COMPOSITION ALGEBRAS AND CYCLIC p-ALGEBRAS IN CHARACTERISTIC 3

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Abstract. Given a central simple algebra A of degree 3 over a field of characteristic 3, we prove that there is a unique symmetric composition algebra extending the commutator operation on the trace-zero part modulo scalars. This is analogous to Okubo's construction of symmetric composition algebras in the case of characteristic not 3. We apply the composition algebra tools to obtain a classification of maximal 3-central spaces and maximal Galois hyperplanes of A, and prove a new common slot lemma for such algebras.

1. Introduction

Composition algebras are a well-known class of nonassociative algebras, characterized by the existence of a multiplicative nondegenerate quadratic norm form. The unital composition algebras are known as Hurwitz algebras: the field itself, quadratic étale algebras, quaternion algebras and Cayley algebras. Any finite dimensional (nonunital) composition algebra has dimension 1, 2, 4 or 8. See [9, Sec. 33] for a thorough overview.

The bilinear form associated to the norm form N of the algebra $(C, \star)$ is defined by $\mathrm{B}(a, b) = \mathrm{N}(a + b) - \mathrm{N}(a) - \mathrm{N}(b)$. If the form is associative, namely $\mathrm{B}(a \star b, c) = \mathrm{B}(a, b \star c)$, the algebra is called symmetric. Equivalently, the algebra is symmetric if

$$t \star (a \star t) = (t \star a) \star t = \mathrm{N}(t)a, \tag{1}$$

for every $t, a \in C$.

In dimension ≥ 2, a symmetric composition algebras is never unital. On the other hand if C has a unit e, then $\bar{a} + a = \mathrm{B}(a, e)e$ defines an involution on C, and the composition algebra $(C, \bar{\star})$, with $a \bar{\star} b = \bar{a} \star \bar{b}$,

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which is always symmetric, is called a para-Hurwitz (para-quaternion, para-Cayley) algebra.

Another construction of symmetric composition algebras was discovered by Okubo [10]: assume $\operatorname{char} F \neq 3$ and assume $\rho \in F$ is a third root of unity. Let A be a central simple algebra A of degree 3, and let $\operatorname{tr}: A \to F$ be the reduced trace. Then, $A_0 = \{a \in A\colon \operatorname{tr}(a) = 0\}$ is a symmetric composition algebra with respect to the operation

$$a \star b = \frac{1-\rho}{3}ab + \frac{1-\rho^2}{3}ba - \frac{1}{3}\operatorname{tr}(ab), \tag{2}$$

where the norm form is a multiple of the trace form. There is an analogous construction for the case where F does not have roots of unity.

Symmetric composition algebras were studied by Petersson, Okubo, Mynug, Osborn, Faulkner, Elduque and Perez, and others, culminating in a complete classification in [3]. A symmetric composition algebra of dimension 8 is either para-Cayley, or an algebra of the form $C_{\lambda,\mu}$, as described in [3, Thm 5.1]. When char $\neq 3$, the $C_{\lambda,\mu}$ are the algebras associated by Okubo to cyclic algebras (see [9, Thm. 34.37]).

In a recent paper [8] we used the work of van der Blij and Springer [12] on the split Cayley algebra to study isotropic spaces in a symmetric composition algebra (isotropicity is always with respect to the norm form). Applying this to the Okubo algebras, we were able to classify the maximal 3-central subspaces of a central simple algebra of degree 3 in the nonmodular case, namely over a field of characteristic not 3.

In this paper we consider the modular case: algebras of degree 3 over a field of characteristic 3. The structure of such algebras, and some problems related to their standard generators, are described in Section 2. The crucial difference from the nonmodular case is that now $F \subseteq A_0$ is the radical of the quadratic trace form, so we are forced to consider the quotient vector space A_0/F, of dimension 7. In Theorem 3.1 we show that there is a unique symmetric composition algebra structure $C(A)$ on $F \oplus (A_0/F)$ whose operation restricts to the commutator on the second component. Twisting the algebra via Kaplansky's trick (see [9, Prop. 33.27]) we obtain the split Cayley algebra, which allows one to show that the symmetric composition algebra constructed before is the split para-Cayley algebra. This is described in Section 4. Thus all the algebras $C(A)$ are isomorphic to each other, in sharp contrast with the fact that A can be recaptured, up to opposite, from its Lie product [6, Thm. X.10].

Nevertheless, the maximal isotropic subspaces of $C(A)$, which are studied in Section 5, shed light on special subspaces of A. Indeed, the maximal 3-central and the maximal Galois hyperplanes of A are fully described in Section 6.

A central simple algebra of degree 3 over a field of characteristic 3 can be presented in the form

$$[\alpha, \beta) = F[x, y \mid x^3 - x = \alpha,\ y^3 = \beta,\ yxy^{-1} = x + 1].$$

Let us say that two presentations of the algebra are neighbors if they have a common slot, namely the same α or β. It was shown in [13] that the distance between two presentations of the same algebra is at most *seven*. In Section 7 we apply information on maximal subspaces in $C(A)$ to show that if $[\alpha, \beta)$ is a given presentation, then every presentation is at distance at most *five* from $[\alpha, \beta)$ or $[-\alpha, \beta^{-1})$ (in fact this is proved, as in [13], at the level of the generators).

