky and Kasparian in 1993. Drensky showed 8 is optimal for $n=3$. The space of central polynomials of degree 8 is described in [V1]: the rank of $\mathrm{id}_8(\mathrm{M}_3(F))$ is 43; the Drensky-Kasparian identity adds 2 to the rank; and the full rank of $\text{c-id}_8(\mathrm{M}_3(F))$ is 47.

In 1994 Drensky and Piacentini found a strict central polynomial for $\mathrm{M}_4(\mathbb{Q})$ of degree 13, also obtainable via weak identities. In 1995 Drensky [D2] discovered a strict central polynomial for arbitrary $\mathrm{M}_n(\mathbb{Q})$ of degree $(n-1)^2+4$, which is minimal for $n=3$ and $n=4$, but its uniqueness is still open for $n=4$, and minimality of degree is open for $n>4$. We treat $n=3$ in Section 8.

4.1. ℓ-weak central polynomials.

Similarly to Definition 3.1, a polynomial p of degree m is an **ℓ-weak central polynomial** of $\mathrm{M}_n(F)$ if it takes central values under the substitutions of $x_1,\dots,x_\ell$ to matrices of trace zero and $x_{\ell+1},\dots,x_m$ to arbitrary matrices. More generally, p is an **I-weak central polynomial**, for $I \subseteq \{1,\dots,m\}$, if it takes central values under substitution of matrices provided that x_i maps to a zero trace matrix for all $i \in I$.

In particular, a 0-weak central polynomial is simply a central polynomial. On the other extreme, if p is m-weak we omit the prefix and say that p is a weak central polynomial.

Also let $\text{c-id}_m^\ell(A)$ be the space of ℓ-weak central polynomials of A, so that

$$\text{c-id}_m(A) = \text{c-id}_m^0(A) \subseteq \text{c-id}_m^1(A) \subseteq \cdots \subseteq \text{c-id}_m^m(A) \tag{2}$$

contains (1) component-wise. A natural question is to ask what is the minimal m for which $\text{id}_m(A) \subset \text{c-id}_m(A)$.

By Razmyslov (cf. [BR, Lemma 1.4.16]), central polynomials can be obtained from 1-weak identities, trading a weak variable in an identity for a strong variable in a central identity. We can copy the proof to get a more general result.

Let $p(x) = \sum a_i x b_i$ be a polynomial which is multilinear in x, where a_i, b_i are monomials over F in some variables other than x. We denote $p^*(x) = \sum b_i x a_i$, which defines an involution. For new variables y, z, consider $q(y,z) = p([y,z]) = \sum(a_i y z b_i - a_i z y b_i)$. Conjugating $q(y,z)$ with respect to y, we have that $q^*(y,z) = \sum(z b_i y a_i - b_i y a_i z) = \sum[z, b_i y a_i] = [z, p^*(y)]$. Therefore $p(x)$ is a weak identity in terms of x if and only if $q(y,z)$ is identically zero, if and only if $q^*(y,z) = [z, p^*(y)]$ is identically zero, if and only if all values of $p^*(y)$ are central. This procedure respects restrictions, such as zero trace, on any other variable involved. We thus proved a major result:

Theorem 4.3 (Razmyslov)**.** *For $\ell \geq 1$, there is a degree-preserving one-to-one correspondence $\text{id}_m^\ell(A) \to \text{c-id}_m^{\ell-1}(A)$ between ℓ-weak identities and $(\ell-1)$-weak central polynomials, given by $f \mapsto f^*$ (pivoting around x_ℓ).*

Consequently, we have a chain of isomorphisms between the components of the chains (1) and (2), albeit with non-commuting squares:

$$\begin{array}{ccccccccccc}
\text{id}_m^0(A) \subseteq & \text{id}_m^1(A) & \subseteq \cdots \subseteq & \text{id}_m^\ell(A) & \subseteq & \text{id}_m^{\ell+1}(A) & \subseteq \cdots \subseteq & \text{id}_m^m(A) & & \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
& \text{c-id}_m^0(A) & \subseteq \cdots \subseteq & \text{c-id}_m^{\ell-1}(A) & \subseteq & \text{c-id}_m^\ell(A) & \subseteq \cdots \subseteq & \text{c-id}_{m-1}^m(A) & \subseteq & \text{c-id}_m^m(A)
\end{array}$$

Moreover, $\text{id}_m^\ell(A)$ is an $(S_\ell \times S_{m-\ell})$-module, and $\text{c-id}_m^{\ell-1}(A)$ is an $(S_{\ell-1} \times S_{m-(\ell-1)})$-module. The groups intersect in the common stabilizer of the pivot variable x_ℓ, which is $S_{\ell-1} \times S_1 \times S_{m-\ell}$, and the isomorphism of Theorem 4.3 is of modules over this group.

5. The connection to the representation theory

We view $\text{id}_m(\text{M}_n(F))$ as a module over S_m, and apply the representation theory of the group to obtain symmetrical identities (the same considerations holds for $\text{id}_m^\ell(\text{M}_n(F))$ over $S_\ell \times S_{m-\ell}$).

5.1. **Identities and the group algebra.** Given a multilinear polynomial

$$\sum_{\sigma\in S_m} a_\sigma x_{\sigma(1)}\dots x_{\sigma(m)} \in P_m,$$

we may associate it with the element

$$\sum_{\sigma\in S_m} a_\sigma \sigma$$

of the group algebra $F[S_m]$.

The action of S_m on P_m translates to the usual multiplication in the group algebra. A natural left action of S_m on $F\{x_1,\dots,x_m\}$ is defined by $\sigma(x_i)=x_{\sigma(i)}$, which induces an action of S_m on P_m by

$$(\sigma\cdot f)(x_1,\dots,x_m)=f(x_{\sigma(1)},\dots,x_{\sigma(m)})$$

for all $\sigma\in S_m$ and $f\in F\{x_1,\dots,x_m\}$, making P_m a cyclic faithful S_m-module. But $F[S_m]$ is semisimple by Maschke's Theorem (assuming $\operatorname{char} F=0$ or $\operatorname{char} F>m$), so the module P_m is semisimple, and decomposes as a direct sum of simple submodules, some of which are generated by PIs of $M_n(F)$.

Each irreducible component of $F[S_m]$ corresponds to a partition λ of m. We denote the matrix subring corresponding to λ by Type_λ. We also denote the irreducible module corresponding to λ by Irr_λ. Notice that while Type_λ is a uniquely defined subset of $F[S_m]$ (and by identification, of P_m), Irr_λ is only defined up to isomorphism, as the decomposition of Type_λ into $\dim(\mathrm{Irr}_\lambda)$ copies of Irr_λ is not unique.

Remark 5.1. *The set* Sp_m *of Spechtian polynomials of degree* m *is a submodule of* P_m*.*

Proof. It is closed under the action. □

Being submodules of P_m, $\mathrm{id}_{m,\mathrm{Sp}}(A)\subseteq \mathrm{id}_m(A)$ both are direct sums of minimal left ideals.

Given a submodule $L\le P_m$, the corresponding subspace $\hat{L}$ of $F[S_m]$ is a left ideal. Since $F[S_m]$ is semisimple, $\hat{L}$ may be written as

$$\hat{L}=\bigoplus_{\lambda\vdash m}(\hat{L}\cap \mathrm{Type}_\lambda).$$

We call each $\hat{L}\cap\mathrm{Type}_\lambda$ the projection of L to λ.

5.2. **Identities and representations.** While we may be able to decompose the weak identities ideal quite nicely using representation theory, it is not obvious that each projection has an "elegant" representative. The following proposition proves the existence of a relatively simple one.

Proposition 5.2. *Let L be a submodule of P_m. Suppose the projection of L on a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash m$ is nonzero. Then there exists a nonzero multilinear polynomial $f(x_1, \dots, x_m) \in L$ which is fixed under the action of*

$$H = S_{\{1,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{\lambda_1+\cdots+\lambda_{r-1}+1,\dots,m\}}.$$

In other words, f is a multilinearization of a polynomial in r (noncommuting) variables $y_1, \dots, y_r$, where the degree of y_i in each monomial is λ_i.

Proof. Recall that $P_m \cong F[S_m]$. Let $\hat{L}$ be the left ideal of $F[S_m]$ corresponding to L, and let $\hat{L}_\lambda = \hat{L} \cap \mathrm{Type}_\lambda$ be the projection of L on λ, which is a left ideal of Type_λ.

Following the notation of [Hu, Section 3.3]), associate to λ the subgroups P_λ and Q_λ of S_m, fixing the rows and columns respectively in the standard tableau corresponding to λ. We also set

$$a_\lambda = \sum_{\sigma \in P_\lambda} \sigma, \quad b_\lambda = \sum_{\sigma \in Q_\lambda} (-1)^\sigma \cdot \sigma, \quad \text{and } c_\lambda = a_\lambda b_\lambda.$$

Then $c_\lambda F[S_m]$ is an irreducible module V_λ of $F[S_m]$, contained in the representation type Type_λ. In particular, $c_\lambda \in \mathrm{Type}_\lambda$. The elements fixed under the action of the above subgroup H of S_m are precisely the elements t such that $a_\lambda t = |H|t$. Since $a_\lambda^2 = |H|a_\lambda$, we conclude that $a_\lambda c_\lambda = |H|c_\lambda$, and thus every element of the right ideal $c_\lambda \mathrm{Type}_\lambda$ of Type_λ is fixed under H. Take any $0 \neq f \in \hat{L}_\lambda \cap c_\lambda \mathrm{Type}_\lambda$, which exists because left and right ideals in the prime ring Type_λ intersect nontrivially. □

6. Weak identities and the case $n = 2$

Our goal in this section is to describe the minimal (and next to minimal) ℓ-weak identities for the matrix algebra $\mathrm{M}_2(F)$, exemplifying the approach described in Remark 3.5.

6.1. **Polynomials of degree $m = 2$.** Write $a \circ b = ab + ba$. Although the PI-degree of $\mathrm{M}_2(F)$ is 4, the Wagner identity provides a weak central polynomial of degree 2, namely $x_1 \circ x_2$. Nevertheless, the space of 1-weak central polynomials of degree 2 is trivial.

6.2. **Weak identities of degree $m = 3$.** The first nonzero instance of the chain (1) occurs for $m = 3$. Let

$$\psi_i = [x_i, x_j \circ x_k],$$

where $\{i, j, k\}$ is a permutation of the index set $\{1, 2, 3\}$. All the ψ_i are 3-weak identities, and ψ_3 is in fact 2-weak. We also observe that

$$\psi_1 + \psi_2 + \psi_3 = 0. \tag{3}$$

Therefore

$$0 = \mathrm{id}_3^0(\mathrm{M}_2(F)) \quad = \quad \mathrm{id}_3^1(\mathrm{M}_2(F)) \quad \subset \quad \mathrm{id}_3^2(\mathrm{M}_2(F)) \quad \subset \quad \mathrm{id}_3^3(\mathrm{M}_2(F)) \;, \tag{4}$$

where $\mathrm{id}_3^3(\mathrm{M}_2(F)) = \langle \psi_1, \psi_2, \psi_3 \rangle$ is 2-dimensional ($\cong \mathrm{Irr}_{\ydiagram{2,1}}$), and $\mathrm{id}_3^2(\mathrm{M}_2(F)) = \langle \psi_3 \rangle$ is 1-dimensional.

Anticipating the computation of $\mathrm{id}_4^\ell(\mathrm{M}_2(F))$ through Remark 3.5, let us further point out specific submodules of P_3. For an even permutation i, j, k of $1, 2, 3$, let

$$g_i = x_i[x_j, x_k], \qquad g_i' = [x_i, x_j]x_k,$$

and $G = \langle g_1, g_2, g_3 \rangle$, $G' = \langle g_1', g_2', g_3' \rangle$ the generated submodules. Observing that $g_1 + g_2 + g_3 = s_3 = g_1' + g_2' + g_3'$ generates the intersection $G \cap G'$, we conclude that

$$G \cong G' \cong \mathrm{Irr}_{\ydiagram{2,1}} \oplus \mathrm{Irr}_{\ydiagram{1,1,1}}$$

(the latter component is the sign representation). It follows that $G + G' = \mathrm{Type}_{\ydiagram{2,1}} \oplus \mathrm{Type}_{\ydiagram{1,1,1}}$ is the complement of $\langle \sum x_{\sigma 1} x_{\sigma 2} x_{\sigma 3} \rangle = \mathrm{Type}_{\ydiagram{3}}$ in P_3.

6.3. **Weak identities of degree $m = 4$.** We now consider the chain

$$\mathrm{id}_4^0(\mathrm{M}_2(F)) \subset \mathrm{id}_4^1(\mathrm{M}_2(F)) \subset \mathrm{id}_4^2(\mathrm{M}_2(F)) \subset \mathrm{id}_4^3(\mathrm{M}_2(F)) \subset \mathrm{id}_4^4(\mathrm{M}_2(F)). \tag{5}$$

For a permutation i, j, a, b of $1, 2, 3, 4$, let

$$h_{ij} = x_i[x_a \circ x_b, x_j], \qquad h_{ij}' = [x_j, x_a \circ x_b]x_i,$$

on which S_4 acts by the natural action on the indices. Both are weak identities, immediate consequences of the Wagner identity ψ_j. Let $H = \langle h_{ij} \,|\, i \neq j \rangle$ and $H' = \langle h_{ij}' \,|\, i \neq j \rangle$ be the generated submodules of P_4.

Proposition 6.1. *The space of weak identities* $\mathrm{id}_4^4(\mathrm{M}_2(F))$ *has dimension* 15, *isomorphic to* $2\mathrm{Irr}_{\ydiagram{2,1,1}} \oplus 2\mathrm{Irr}_{\ydiagram{3,1}} \oplus \mathrm{Irr}_{\ydiagram{2,2}} \oplus \mathrm{Irr}_{\ydiagram{1,1,1,1}}$. *It is generated as a module by* s_4, $h_{34} = x_3[x_1 \circ x_2, x_4]$, *and* $h_{34}' = [x_4, x_1 \circ x_2]x_3$.

Proof. We apply a computer program to find the dimension as described in [V1], which is indeed 15. We then guess and verify easy-to-describe identities in this space; and analyze the submodule they generate to the extent that its dimension becomes apparent, until we obtain a set of generators.