Finally in Section 8 we interpret the strong orthogonality relation, which was introduced for an arbitrary symmetric composition algebra in [8], for the algebras $C(A)$, showing that every two isotropic elements are at distance at most 4 with respect to this relation.

2. Central simple algebras of degree 3

Let F be a field of prime characteristic p, and A a central simple algebra of degree p over F. Thus A is either a division algebra, or the matrix algebra $\mathrm{M}_p(F)$. For general background on p-algebras (i.e. central simple algebras of degree a power of p over a field with characteristic p), see [7].

A pair of elements $x, y \in A$, with y invertible, and such that

$$yxy^{-1} = x + 1 \tag{3}$$

is called a *standard pair of generators*. Indeed in this case $A = F[x, y]$, $F[x]$ is a Galois extension of F with $x^p - x \in F$, $F[y]$ is an nonseparable subalgebra with $y^p \in F$, and A is determined, up to isomorphism, by the constants $\theta = x^p - x$ and $\beta = y^p$. A noncentral element $y \in A$ such that $y^p \in F$ is called *p-central*. Every nonseparable subfield is of the form $F[y]$ where y is p-central, and every Galois subfield is of the form $F[x]$ where $x^p - x \in F$; we call such x a *standard Galois element*.

These two types complement each other. For every standard Galois element x there is an element y, and for every p-central y there is an element x, such that (3) holds.

If such a pair exists then the algebra is *cyclic*. For $p = 2, 3$ every p-algebra of degree p is cyclic; for $p > 3$, no counterexamples are known. Since we will take $p = 3$, we assume from now on that the algebra is cyclic. The presentation of A as a so-called p-symbol $[\theta, \beta)$ is by no means unique. In order to solve the isomorphism problem for cyclic p-algebras of degree p, one has to understand the various presentation of a given algebra, or better yet, the various standard pairs of generators.

To study p-central elements in A, we consider p-central subspaces, which are by definition subspaces $V \subseteq A$ such that $v^p \in F$ for every $v \in V$. Since $\operatorname{char} F = p$, if V is p-central then so is $F + V$, so we may assume a p-central space contains the center. The tools used to study p-central elements will be used to study standard Galois elements as well.

When $p = 2$ there is not much to do. The dimension is $[A\!:\!F] = 4$, and $A_0 = \{a \in A\colon \operatorname{tr}(a) = 0\}$ is a 3-dimensional space, which is clearly the unique maximal 2-central subspace in A. Taking any standard pair of generators x, y, $A_0 = F + Fy + Fxy$, and every element x' in the translation $x + A_0$ is standard Galois, namely $x'^2 - x' \in F$.

From now on we assume $p = 3$. Let us write

$$\mathrm{X}_A = \left\{x \in A\colon\ x^3 - x \in F,\ x \notin F\right\}$$

for the set of standard Galois elements,

$$\mathrm{Y}_A = \left\{y \in A^\times\colon\ y^3 \in F,\ y \notin F\right\}$$

for the set of 3-central elements, and

$$\mathrm{XY}_A = \left\{(x, y) \in \mathrm{X}_A \times \mathrm{Y}_A\colon\ yxy^{-1} = x + 1\right\}$$

for the set of standard pairs of generators of A.

We immediately observe that $\mathrm{X}_A, \mathrm{Y}_A \subseteq A_0$. Moreover for $x, y \in A_0$ ($x \notin F$), $x^3 - x \in F$ iff $\operatorname{tr}(x^2) = -1$, and $y^3 \in F$ iff $\operatorname{tr}(y^2) = 0$. So we are led to consider the quadratic trace form, which is degenerate, with $F \subseteq A_0$ as the radical. Indeed, the trace form is well defined on the quotient vector space A_0/F.

3. Composition algebras from central simple algebras

Let F be a field of characteristic 3, and let A be a central simple algebra of degree 3 over F. Let $A_0 = \{u \in A\colon \operatorname{tr}(u) = 0\}$. As explained above, the quadratic trace form is well defined and nondegenerate on the quotient vector space $W_0 = A_0/F$. We study solutions to $\operatorname{tr}(x^2) = -1$ by homogenization to the form $\alpha^2 + \operatorname{tr}(x^2)$ on the 8-dimensional space $C = F\zeta \oplus W_0$, where ζ is a formal basis element spanning the first component. The goal is to define a binary operation $\star$ so that $(C, \star)$ is a symmetric composition algebra, similarly to Okubo's construction in the case $\operatorname{char} F \neq 3$. As we will see in (5) below, the norm form of this new algebra is indeed $\alpha^2 + \operatorname{tr}(x^2)$.