For every i, it follows from (3) that $\sum_{j \neq i} h_{ij} = \sum_{j \neq i} h_{ij}' = 0$. There are no other relations, so $\dim H = \dim H' = 8$. But since $8 + 8 > 15$, the spaces must intersect. The intersection is most easily computed by passing to the dual space. Elements $\sum \alpha_\sigma \sigma \in H$ are characterized by the "right transposition condition" $\alpha_{ijk\ell} + \alpha_{i\ell kj} = 0$ and the condition $\alpha_{ij_0j_1j_2} + \alpha_{ij_1j_2j_0} + \alpha_{ij_2j_0j_1} = 0$. Likewise H' is characterized by the "left transposition condition" $\alpha_{ijk\ell} + \alpha_{jki\ell} = 0$ and the condition $\alpha_{i_0i_1i_2j} + \alpha_{i_1i_2i_0j} + \alpha_{i_2i_0i_1j} = 0$. So $H \cap H'$ is characterized by the transposition conditions, as well as $\alpha_{ijk\ell} = \alpha_{ji\ell k}$ and $\alpha_{1234} + \alpha_{2314} + \alpha_{3124} = 0$; computation then indicates that $\dim(H \cap H') = 2$. Indeed, acting with $\sum_{\sigma \in K_4} \sigma$, where K_4 is the Klein 4-group, we find the equality $h_{ij} + h_{ji} + h_{k\ell} + h_{\ell k} = h_{ij}' + h_{ji}' + h_{k\ell}' + h_{\ell k}'$ for any partition $ij|k\ell$ of the index set. These are three equalities, each defining an element of $H \cap H'$, whose sum is zero. Thus $H \cap H' \cong \mathrm{Irr}_{\ydiagram{2,2}}$. The characters of H, H' can be computed from the action on the basis, and knowing the characters of S_4 we conclude that $H \cong H' \cong \mathrm{Irr}_{(2,2)} \oplus \mathrm{Irr}_{(2,1,1)} \oplus \mathrm{Irr}_{(3,1)}$ (of dimensions $2+3+3$). It follows that $\langle s_4 \rangle \cong \mathrm{Irr}_{(1,1,1,1)}$ cannot intersect $H + H'$, so that $H + H' + \langle s_4 \rangle$ is of dimension 15, and thus equal to the full space of identities. □

Remark 6.2. *The dimensions in the chain (5) are* $1 < 3 < 8 < 12 < 15$*. The* ℓ*-weak identity spaces are given as follows.*

(3) *The space* $\mathrm{id}_4^3(\mathrm{M}_2(F))$ *of* 3*-weak identities has dimension* 12*, spanned as an* $S_{\{1,2,3\}}$*-module by* $\{[s_3, x_4],\ h_{43},\ h_{34},\ t\}$*, where*

$$t = [x_1 \circ [x_2, x_4], x_3].$$

We have a direct sum decomposition, $\langle [s_3, x_4] \rangle \oplus \langle h_{43} \rangle \oplus \langle h_{34} \rangle \oplus \langle t \rangle$*, with the components isomorphic to* $\mathrm{Irr}_{(1,1,1)}$*,* $\mathrm{Irr}_{(2,1)}$ *(as* $h_{43} + h_{42} + h_{41} = 0$*),* $\mathrm{Irr}_{(3)} \oplus \mathrm{Irr}_{(2,1)}$*, and the regular representation, respectively. Namely,* $\mathrm{id}_4^3(\mathrm{M}_2(F))$ *is twice the regular representation. We also note that* $[s_3, x_4] = \frac{1}{2}(1 + (123) + (132))(34)t$*.*

(2) *The space* $\mathrm{id}_4^2(\mathrm{M}_2(F))$ *of* 2*-weak identities has dimension* 8*, spanned as an* $S_{\{1,2\}}S_{\{3,4\}}$*-module by* $\{s_4, t, h_{34}, q\}$*, where* $q = [x_1 \circ x_3, x_2 \circ x_4] + [x_2 \circ x_3, x_1 \circ x_4]$*. In fact,* $\mathrm{id}_4^2 = \langle s_4 \rangle \oplus \langle t \rangle \oplus \langle h_{34} \rangle \oplus \langle q \rangle$*, of dimensions* $1+4+2+1$ *respectively.*

(1) $\mathrm{id}_4^1(\mathrm{M}_2(F))$ *is the* 3*-dimensional space spanned as an* $S_{\{2,3,4\}}$*-module by* $(34)t = [x_1 \circ [x_2, x_3], x_4]$*. This is a* 1*-weak identity,* $x_1 \circ [x_2, x_3]$ *being central when* $\mathrm{tr}(x_1) = 0$*. In fact,* $(34)t + (24)t + (23)t = s_4$*, explaining how* $\mathrm{id}_4^0 \subset \mathrm{id}_4^1$*.*

(0) $\mathrm{id}_4^0(\mathrm{M}_2(F)) = F \cdot s_4$ *is the well-known* 1*-dimensional space of degree* 4 *identities.*

Remark 6.3. *The spaces of* ℓ*-weak central polynomials of* $\mathrm{M}_2(F)$ *in degree* 4*, for* $\ell = 0, 1, 2, 3, 4$*, have dimensions* $3, 8, 12, 15$ *and* 18*, respectively.*

(The dimensions $3 < 8 < 12 < 15$ follow from Remark 6.2 by Theorem 4.3; and the dimension 18 for the space of weak central polynomials was found, once more, by a computer program).

7. The weak PI-degree of $\mathrm{M}_3(F)$

This section is concerned with weak identities of degree 5 for $\mathrm{M}_3(F)$. We show that there are none if $\mathrm{char}\, F \neq 3$, and describe the weak identities in degree 5 when $\mathrm{char}\, F = 3$.

7.1. **Fields of characteristic not** 3.

Proposition 7.1. *The algebra* $\mathrm{M}_3(F)$ *has no weak identities of degree* 5 *when* $\mathrm{char}\, F \neq 3$.

Proof. Suppose that

$$f(x_1,\dots,x_5)=\sum_{\sigma\in S_5} a_\sigma x_{\sigma(1)}\dots x_{\sigma(5)}$$

is a weak identity for $\mathrm{M}_3(F)$. Note that for all $\pi\in S_5$,

$$f(x_{\pi(1)},\dots,x_{\pi(5)})=\sum_{\sigma\in S_5} a_\sigma x_{\pi(\sigma(1))}\dots x_{\pi(\sigma(5))}=\sum_{\tau\in S_5} a_{\pi^{-1}\tau} x_{\tau(1)}\dots x_{\tau(5)},$$

so permutation of the variables acts on the coefficients from the right by $a_\sigma\cdot\pi=a_{\pi^{-1}\sigma}$. We write permutations by the cycle decomposition.

Substituting $x_1,\dots,x_5=e_{12},e_{23},e_{32},e_{23},e_{31}$, the resulting matrix satisfies

$$f(e_{12},e_{23},e_{32},e_{23},e_{31})_{1,1}=a_1+a_{(2,4)}.$$

Hence $a_{(2,4)}=-a_1$. Applying a permutation $\pi\in S_5$ yields

$$a_{\pi(2,4)}=-a_\pi \tag{6}$$

for every $\pi\in S_5$.