Not only Okubo's coefficients $\frac{1-\rho}{3}$ and $\frac{1-\rho^2}{3}$ in (2) have no meaning in characteristic 3, the multiplication and its reverse are not defined on A_0/F. However the commutator is well defined on A_0/F, since $[u+\delta, u'+\delta'] = [u, u']$ for every $\delta, \delta' \in F$, and up to a scalar multiple this is the only linear combination of $(u, u') \mapsto uu'$ and $(u, u') \mapsto u'u$ which is well defined there.

Theorem 3.1. *Let $W_0 = A_0/F$ as above. Let C be the vector space $C = F\zeta \oplus W_0$. Then there is a unique symmetric composition algebra structure $\star\colon C \times C \to C$ such that the diagram*

$$\begin{tikzcd} C \times C \arrow[r, "\star"] & C \arrow[d, two heads] \\ W_0 \times W_0 \arrow[u, hook] \arrow[r, "{[\,,\,]}"] & W_0 \end{tikzcd}$$

commutes, where the vertical arrows are the natural embedding and projection.

The multiplication in this algebra, which we denote $C = C^-(A)$, is given by

$$(\alpha\zeta + u) \star (\alpha'\zeta + u') = (\alpha\alpha' - \mathrm{tr}(uu'))\zeta + (-\alpha u' - \alpha' u + [u, u']), \tag{4}$$

and the norm form is

$$\mathrm{N}(\alpha\zeta + u) = \alpha^2 + \mathrm{tr}(u^2). \tag{5}$$

In particular $F\zeta$ is the orthogonal complement of W_0 in (C, N).

An interesting feature of this result is that the trace form, which plays a crucial role in the applications, is completely absent from the assumptions. This mystery is somewhat explained by the following consequence of the Cayley-Hamilton identity

$$u^3 - \mathrm{tr}(u)u^2 + \frac{1}{2}(\mathrm{tr}(u)^2 - \mathrm{tr}(u^2))u = \det(u) \tag{6}$$

for 3×3 matrices (assuming characteristic not 2). Linearization gives the weak trace central identity

$$2(uvv + vuv + vvu) - 2\mathrm{tr}(uv)v - \mathrm{tr}(v^2)u; \tag{7}$$

namely, for every field K and every $u, v \in \mathrm{M}_3(K)$ of trace zero, the quantity in (7) is central. Extending scalars, this identity holds in any central simple algebra of degree 3. Our key observation is that since $[v, [u, v]] = 3vuv - (uvv + vuv + vvu)$ and we specialize to characteristic 3, the central identity (7) gives the equality

$$[v, [u, v]] = \mathrm{tr}(v^2)u - \mathrm{tr}(uv)v, \tag{8}$$

in W_0, for every $u, v \in W_0$.

Proof of Theorem 3.1. Let us write down the most general binary operation $\star : C \times C \to C$. There are eight separate parts: $F\zeta \times F\zeta \to F\zeta$, $F\zeta \times F\zeta \to W_0$, $F\zeta \times W_0 \to F\zeta$, and so on. We thus have

(a) A scalar $\theta_0 \in F$ and a vector $m_0 \in W_0$;
(b) Linear functionals $\phi, \phi' : W_0 \to F$;
(c) Linear operators $T, T' : W_0 \to W_0$; and
(d) A bilinear form $b : W_0 \times W_0 \to F$;

so that the product in C is

$$\begin{aligned}(\alpha\zeta + u) \star (\alpha'\zeta + u') &= (\theta_0\alpha\alpha' + \alpha\phi(u') + \alpha'\phi'(u) + b(u, u'))\zeta \\ &\quad +(\alpha\alpha' m_0 + \alpha T(u') + \alpha' T'(u) + [u, u']).\end{aligned}$$

Notice that the heaviest component, the map $W_0 \times W_0 \to W_0$, is by assumption the commutator.

Let L_v and R_v denote the operations of left and right multiplication by v. Then (1) can be written as

$$L_t R_t = R_t L_t = \mathrm{N}(t)\mathrm{id}_C. \tag{9}$$

We need to solve this symmetry condition for the ingredients displayed above. Taking $t = v \in W_0$, (9) becomes

$$L_v R_v(\alpha\zeta + u) = R_v L_v(\alpha\zeta + u) = \mathrm{N}(v)(\alpha\zeta + u)$$

for every $\alpha \in F$ and $u \in W_0$. Using the multiplication formulas

$$L_v(\alpha\zeta + u) = v \star (\alpha\zeta + u) = (\alpha\phi'(v) + b(v, u))\zeta + (\alpha T'(v) + [v, u])$$

and

$$R_v(\alpha\zeta + u) = (\alpha\zeta + u) \star v = (\alpha\phi(v) + b(u, v))\zeta + (\alpha T(v) + [u, v]),$$

this condition spreads out to

$$\phi(v)\phi'(v) + b(v, T(v)) = \mathrm{N}(v) \tag{10}$$
$$\phi(v)\phi'(v) + b(T'(v), v) = \mathrm{N}(v) \tag{11}$$
$$b(u, v)\phi'(v) + b(v, [u, v]) = 0 \tag{12}$$
$$b(v, u)\phi(v) + b([v, u], v) = 0 \tag{13}$$
$$\phi(v)T'(v) + [v, T(v)] = 0 \tag{14}$$
$$\phi'(v)T(v) + [T'(v), v] = 0 \tag{15}$$
$$b(u, v)T'(v) + [v, [u, v]] = \mathrm{N}(v)u \tag{16}$$
$$b(v, u)T(v) + [[v, u], v] = \mathrm{N}(v)u. \tag{17}$$