Next, we substitute $x_1,\dots,x_5=e_{13},e_{31},e_{12},e_{23},e_{32}$, and the $(1,2)$ entry of the resulting matrix is

$$a_1+a_{(2,5,3,4)}+a_{(1,3,2,4)}=0.$$

Using (6) and acting with an arbitrary $\pi\in S_5$, we get

$$a_\pi-a_{\pi(3,4,5)}-a_{\pi(1,3,2)}=0. \tag{7}$$

Tracing this equation over $(3,4,5)$ (that is applying $(3,4,5)$ and $(3,5,4)$, then summing the three equations) and applying $(1,2,3)$ yields the equation

$$a_1+a_{(1,4,5)}+a_{(1,5,4)}=0. \tag{8}$$

We now substitute $x_1,\dots,x_5=e_{13},e_{32},e_{23},e_{22}-e_{33},e_{31}$. The $(1,1)$ entry of the resulting matrix is

$$-a_1+a_{(3,4)}-a_{(2,4,3)}=0.$$

Using (6), we see that

$$a_1-a_{(3,4)}-a_{(2,3)}=0.$$

Applying $(1,3)$ yields the equation $a_{(1,3,2)}=a_{(1,3)(2,3)}=a_{(1,3)}-a_{(1,3,4)}$. We substitute this expression in (7) (with $\pi=\mathrm{Id}$) to achieve

$$a_1-a_{(3,4,5)}-a_{(1,3)}+a_{(1,3,4)}=0.$$

By applying $(1,3)$ on the last equation, we get

$$a_{(1,3)}-a_{(1,3,4,5)}-a_1+a_{(3,4)}=0.$$

Summing up the last two equations, we get

$$-a_{(3,4,5)}+a_{(1,3,4)}-a_{(1,3,4,5)}+a_{(3,4)}=0.$$

Applying $(3,4)$ means

$$a_1+a_{(1,4)}-a_{(4,5)}-a_{(1,4,5)}=0.$$

Applying $(1,5)$ yields the equation

$$a_{(1,5)} + a_{(1,4,5)} - a_{(1,5,4)} - a_{(1,4)} = 0.$$

Subtracting the second equation from the first, we see that

$$a_1 - 2a_{(1,4,5)} + a_{(1,5,4)} = -2a_{(1,4)} + a_{(1,5)} + a_{(4,5)}.$$

So, using (8),

$$3a_{(1,4,5)} = 3a_{(1,4)},$$

and $a_1 = a_{(4,5)}$ since we assume $\operatorname{char} F \neq 3$. We may again apply $\pi \in S_5$ to get

(9) $$a_{\pi(4,5)} = a_\pi.$$

We now see that using (6) and (9),

$$a_{\pi(2,5)} = a_{\pi(2,4)(4,5)(2,4)} = a_\pi,$$

but also

$$a_{\pi(2,5)} = a_{\pi(4,5)(2,4)(4,5)} = -a_\pi,$$

implying that $a_\pi = 0$ for all $\pi \in S_5$. Hence $f = 0$, as required. □

Since there are identities of degree 6, we conclude that the “weak PI degree” of $\mathrm{M}_3(F)$ is 6:

Corollary 7.2. *The minimal degree of a weak identity of* $\mathrm{M}_3(F)$ *is* 6*.*

In Section 8 we indicate that in degree 6 there are weak identities other than the standard identity, so the “strict weak PI degree” of $\mathrm{M}_3(F)$ is 6 as well.

7.2. **The case** $\operatorname{char} F = 3$**.** Proposition 7.1 holds when $\operatorname{char} F \neq 3$. Interestingly, the situation is quite different in characteristic 3.

Proposition 7.3. *Assume* $\operatorname{char} F = 3$*. The standard identity* s_4 *is a weak central identity of* $\mathrm{M}_3(F)$*. In particular* $\mathrm{M}_3(F)$ *has* 4*-weak identity of degree* 5*, namely*

$$[s_4(x_1,\dots,x_4),x_5].$$

Proof. The value of $s_4(x_1,\dots,x_4)$ under substitution of matrix units e_{ij} $(i \neq j)$ or matrices of the form $e_{ii} - e_{jj}$, results in either $\pm 3e_{ij}$ $(i \neq j)$ or $\pm(1 - 3e_{ii})$. Over a field of characteristic 3, this implies all values of s_4 under weak substitutions are central. Hence $[s_4(x_1,\dots,x_4),x_5]$ is a 4-weak identity.

(Incidentally, if even one variable is strong, the $\mathbb{Z}$-span of $s_4(x_1,\dots,x_4)$ is the zero-trace part of $\mathrm{M}_3(\mathbb{Z})$; so $[s_4(x_1,\dots,x_4),x_5]$ is not 3-weak). □

For any m, let $\psi_m = [s_{m-1}(x_1,\dots,x_{m-1}),x_m]$. Let $F \oplus N_0$ be the natural representation of S_m, decomposed into the trivial module and its irreducible complement.

Proposition 7.4. *The* S_m*-module generated by* ψ_m *is:*

1. $F[S_m]\psi_m \cong N_0 \otimes \operatorname{sgn}$ *when* m *is odd.*
2. $F[S_m]\psi_m \cong (F \oplus N_0) \otimes \operatorname{sgn}$ *when* m *is even.*

Proof. Fix $\sigma = (1\,2\,3\,\dots m)$. Since S_{m-1} alternates ψ_m, the module is generated by the cyclic permutations $\sigma^j \psi_m$.

Every monomial appears in exactly two of the polynomials $\sigma^j \psi_m$. When m is odd, the signs are opposite. Therefore $\sum \sigma^j \psi_m = 0$ and there are no other relations, so the module is $N_0 \otimes \operatorname{sgn}$. When m is even, the signs are equal (opposite) when the difference of the indices of the first and last variables is even (odd); so the $\sigma^j \psi_m$ are linearly independent, and the module is $N \otimes \operatorname{sgn}$. $\square$

Going back to the case $m = 5$ when $\operatorname{char} F = 3$,

$$U = F[S_5] \cdot [s_4(x_1, x_2, x_3, x_4), x_5] \tag{10}$$

is 4-dimensional, isomorphic as an S_5-module to the nontrivial irreducible component of the natural representation, tensored with the sign character.

Proposition 7.5. *Assume* $\operatorname{char} F = 3$. *The space* $\mathrm{id}_5^5(\mathrm{M}_3(F))$ *of weak identities of degree* 5 *has dimension* 5. *As an* S_5*-module, the representation space is uniquely an extension*

$$0 \longrightarrow U \longrightarrow \mathrm{id}_5^5(\mathrm{M}_3(F)) \longrightarrow F \longrightarrow 0$$

where U *is given in (10) and* F *denotes the trivial module.*

Proof. The dimension is based on a Sage program. We find the 4-weak identity

$$\begin{aligned} \varphi \;\; = \;\; & [x_1[x_2, x_3 \circ x_4] + x_2[x_1, x_3 \circ x_4] - x_3[x_4, x_1 \circ x_2] - x_4[x_3, x_1 \circ x_2],\ x_5] + \\ & + \sum_{\sigma \in S_4} x_{\sigma(1)}[x_5,\, x_{\sigma(2)} x_{\sigma(3)}] x_{\sigma(4)}, \end{aligned}$$

generating $\mathrm{id}_5^5(\mathrm{M}_3(F))$ as a module; indeed, $\psi_5 = (1 - (23))\varphi$. Notice that $(12)\varphi = (34)\varphi = \varphi$, showing that $\mathrm{id}_5^5(\mathrm{M}_3(F))/U$ is the trivial (and not the sign) module. $\square$

A Sage computation also shows that (when $\operatorname{char} F = 3$) $\mathrm{id}_5^3(\mathrm{M}_3(F)) = 0$, and $\mathrm{id}_5^4(\mathrm{M}_3(F))$ is 2-dimensional, spanned by φ and ψ_5. Again $F\psi_5$ is the unique irreducible S_4-submodule, and $(F\varphi + F\psi_5)/(F\psi_5)$ is the trivial S_4-module.

8. Weak identities for $\mathrm{M}_3(F)$ in degree 6

Assuming $\operatorname{char} F = 0$, in this section we describe the sets $\mathrm{id}_6^\ell(\mathrm{M}_3(F))$ of ℓ-weak identities of $\mathrm{M}_3(F)$ in degree 6, which by Corollary 7.2 is the minimal degree of weak identities.

In [DR] the authors study weak identities (when all variables are weak, namely the case $\ell = 6$) of $\mathrm{M}_3(F)$. Decomposing the S_6-module $\mathrm{id}_6^6(\mathrm{M}_3(F))$ into the representation components, their computations indicate that there are four nonzero summands, whose Young diagrams are $\yng(4,2)$, $\yng(4,1,1)$, $\yng(3,1,1,1)$ and $\yng(1,1,1,1,1,1,1)$.

We correct a minor omission in the literature by observing the following:

Proposition 8.1. *The space* $\mathrm{id}_6^6(\mathrm{M}_3(F))$ *has five nonzero components, namely the above four, as well as* $\yng(2,2,2)$.

In the first subsection we supply complete details on the dimensions of the spaces of weak identities, and in the second subsection we present explicit 4-weak identities and use the Okubo algebra to prove that they indeed have this property.

8.1. **Weak identities of $\mathrm{M}_3(F)$.** We used a Sage program to find an F-basis for each weak identity space $\mathrm{id}_6^{\ell}(\mathrm{M}_3(F))$, and compute the intersection with each representation ideal Type_{λ}. The dimensions of the intersections $\mathrm{id}_6^{\ell}(\mathrm{M}_3(F)) \cap \mathrm{Type}_{\lambda}$ (for the partitions λ with nonzero intersection) are listed in the table below. In all participating representations, $\mathrm{id}_6^{6}(\mathrm{M}_3(F)) \cap \mathrm{Type}_{\lambda}$ happens to have rank 1, so the dimension of the representation is equal to the dimension of the intersection at the bottom line.

ℓ	$\dim \mathrm{id}_6^{\ell}(\mathrm{M}_3(F))$	$(4,2)$	$(4,1,1)$	$(3,1,1,1)$	$(2,2,2)$	(1^7)
0	1	0	0	0	0	1
1	1	0	0	0	0	1
2	1	0	0	0	0	1
3	2	0	0	1	0	1
4	6	1	1	3	0	1
5	15	4	4	6	0	1
6	35	9	10	10	5	1

It follows that there are no 2-weak identities except for the standard identity; and there is a unique 3-weak identity modulo the standard identity (whose explicit description, in an appealing form, remains a challenge). The bottom line proves Proposition 8.1.

8.2. **Halpin's identity and its projections.** For $n = 3$, Halpin's identity from Lemma 4.1 is

$$f(x, z) = [[x, z], [x^3, z]], \tag{11}$$

which (when multilinearized) is a 4-weak identity of $\mathrm{M}_3(F)$, namely we restrict x to have zero trace.

Proposition 8.2. *The (multilinearization of the) polynomials*

$$f'(x, z_1, z_2) = [[x, z_1], [x^3, z_2]] + [[x, z_2], [x^3, z_1]], \tag{12}$$

and

$$f''(x, z_1, z_2) = [x, z_1] \circ [x^3, z_2] - [x, z_2] \circ [x^3, z_1], \tag{13}$$

are the unique (up to scalar) 4*-weak identities of degree* 6 *of* $\mathrm{M}_3(F)$ *corresponding to the components* $(4,2)$ *and* $(4,1,1)$*, respectively.*

Proof. The representation type follows from symmetries, so uniqueness follows from the line $\ell = 4$ in the table above. It remains to show that these are indeed 4-weak identities.

Linearizing z in (11), we get the 4-weak identity f' defined in (12), which can be decomposed as $f' = f_1 + f_2$ where $f_1(x, z_1, x_2) = [x, z_1][x^3, z_2] - [x^3, z_1][x, z_2]$ is the sum of monomials in which z_1 precedes z_2, and $f_2(x, z_1, z_2) = f_1(x, z_2, z_1)$ is the sum of monomials in which z_2 precedes z_1. By [DR, Theorem 1.3(ii)], both f_1 and f_2 are 4-weak identities for $\mathrm{M}_3(F)$. It is easy to verify that $f'' = f_1 - f_2$ is the is the polynomial f'' defined in (13). □

8.3. **Identities from the Okubo algebra.** Some surprising identities of $\mathrm{M}_3(F)$ arise from the Okubo algebra, which we now describe. A nonassociative F-algebra $(A, \star)$ is a **composition algebra** if it is endowed with a nondegenerate quadratic form $N : A \to F$ such that $N(x \star y) = N(x)N(y)$. The algebra is **symmetric** if it further satisfies

$$y \star (x \star y) = (y \star x) \star y = N(y)x. \tag{14}$$

A major example of a symmetric composition algebra is the **Okubo algebra** [MVS], whose underlying vector space is the space $\mathrm{M}_3(F)_0$ of zero-trace matrices. Assuming F has a cubic root of unity which we denote ρ, the multiplication is defined by

$$x \star y = \frac{1-\rho}{3}xy + \frac{1-\rho^2}{3}yx - \frac{1}{3}\,\mathrm{tr}(xy).$$

(There is an analogous description for the case $\rho \notin F$, which does not concern us here). The norm form is $N(x) = -\frac{1}{3}s_2(x)$, where $s_2(x)$ is the second coefficient of the characteristic polynomial of x.