Substituting (8) in (16) we get

$$b(u, v)T'(v) - \mathrm{tr}(uv)v = (N(v) - \mathrm{tr}(v^2))u,$$

so fixing v we conclude that

$$N(v) = \mathrm{tr}(v^2) \tag{18}$$

by taking u outside the linear subspace spanned by $T'(v)$ and v. We then have

$$b(u, v)T'(v) = \mathrm{tr}(uv)v.$$

For every $v \neq 0$ there is some u such that $\mathrm{tr}(uv) \neq 0$, forcing $0 \neq T'(v) \in Fv$. But a linear operator which preserves all the directions must be multiplication by a scalar, so for some $\gamma \in F^\times$,

$$T'(v) = \gamma^{-1}v, \tag{19}$$

and then

$$b(u, v) = \gamma\mathrm{tr}(uv). \tag{20}$$

In the same manner we get from (17) that

$$b(v, u)T(v) = \mathrm{tr}(uv)v$$

so T is also multiplication by a scalar, but taking $u = v$ in (16) and (17) shows that this is the same scalar, namely

$$T(v) = \gamma^{-1}v. \tag{21}$$

In particular $T(v)$ and $T'(v)$ commute with v, so (14) and (15) prove that

$$\phi(v) = \phi'(v) = 0. \tag{22}$$

Collecting all this information, the multiplication formula becomes

$$\begin{aligned}(\alpha\zeta + u) \star (\alpha'\zeta + u') &= (\theta_0\alpha\alpha' + \gamma\mathrm{tr}(uu'))\zeta \\ &\quad +(\alpha\alpha' m_0 + \alpha\gamma^{-1}u' + \alpha'\gamma^{-1}u + [u, u']).\end{aligned}$$

It remains to determine θ_0 and m_0 (and γ), so let us apply (1) to the vector $t = \zeta + u$. Using (8) and the obvious fact that $\mathrm{tr}([u, w]u) = 0$, the symmetry condition

$$((\zeta + u) \star (\alpha\zeta + w)) \star (\zeta + u) = \mathrm{N}(\zeta + u) \cdot (\alpha\zeta + w)$$

gives the four equations

$$(1 + \theta_0\gamma)\mathrm{tr}(uw) = 0, \tag{23}$$

$$\theta_0^2 + \gamma\mathrm{tr}(m_0u) + \mathrm{tr}(u^2) = \mathrm{N}(\zeta + u), \tag{24}$$

$$(\theta_0 + \gamma^{-1})(m_0 + \gamma^{-1}u) + [m_0, u] = 0, \tag{25}$$

$$\gamma\mathrm{tr}(uw)m_0 + \gamma^{-2}w + \mathrm{tr}(u^2)w = \mathrm{N}(\zeta + u)w. \tag{26}$$

Choosing some u and w with $\mathrm{tr}(uw) \neq 0$, Equation (23) gives

$$\theta_0 = -\gamma^{-1}, \tag{27}$$

but then by (25) we have that $[m_0, u] = 0$ for every $u \in W_0$. Lifting this to A_0, we can define a functional $\varphi : A_0 \to F$ by $\varphi(u) = [m_0, u] \in F$, whose kernel has dimension at least 7, forcing m_0 to commute with a 7-dimensional subspace of A, so $m_0 \in F$. Back to W_0, we have

$$m_0 = 0. \tag{28}$$

Replacing ζ by $-\gamma\zeta$, the multiplication formula scales to the one given in (4), proving the main statement.

From (24) we then obtain the norm formula (5), so the associated bilinear form is

$$\mathrm{B}(\alpha\zeta + u, \alpha'\zeta + u') = -\alpha\alpha' - \mathrm{tr}(uu'). \tag{29}$$

The multiplication formula (4), or equivalently the ingredients defined in (18)–(21), (27), (28) with $\gamma = -1$, satisfy the conditions (10)–(17) and (23)–(26). This proves that $(C, \star, N)$ is in fact a symmetric composition algebra, and completes the proof of Theorem 3.1. $\square$

Inside $C^-(A)$, W_0 can be recaptured as the orthogonal complement of the center $F\zeta$. Of course it is not a subalgebra: $u \star u' = -\mathrm{tr}(uu')\zeta + [u, u']$.

Remark 3.2. *The multiplicativity of the norm (namely $N(t \star t') = \mathrm{N}(t)N(t')$ for every $t, t' \in C$) is equivalent to the identity*

$$\mathrm{tr}(uu')^2 + \mathrm{tr}([u, u']^2) = \mathrm{tr}(u^2)\mathrm{tr}(u'^2) \tag{30}$$

for every two trace-zero elements u, u'. To verify (30) directly, note as above that over any field K of characteristic not 2*, by the Cayley-Hamilton theorem $2z^3 - \mathrm{tr}(z^2)z$ is scalar for every $z \in \mathrm{M}_3(K)$ with trace zero; therefore if $\mathrm{tr}(u') = 0$ then $\mathrm{tr}((2z^3 - \mathrm{tr}(z^2)z)u') = 0$. Taking $z = \lambda u + u'$, the coefficient of λ^2 is*

$$4\mathrm{tr}(u^2u'^2) + 2\mathrm{tr}((uu')^2) - \mathrm{tr}(u^2)\mathrm{tr}(u'^2) - 2\mathrm{tr}(uu')^2 = 0,$$

which in characteristic 3 *is nothing but (30).*

4. $C^-(A)$ and its relatives

As before, let A be a central simple algebra of degree 3 over F, where $\mathrm{char}\, F = 3$. Let $C^-(A) = (F\zeta \oplus A_0/F, \star)$ be the algebra constructed in Theorem 3.1.

4.1. **Unital twist.** Kaplansky showed how to twist a composition algebra into a unital one [9, Prop. 33.27]. Since $\mathrm{N}(\zeta) = 1$, we define

$$a \diamond b = (\zeta \star a) \star (b \star \zeta), \tag{31}$$

and then $(C, \diamond)$ becomes a unital composition algebra, with $\zeta \star \zeta = \zeta$ its unit. Let us denote this algebra by $C^+(A) = (C, \diamond)$. The multiplication in this new algebra is given by

$$(\alpha\zeta + u) \diamond (\alpha'\zeta + u') = (\alpha\alpha' - \mathrm{tr}(uu'))\zeta + (\alpha u' + \alpha' u + [u, u']), \tag{32}$$

with the same norm and bilinear form as in $C^-(A)$.

Remark 4.1. *The idempotents in $C^+(A)$, besides 0 and ζ, are the elements $-\zeta + x$ where $x \in W_0$ has $\mathrm{tr}(x^2) = -1$.*

In particular, since $C^+(A)$ has a unit and nontrivial idempotents:

Corollary 4.2. *$C^+(A)$ is the split Cayley algebra over F.*

The standard involution on $(C, \diamond)$ is defined by

$$\bar{t} + t = \mathrm{B}(t, \zeta) \cdot \zeta,$$

which gives

$$\overline{\alpha\zeta + u} = \alpha\zeta - u. \tag{33}$$

Theorem 4.3. *For every A, $C^-(A)$ is the split para-Cayley algebra over F, and in particular it is independent of A.*

Proof. As we observed that $C^+(A)$ is the split Cayley algebra, let us compute the split para-Cayley algebra $(C, \bar{\diamond})$ in these terms. The operation is defined by

$$t \bar{\diamond} t' = \bar{t} \diamond \overline{t'},$$

which by (33) computes out to be

$$\begin{aligned} (\alpha\zeta + u)\bar{\diamond}(\alpha'\zeta + u') &= (\alpha\zeta - u) \diamond (\alpha'\zeta - u') \\ &= (\alpha\alpha' - \mathrm{tr}(uu'))\zeta + (-\alpha u' - \alpha' u + [u, u']), \end{aligned}$$

so $\bar{\diamond} = \star$ is the operation given in (4). □

It follows that the involution (33) of $(C, \diamond)$ serves as an involution for $(C, \star)$ as well. Let us record two useful properties:

Remark 4.4. *For every $t \in C$,*

1. *$\zeta \star t = t \star \zeta = \bar{t}$;*
2. *$t \star \bar{t} = \bar{t} \star t = \mathrm{N}(t)\zeta$.*

4.2. **Variations on $\star$.** The two composition algebras constructed above, one symmetric and one unital, have similar multiplication operations. In this subsection we verify that there are no other composition algebras of this form on the space $F\zeta \oplus A_0/F$.

Let $\theta, \pi_1, \pi_2, \pi \in F$ be arbitrary parameters, and consider the algebra on $F\zeta \oplus (A_0/F)$ with multiplication

(34)
$$(\alpha\zeta + u) \circ (\alpha'\zeta + u') = (\theta\alpha\alpha' + \pi_1 \mathrm{tr}(uu'))\zeta + (\pi_2(\alpha u' + \alpha' u) + \pi[u, u']).$$

Similarly we assume $\mathrm{N}(\alpha\zeta + u)$ is a linear combination of α^2 and $\mathrm{tr}(u^2)$. Using (30), one can verify that the only combination that serves as a nondegenerate multiplicative norm, is

$$N(\alpha\zeta + u) = \theta^2\alpha^2 + \pi^2 \mathrm{tr}(u^2)$$

which also requires

$$\begin{aligned} \pi_2 &= \pm\theta, \\ \pi_1 &= -\theta^{-1}\pi^2 \end{aligned}$$

where $\theta, \pi_1 \neq 0$.