We can now prove the following trace identity:

Proposition 8.3. *Assume* $x, y \in \mathrm{M}_3(F)_0$. *Then*

$$[x^2, y^2] - [y, xyx] = \mathrm{tr}(xy)[x, y].$$

Proof. Write $\alpha = \frac{1-\rho}{3}$ and $\alpha' = \frac{1-\rho^2}{3}$, so that $\alpha + \alpha' = 1$ and $\alpha^2 = \alpha - \frac{1}{3}$, and therefore $\alpha^2 + \alpha'^2 = \alpha\alpha' = \frac{1}{3}$. By assumption,

$$x \star y = \alpha xy + \alpha' yx - \frac{1}{3}\,\mathrm{tr}(xy).$$

Multiplying by y from left, we have

$$\begin{aligned}
y \star (x \star y) &= y \star (\alpha xy + \alpha' yx - \frac{1}{3}\,\mathrm{tr}(xy)) = \\
&= \alpha y(\alpha xy + \alpha' yx - \frac{1}{3}\,\mathrm{tr}(xy)) \\
&\qquad + \alpha'(\alpha xy + \alpha' yx - \frac{1}{3}\,\mathrm{tr}(xy))y - \frac{1}{3}\,\mathrm{tr}(y(\alpha xy + \alpha' yx - \frac{1}{3}\,\mathrm{tr}(xy))) = \\
&= (\alpha^2 + \alpha'^2)yxy + \alpha\alpha'(y^2x + xy^2) - (\alpha + \alpha')\frac{1}{3}\,\mathrm{tr}(xy)y - \frac{1}{3}\,\mathrm{tr}(y(\alpha xy + \alpha' yx)) \\
&= \frac{1}{3}yxy + \frac{1}{3}(y^2x + xy^2) - \frac{1}{3}\,\mathrm{tr}(xy)y - \frac{1}{3}\,\mathrm{tr}(\alpha yxy + \alpha' y^2x).
\end{aligned}$$

Since $y \star (x \star y) = N(y)x$, the above expression commutes with x. Hence

$$\begin{aligned} 0 &= [x, yxy + y^2x + xy^2 - \mathrm{tr}(xy)y] = \\ &= xyxy - yxyx + x^2y^2 - y^2x^2 - \mathrm{tr}(xy)[x, y] \\ &= -[y, xyx] + [x^2, y^2] - \mathrm{tr}(xy)[x, y]. \end{aligned}$$

□

Taking $y = [z, x]$ we get $y \in \mathrm{M}_3(F)_0$ and $\mathrm{tr}(xy) = \mathrm{tr}(x[z, x]) = \mathrm{tr}([xz, x]) = 0$ so Proposition 8.3 gives the 4-weak identity

$$[[z, x], x[z, x]x] - [x^2, [z, x]^2] = 0;$$

but we already know the 4-weak identities, and this is indeed Halpin's identity (11):

Remark 8.4. *We have the tautological identity*

$$[[z, x], x[z, x]x] - [x^2, [z, x]^2] = [[x, z], [x^3, z]]. \tag{15}$$

Indeed, let $y = [x, z]$. Then $xy + yx = [x^2, z]$, and the left hand side is equal to

$$\begin{aligned} [y, xyx] - [x^2, y^2] &= y(xy + yx)x - x(yx + xy)y \\ &= y[x^2, z]x - x[x^2, z]y \\ &= zx^3zx + xz^2x^3 - zxzx^3 + x^3zxz - x^3z^2x - xzx^3z \\ &= [zx^3, zx] - [zx^3, xz] + [x^3z, xz] - [x^3z, zx] \\ &= [[x, z], [x^3, z]]. \end{aligned}$$

9. Matrices of size $n \geq 4$

In Sections 6 and 8 we have seen that $\mathrm{M}_n(F)$ has properly weak identities of degree $2n$ when $n = 2, 3$. Here we show that for $n \geq 4$ the only weak identity of $\mathrm{M}_n(F)$ in degree $2n$ is the standard identity, slightly improving on Amitsur-Levizki [AmL] who proved that s_{2n} is the only identity of $\mathrm{M}_n(F)$ in this degree.

An easy argument, similar to that of [GZ, Lemma 1.10.7], rules out identities of degree $2n - 2$:

Proposition 9.1. *The minimal degree of a weak identity of $\mathrm{M}_n(F)$ is $\geq 2n - 1$.*

Proof. There is a vector space embedding $\mathrm{M}_{n-1}(F) \subseteq \mathrm{M}_n(F)_0$ by sending $a \mapsto (a, -\mathrm{tr}(a))$, which preserves multiplication in the first component. It follows that s_{2n-2} is the only possible identity of degree $< 2n - 1$. But the standard identity s_{2n-2} is ruled out as a weak identity for $\mathrm{M}_n(F)$ by the path $1 \to 2 \to \cdots \to n \to \cdots \to 2 \to 1$. □

9.1. **Shadows of identities.** We begin by developing a simple decomposition technique for multilinear identities.

Definition 9.2. *Let $f \in P_m$ be a multilinear polynomial. Writing*

$$f = \sum_{i \neq j} x_i f_{i,j}(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_m) x_j,$$

for strong variables x_i, x_j, we call each $f_{i,j}$ a ***shadow*** *of f.*

As usual, $\widehat{x_i}$ denotes omission of x_i from the list. Each $f_{i,j}$ is an $(m-2)$-multilinear polynomial (on the variables $\{x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_m\}$). The action of S_m on P_m induces an action on the shadows by

$$(\sigma f)_{\sigma(i),\sigma(j)} = f_{i,j}. \tag{16}$$

Proposition 9.3. *Suppose $f \in P_m$ is an I-weak identity for $\mathrm{M}_n(F)$. Then the shadow $f_{i,j}$ is an $(I \setminus \{i,j\})$-weak identity for $\mathrm{M}_{n-1}(F)$.*

In particular, if $f \in P_m$ is a (weak) identity for $\mathrm{M}_n(F)$, then each $f_{i,j}$ is a (resp. weak) identity for $\mathrm{M}_{n-1}(F)$.

Proof. The latter statement follows from the former by taking $I = \emptyset$ (resp. $\ell = \{1, \dots, m\}$). We view $\mathrm{M}_{n-1}(F) \subseteq \mathrm{M}_n(F)$ in the natural way, embedded in the upper-left corner. Fix $u, v = 1, \dots, n-1$, and substitute $x_i \mapsto e_{nu}$ and $x_j \mapsto e_{vn}$. By substituting matrices from $\mathrm{M}_{n-1}(F)$ into the other variables, we see that

$$f(x_1, \dots, e_{nu}, \dots, e_{vn}, \dots, x_m)_{nn} = f_{i,j}(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_m)_{uv},$$

since any monomial is zero unless e_{nu} appears first and e_{vn} last in the product.

By assumption we are forced to assume the variables whose indices are in I are weak, and this condition for the variables other than x_i, x_j remains on the substitution in $f_{i,j}$. □

For distinct $i, j = 1, \dots, m$, let $[i,j]\ell$ denote the quantity $|\{1, \dots, \ell\} - \{i,j\}|$. Thus $[i,j]\ell \in \{\ell-2, \ell-1, \ell\}$. By Proposition 9.3, if $f \in P_m$ is an ℓ-weak identity for $\mathrm{M}_n(F)$, then $f_{i,j}$ is an $[i,j]\ell$-weak identity for $\mathrm{M}_{n-1}(F)$.

Corollary 9.4. *For every ℓ there is an injective map*

$$\mathrm{id}_m^{\ell}(\mathrm{M}_n(F)) \hookrightarrow \bigoplus_{u \neq v} \mathrm{id}_{m-2}^{[u,v]\ell}(\mathrm{M}_{n-1}(F)).$$

In particular there are injective maps for identities,

$$\mathrm{id}_m^0(\mathrm{M}_n(F)) \hookrightarrow \mathrm{id}_{m-2}^0(\mathrm{M}_{n-1}(F))^{m(m-1)},$$

and for weak identities,

$$\mathrm{id}_m^m(\mathrm{M}_n(F)) \hookrightarrow \mathrm{id}_{m-2}^{m-2}(\mathrm{M}_{n-1}(F))^{m(m-1)}. \tag{17}$$

Corollary 9.5. $\mathrm{PIdeg}^{\infty}(\mathrm{M}_n(F)) \geq 2 + \mathrm{PIdeg}^{\infty}(\mathrm{M}_n(F))$. *Indeed, if we have $\mathrm{id}_{m-2}^{m-2}(\mathrm{M}_{n-1}(F)) = 0$ then $\mathrm{id}_m^m(\mathrm{M}_n(F)) = 0$ by (17).*

Proposition 9.6. *The matrix algebra $\mathrm{M}_4(F)$ has no weak identities of degree 7.*

Proof. For fields of characteristic different than 3, $\mathrm{M}_3(F)$ has no weak identities of degree 5 by Corollary 7.2, so $\mathrm{M}_4(F)$ has no weak identities of degree 7 by Corollary 9.5. For the remaining case of fields of characteristic 3, the claim was verified by a Sage program (computing over $\mathbb{F}_3$). □

Corollary 9.7. *The weak PI degree of* $\mathrm{M}_n(F)$ *is* $2n$ *for all* $n \geq 3$.

Proof. We have that $\mathrm{PIdeg}^{\infty}(\mathrm{M}_n(F)) \leq \mathrm{PIdeg}(\mathrm{M}_n(F)) = 2n$ by Amitsur-Levizki. The lower bound $2n \leq \mathrm{PIdeg}^{\infty}(\mathrm{M}_n(F))$ is given for $n = 4$ in Proposition 9.6, and follows for $n > 4$ by induction applying Corollary 9.5. □

9.2. **Weak identities degree $2n$.** We will now strengthen this result, and show that in the minimal degree $2n$, the standard identity is the only weak identity, namely $\mathrm{id}_{2n}^{2n}(\mathrm{M}_n(F))$ is one dimensional for all $n \geq 4$.

Theorem 9.8. *Let F be a field of characteristic zero. For $n \geq 4$,*

$$\mathrm{id}_{2n}^{2n}(\mathrm{M}_n(F)) = F s_{2n},$$

where s_{2n} is the standard identity.

Proof. We prove this theorem by induction. The case $n = 4$ was verified using a Sage program (computing over $\mathbb{Q}$).

Suppose the proposition is true for some $n \geq 4$. We consider a weak identity $f \in \mathrm{id}_{2n+2}^{2n+2}(\mathrm{M}_{n+1}(F))$. Since this is an S_{2n+2}-module, we may assume f lies in the λ-component of $\mathrm{id}_{2n+2}^{2n+2}(\mathrm{M}_{n+1}(F))$, for some partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash 2n + 2$.

By (17) we have an embedding $\mathrm{id}_{2n+2}^{2n+2}(\mathrm{M}_{n+1}(F)) \hookrightarrow \mathrm{id}_{2n}^{2n}(\mathrm{M}_n(F))^{(2n+2)(2n+1)}$. Let us denote the right-hand side by M. As an S_{2n+2}-module, M is isomorphic to the induced representation $\mathrm{Ind}_{S_{2n}}^{S_{2n+2}}(\mathrm{sgn})$. The irreducible subrepresentations of M are, by Frobenius reciprocity, those whose restriction from S_{2n+2} to S_{2n} is the sign representation of degree $2n$, namely, by the Branching Theorem [GZ, Theorem 2.3.1], the representations $[3^1 1^{2n-1}]$, $[2^2 1^{2n-2}]$, $[2^1 1^{2n}]$ and the sign representation $[1^{2n+2}]$.

By Proposition 5.2, we may assume that f is fixed under the action of

$$H = S_{\{1,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{\lambda_1+\cdots+\lambda_{r-1}+1,\ldots,2n+2\}}.$$

In particular, each shadow $f_{i,j}$ is symmetric under the stabilizer of i, j in H, namely under $H_{ij} = \{\sigma \in H \mid \sigma(i) = i,\ \sigma(j) = j\}$.

On the other hand, by Proposition 9.3, each shadow $f_{i,j}$ is a weak identity for $\mathrm{M}_n(F)$ of degree $2n$. According to the induction hypothesis, this is only possible if

$$f_{i,j} = \alpha_{i,j} \cdot s_{2n}(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{2n+2})$$

for some $\alpha_{i,j} \in F$, and so the shadow is antisymmetric. We conclude that if H_{ij} contains odd permutations, then necessarily $f_{ij} = 0$. In other words for $f_{ij} \neq 0$ it is necessary that removing i and j will leave no more than a single point in each part of λ (reaffirming the list of possible partitions).

CASE I. $\lambda = [31^{2n-1}]$. Here the only nonzero shadows $f_{i,j}$ of f must be those where $1 \leq i,j \leq 3$. Since f must be symmetric with respect to x_1, x_2, x_3, their coefficients $\alpha_{i,j}$ must also be equal to each other, so up to multiplication by a scalar, f has to be

$$f = \sum_{1\leq i,j\leq 3} x_i \cdot s_{2n}(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{2n+2}) \cdot x_j.$$

In other words, f is the multilinearization of

$$\hat{f}(x, x_4, \dots, x_{2n+2}) = x \cdot s_{2n}(x, x_4, \dots, x_{2n+2}) \cdot x.$$

Substitute $x \mapsto e_{11} - e_{22}$ and for the variables $x_4, x_5, \dots, x_{2n+2}$ take the "ladder" matrix units $e_{12}, e_{23}, \dots, e_{n,n+1}, e_{n+1,n}, \dots, e_{32}$. By direct computation, one can verify that

$$\hat{f}(x, x_4, \dots, x_{2n+2})_{1,2} = s_{2n}(x, x_4, \dots, x_{2n+2})_{1,2} = 3,$$

which proves that $\hat{f}$ is not a weak identity for $\mathrm{M}_{n+1}(F)$.

CASE II. $\lambda = (2, 2, 1^{2n-2})$. In this case, f is symmetric with respect to x_1 and x_2 and with respect to x_3 and x_4. The possible nonzero shadows are $f_{i,j}$ where $i \in \{1,2\}$ and $j \in \{3,4\}$, or vice versa. A similar explanation shows that f is the multilinearization of an identity of the form

$$\hat{f} = \alpha \cdot x \cdot s_{2n}(x, y, x_5, \dots, x_{2n+2}) \cdot y + \beta \cdot y \cdot s_{2n}(x, y, x_5, \dots, x_{2n+2}) \cdot x$$

for some $\alpha, \beta \in F$. Set

$$x, y, x_5, \dots, x_{2n+2} = e_{12}, e_{21}, e_{13}, e_{31}, \dots, e_{1,n+1}, e_{n+1,1}.$$

A simple calculation shows that $s_{2n}(e_{12}, e_{21}, e_{13}, e_{31}, \dots, e_{1,n+1}, e_{n+1,1}) = n!e_{11} - \sum_{k=2}^{n+1}(n-1)!e_{kk}$. Hence $\hat{f}(e_{12}, e_{21}, e_{13}, e_{31}, \dots, e_{1,n+1}, e_{n+1,1}) = -\alpha(n-1)!e_{11} + \beta n!e_{22}$, showing that $\alpha = \beta = 0$.