Thus, for every $\theta, \pi \in F^\times$ and $\epsilon = \pm 1$, let the algebra $C^{\epsilon}_{\theta,\pi}(A)$ be the vector space $F\zeta \oplus A_0/F$ with the operation

$$(\alpha\zeta + u) \circ (\alpha'\zeta + u') = (\theta\alpha\alpha' - \theta^{-1}\pi^2 \mathrm{tr}(uu'))\zeta + (\epsilon\theta(\alpha u' + \alpha' u) + \pi[u, u']);$$

this is always a composition algebra with respect to the norm

$$N(\alpha\zeta + u) = \theta^2\alpha^2 + \pi^2 \mathrm{tr}(u^2).$$

Scaling the first and second components provides the isomorphisms $C^{\epsilon}_{\theta,\pi}(A) \cong C^{\epsilon}_{\theta',\pi'}(A)$ for every nonzero θ, θ' and π, π'. Comparing the multiplication formulas, (4) of the symmetric composition and (32) of the unital one, with the general formula (34), taking $\theta = \pi = 1$, we conclude that there are only two algebras up to isomorphism in this family, $C^+ = C^+_{1,1}(A)$ and $C^- = C^-_{1,1}(A)$.

5. Maximal isotropic subspaces in C

In this section we specialize to $C = C^-(A)$ some properties of isotropic elements and spaces in symmetric composition algebras, which were established in [8]. In turn, these follow from the work of van der Blij and Springer on isotropic subspaces of the split Cayley algebra in [12].

5.1. **maximal isotropic subspaces.** By [8, Thm. 3.1], every maximal isotropic subspace has the form $C \star t$ or $t \star C$, where t is isotropic, and those spaces uniquely characterize t up to scalar multiples. Moreover every maximal isotropic space has a unique kind, either left ($C \star t$) or right ($C \star t$).

Remark 5.1. *There are two types of isotropic elements in C.*

- *An isotropic element t with $t \star t = 0$ has the form $t = y$ where $\mathrm{tr}(y^2) = 0$ and $y \in A_0$ invertible. We call these isotropic elements* **nonseparable**.
- *An isotropic element t with $t \star t \neq 0$ is a multiple of $\zeta + x$, where $\mathrm{tr}(x^2) = -1$. We call these isotropic elements* **separable**.

For separable isotropic elements, note that $t = \zeta + x$ satisfies $t \star t = -\bar{t}$.

Remark 5.2. *In particular, the nonzero elements satisfying $t \star t = 0$ are precisely the nonseparable isotropic elements. Their existence shows that $C^-(A)$ is not reduced, again in contrast with the Okubo algebras constructed from a central division algebra of degree* 3 *in characteristic not* 3.

Fix a standard pair of generators x, y. Then $A = \sum F x^i y^j$, and the only basis element with nonzero trace is x^2. Therefore $W_0 = A_0/F = \operatorname{span}\{x, y, xy, x^2y, y^2, xy^2, x^2y^2\}$ (in fact their images in A/F), and C is spanned by

$$\zeta,\ x,\ y,\ xy,\ x^2y,\ y^2,\ xy^2,\ x^2y^2.$$

Lemma 5.3. *Every maximal isotropic space in C has one of the forms*

$$\begin{aligned} y \star C &= F(\zeta - x) + Fy + Fy^2 + Fxy^2, \\ C \star y &= F(\zeta + x) + Fy + Fy^2 + Fxy^2, \\ (\zeta + x) \star C &= F(\zeta - x) + Fy + Fxy + Fx^2y, \\ C \star (\zeta + x) &= F(\zeta - x) + Fy^2 + Fxy^2 + Fx^2y^2, \end{aligned}$$

for a suitable standard pair of generators x, y.

Corollary 5.4. *Every maximal isotropic subspace U_0 of W_0 is of one of the forms*

$$Fy + Fy^2 + Fxy^2 \qquad \text{or} \qquad F[x]y,$$

where x, y is a standard pair of generators, and is contained in exactly two maximal isotropic subspaces of C which have the form $F(\zeta \pm x) + U_0$.

Proof. The dimension is 3 because the trace form is nondegenerate on the 7-dimensional space W_0. Let U be a maximal isotropic space in C such that $U \supseteq U_0$; then $U_0 = U \cap W_0$, and can be computed from the lemma. In both cases $F(\zeta \pm x) + U_0$ are seen to be maximal isotropic spaces in C, and they are unique by general facts on quadratic forms. □

In any symmetric composition algebra, the intersection of maximal isotropic spaces is even dimensional if they have the same kind, and odd dimensional otherwise. More explicitly, we have:

Proposition 5.5 ([8, Prop. 3.7]). *Let t and t' be linearly independent isotropic elements. The intersection of $C\star t$ and $C\star t'$ has dimension 2 if $\mathrm{B}(t,t')=0$, and dimension 0 otherwise; likewise for $t\star C$ and $t'\star C$.*

Proposition 5.6 ([8, Prop. 3.8]). *Let t, t' be isotropic elements. The intersection of $t\star C$ and $C\star t'$ is one dimensional and spanned by $t\star t'$ if this is nonzero, and three dimensional otherwise.*

Corollary 5.7. *Let $t \in C^-(A)$ be an isotropic element. The spaces $C\star t$ and $\bar{t}\star C$ are conjugate to each other, and since $t\star\bar{t} = \bar{t}\star t = 0$, by Remark 4.4.(2), their intersection is 3-dimensional. We notice that the intersection is always contained in W_0: in the notation of Lemma 5.3,*

- $C\star y \cap y\star C = Fy + Fy^2 + Fxy^2$;
- $C\star(\zeta - x) \cap (\zeta + x)\star C = F[x]y$.

5.2. **Products of related subspaces.** Given a maximal isotropic subspace of $C = C^-(A)$, we wish to identify the type of its generator, be it separable or nonseparable. To this end we note the following multiplication properties.

Remark 5.8. *Let t be an isotropic element, and take $U_\ell = t\star C$ and $U_r = C\star t$, the corresponding left and right maximal isotropic subspaces.*

(1) *Assume t is nonseparable. By Lemma 5.3, $U_\ell \cap W_0 = U_r \cap W_0$; let us denote this subspace by U_0. Then we have the following multiplication table of subspaces:*

$\star$	Ft	U_0	U_ℓ	U_r
Ft	0	0	Ft	0
U_0	0	Ft	Ft	U_0
U_ℓ	0	U_0	U_r	U_0
U_r	Ft	Ft	Ft	U_ℓ

so in particular, $Ft = (U_\ell\star U_\ell)\star U_\ell = U_r\star(U_r\star U_r)$, and $U_0 = (U_r\star U_r)\star U_r = U_\ell\star(U_\ell\star U_\ell)$.

(2) *Assume that t is separable. By Lemma 5.3, $Ft + (U_\ell \cap W_0) + (U_r \cap W_0)$ is a 7-dimensional space; but clearly this space is*

orthogonal to t with respect to the form $\mathrm{B}(\cdot,\cdot)$*, so equal to* $t^\perp$*. The products are*

$$\begin{array}{c|cc} \star & U_\ell & U_r \\ \hline U_\ell & t^\perp & C \\ U_r & Ft & t^\perp \end{array}$$

and moreover, $(U_\ell \star U_\ell) \star U_\ell = U_r \star (U_r \star U_r) = t^\perp$*. At the same time,* $(U_r \star U_r) \star U_r = U_\ell$ *and* $U_\ell \star (U_\ell \star U_\ell) = U_r$*.*

This gives a neat algorithm to identify the generator and the kind of a given subspace:

Remark 5.9. *Let U be a maximal isotropic subspace. Then* $\dim(U \star U)$ *is* 4*, respectively* 7*, if U is generated by a nonseparable, respectively separable, element.*

In the nonseparable case, exactly one of $(U \star U) \star U$ and $U \star (U \star U)$ is one-dimensional: the first if U is of the left kind, and the second if U is of the right kind. In both cases, the generator of U spans this space.

In the separable case, exactly one of $(U \star U) \star U$ and $U \star (U \star U)$ is seven-dimensional: the first if U is of the left kind, and the second if U is of the right kind. In both cases, the generator of U spans the orthogonal of this space.

5.3. **Intersection of maximal isotropic subspaces.** When Proposition 5.5 is applied to $C^-(A)$, t and t' can each take either form of the elements described in Remark 5.1.

Corollary 5.10. *Fix two standard pairs x, y and x', y' of generators. In the following table,* $\dim(U_1 \cap U_2) = 2$ *if the condition holds, and the intersection is trivial otherwise.*

U_1	U_2	*the condition*
$\mathrm{span}\,\{\zeta - x, y, y^2, xy^2\}$	$\mathrm{span}\,\{\zeta - x', y', y'^2, x'y'^2\}$	$\mathrm{tr}(yy') = 0$
$\mathrm{span}\,\{\zeta + x, y, y^2, xy^2\}$	$\mathrm{span}\,\{\zeta + x', y', y'^2, x'y'^2\}$	$\mathrm{tr}(yy') = 0$
$\mathrm{span}\,\{\zeta - x, y, y^2, xy^2\}$	$F(\zeta - x') + F[x']y'$	$\mathrm{tr}(yx') = 0$
$\mathrm{span}\,\{\zeta + x, y, y^2, xy^2\}$	$F(\zeta - x') + F[x']y'^2$	$\mathrm{tr}(yx') = 0$
$F(\zeta - x) + F[x]y$	$F(\zeta - x') + F[x']y'$	$\mathrm{tr}(xx') = -1$
$F(\zeta - x) + F[x]y^2$	$F(\zeta - x') + F[x']y'^2$	$\mathrm{tr}(xx') = -1$