CASE III. $\lambda = (2, 1^{2n})$. In a similar manner, one may see that f must be a multilinearization of a weak identity of the form

$$\begin{aligned} \hat{f}(x, x_1, \dots, x_{2n}) &= \sum_{i=1}^{2n} \alpha_i \, x \, s_{2n}(x, x_1, \dots, \widehat{x_i}, \dots, x_{2n}) \, x_i + \\ &+ \sum_{i=1}^{2n} \beta_i \, x_i \, s_{2n}(x, x_1, \dots, \widehat{x_i}, \dots, x_{2n}) \, x + \\ &+ \gamma \, x \, s_{2n}(x_1, \dots, x_{2n}) \, x. \end{aligned}$$

Fixing $1 \leq j < 2n$, we substitute x_j in place of x_{j+1} and keep all the other variables in place. Most summands vanish, and the resulting polynomial is

$$\begin{aligned} &(\alpha_j + \alpha_{j+1}) \, x \, s_{2n}(x, x_1, \dots, \widehat{x_{j+1}}, \dots, x_{2n}) \, x_j + \\ &+(\beta_j + \beta_{j+1}) \, x_j \, s_{2n}(x, x_1, \dots, \widehat{x_{j+1}}, \dots, x_{2n}) \, x. \end{aligned}$$

This must be a weak identity for $\mathrm{M}_{n+1}(F)$. Since its multilinearization is symmetric with respect to two pairs of variables, it lies in the component of $(2, 2, 1^{2n-2})$, hence must be zero by CASE II. This shows that $\alpha_{j+1} = -\alpha_j$ and $\beta_{j+1} = -\beta_j$. But the argument holds for all j, so $\alpha_i = (-1)^{i-1}\alpha_1$ and $\beta_i = (-1)^{i-1}\beta_1$.

Next we substitute $x_1 = x$. Again, most terms become zero, and the result is

$$(\alpha_1 + \beta_1 + \gamma)\, x\, s_{2n}(x, x_2, \dots, x_{2n})\, x.$$

This should be a weak identity for $\mathrm{M}_{n+1}(F)$ lying in the component of $(3, 1^{2n-1})$, and by CASE I must be zero. This proves that $\alpha_1 + \beta_1 + \gamma = 0$.

We have therefore shown that our weak identity has the form

$$\begin{aligned}
\hat{f} &= \alpha \sum_{i=1}^{2n} (-1)^{i-1}\, x\, s_{2n}(x, x_1, \dots, \widehat{x_i}, \dots, x_{2n})\, x_i + \\
&+ \beta \sum_{i=1}^{2n} (-1)^{i-1}\, x_i\, s_{2n}(x, x_1, \dots, \widehat{x_i}, \dots, x_{2n})\, x - \\
&- (\alpha + \beta)\, x\, s_{2n}(x_1, \dots, x_{2n})\, x = \\
&= \alpha \sum_{i=1}^{2n} x\, s_{2n}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{2n})\, x_i + \\
&+ \beta \sum_{i=1}^{2n} x_i\, s_{2n}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{2n})\, x - \\
&- (\alpha + \beta)\, x\, s_{2n}(x_1, \dots, x_{2n})\, x
\end{aligned}$$

for appropriate $\alpha, \beta \in F$.

We substitute

$$x, x_1, x_2, \dots, x_{2n} = e_{12} + e_{23}, e_{12}, e_{21}, \dots, e_{1,n+1}, e_{n+1,1}.$$

We know that $s_{2n}(x_1, \dots, x_{2n}) = n! e_{11} - (n-1)! \sum_{k=2}^{n+1} e_{kk}$, so $x\, s_{2n}(x_1, \dots, x_{2n})\, x = -(n-1)! e_{13}$. We next compute $s_{2n}(x_1, \dots, x_{i-1}, e_{23}, x_{i+1}, \dots, x_{2n})$. Consider the directed graph G_i on the vertices $1, 2, \dots, n+1$, with an edge $j \to j'$ if and only if $e_{j,j'}$ appears in the list $x_1, \dots, \widehat{x_i}, \dots, x_{2n}, e_{23}$ after the substitution above. Any nonzero summand in the expression $s_{2n}(x_1, \dots, x_{i-1}, e_{23}, x_{i+1}, \dots, x_{2n})$ corresponds to an Eulerian path in G_i. We consider the following cases:

- $i = 2\ell - 1$ is odd, in which case $x_i = e_{1,\ell+1}$. Then $\deg^-(1) - \deg^+(1) = 1$, so any Hamiltonian path must end at 1. But if $\ell \neq 2$, we also have $\deg^-(\ell+1) - \deg^+(\ell+1) = 1$, so G_i has no hamiltonian path. There are two types of Hamiltonian paths in G_3: those that begin with $2 \to 3 \to 1$, and those that begin with $2 \to 1$. One can see that each path of the first type contributes $+1$ to the sum, and each path of the second type contributes -1 to the sum. Since their number is identical, the result is 0.
- $i = 3$. We want to compute $s_{2n}(e_{12}, e_{21}, e_{23}, e_{31}, e_{14}, \dots, e_{n+1,1})$. Using the same considerations, every Hamiltonian path must start at 2 and end at 1.
- $i = 2\ell$ is even, in which case $x_i = e_{\ell+1,1}$. But then $\deg^-(1) - \deg^+(1) = -1$, and also $\deg^-(3) - \deg^+(3) = -1$ (or -2 if $i = 4$), which again shows that G_i has no Hamiltonian path.

To conclude, we know that $s_{2n}(x_1, \dots, x_{i-1}, e_{23}, x_{i+1}, \dots, x_{2n}) = 0$ for all i. Hence, for $i > 1$ we have

$$\begin{aligned} s_{2n}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{2n}) &= s_{2n}(e_{12}, x_2, \dots, x_{i-1}, e_{12}, x_{i+1}, \dots, x_{2n}) + \\ &+ s_{2n}(x_1, \dots, x_{i-1}, e_{23}, x_{i+1}, \dots, x_{2n}) = 0, \end{aligned}$$

and for $i = 1$ we have

$$\begin{aligned} s_{2n}(x, x_2, \dots, x_{2n}) &= s_{2n}(e_{12}, x_2, \dots, x_{2n}) + s_{2n}(e_{23}, x_2, \dots, x_{2n}) = \\ &= s_{2n}(x_1, x_2, \dots, x_{2n}) = n! e_{11} - (n-1)! \sum_{k=2}^{n+1} e_{kk}. \end{aligned}$$

The appropriate summands are thus

$$\begin{aligned} x\, s_{2n}(x, x_2, \dots, x_{2n})\, x_1 &= (e_{12} + e_{23})\, s_{2n}(x, x_2, \dots, x_{2n})\, e_{12} = 0 \\ x_1\, s_{2n}(x, x_2, \dots, x_{2n})\, x &= e_{12}\, s_{2n}(x, x_2, \dots, x_{2n})\, (e_{12} + e_{23}) = -(n-1)! e_{13}. \end{aligned}$$

Therefore, the substitution above in $\hat{f}$ yields a matrix whose $(1, 3)$ component is $(n-1)!\alpha$, hence $\alpha = 0$. Similarly, one may show that $\beta = 0$, so $\hat{f} = 0$ as required.

In conclusion, we are left with the case where $\lambda = (1^{2n+2})$, which indeed corresponds to the standard identity s_{2n+2}. □

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