Proof. In Proposition 5.5, take $t = y$ and $t' = y'$ for the first two lines, $t = y$ and $t' = \zeta + x'$ for the intermediate two, and $t = \zeta + x$ and $t' = \zeta + x'$ for the last two, plugging in the spaces from Lemma 5.3 in each case. □

Similarly, following Proposition 5.6 for the intersection of maximal spaces of opposing kinds, we have:

Corollary 5.11. *Fix two standard pairs x, y and x', y' of generators. In the following table, $U_1 \cap U_2$ is spanned by the given vector if it is nonzero, and is three-dimensional otherwise.*

U_1	U_2	*the vector*
$\operatorname{span}\{\zeta - x, y, y^2, xy^2\}$	$\operatorname{span}\{\zeta + x', y', y'^2, x'y'^2\}$	$-\operatorname{tr}(yy')\zeta + [y, y']$
$\operatorname{span}\{\zeta - x, y, y^2, xy^2\}$	$F(\zeta - x') + F[x']y'^2$	$-\operatorname{tr}(yx')\zeta + [y, x'] - y$
$F(\zeta - x) + F[x]y$	$F(\zeta + x') + F[x']y'$	$(1 + \operatorname{tr}(xx'))\zeta + x' - x - [x, x']$

Proof. In Proposition 5.6, take $t = y$ and $t' = y'$ for the first line, $t = y$ and $t' = \zeta + x'$ for the second, and $t = \zeta + x$ and $t' = \zeta - x'$ for the third.

We remark that in all cases, if the second component of the vector is zero (in A_0/F) then, being isotropic, the whole vector is zero. This happens in the respective cases if $y' \in F[y] + Fxy^2$, $x' \in x + F[y] + Fxy^2$ or $x' \in x + F + F[x]y^2$ (inclusions in A rather than A/F). $\square$

6. Maximal 3-central spaces and Galois hyperplanes in A

Let A be a central simple algebra of degree 3 over a field F of characteristic 3. After analyzing the maximal isotropic subspaces of the symmetric composition algebra $C = C^-(A) = F\zeta \oplus W_0$ in the previous section, where $W_0 = A_0/F$ and $A_0 = \{x \in A\colon \operatorname{tr}(x) = 0\}$, in this section we classify the maximal 3-central subspaces and maximal Galois hyperplanes of A (see Definition 6.3). We denote by $\pi\colon A_0 \to A_0/F$ the standard projection.

For every maximal isotropic space $U \subseteq C$, the projective space $\mathbb{P}U$ can be decomposed into the disjoint union of two subsets, $\mathbb{P}U = \mathbb{P}U_Y \cup F^\times U_X/F^\times$, according to the coefficient of ζ. Namely, $U_Y = U \cap W_0$ are the elements $\pi(u) \in U$ where $u \in A$ is 3-central element, and $U_X = U \cap (\zeta + W_0)$ are the elements $\zeta + \pi(x) \in U$ such that $x \in A$ is a standard Galois element. This can be exploited to classify the maximal linear ensembles of both types in A.

Theorem 6.1. *Every maximal 3-central space of A is of one of the forms*
$$F[y] + Fxy$$
or
$$F + F[x]y$$
for a suitable standard pair of generators x, y.

Proof. Let V be a maximal 3-central space in A. Then $F \subseteq V$, and $\pi(V) \subseteq W_0$ is an isotropic space in C. Therefore, $\pi(V)$ is given by Corollary 5.4. $\square$

Remark 6.2. *One can assign a* type *to maximal* 3*-central subspaces by appealing to the type of the generator of the associated isotropic space in* C*. Alternatively, note that for* $V = F[y] + Fxy$ *we have* $[V, V] = F + Fy^2$*, while for* $V = F + F[x]y$*,* $[V, V] = F[x]y^2$*.*

Recall that a noncentral element $x \in A$ is standard Galois if $x^3 - x \in F$; equivalently, if $\mathrm{tr}(x) = 0$ and $\mathrm{tr}(x^2) = -1$. Notice that if x is standard Galois, then so is every element in $F + x$.

Definition 6.3. *A* Galois hyperplane *of* A *is a translation of a linear subspace, all of whose elements are standard Galois.*

Proposition 6.4. *Let* V *be a linear subspace of* A*, and* $x \in A$*. Then* $x + V$ *is a Galois hyperplane iff* V *is* 3*-central,* x *is standard Galois, and* $\mathrm{tr}(xV) = 0$*.*

Proof. The condition for the elements of $x + V$ to be standard Galois is that $\mathrm{tr}(x + y) = 0$ and $\mathrm{tr}((x + y)^2) = -1$ for every $y \in V$. Replacing y by λy for $\lambda \in F$, this is equivalent to x being standard Galois, $\mathrm{tr}(y) = \mathrm{tr}(xy) = 0$ and $\mathrm{tr}(y^2) = 0$ for every $y \in V$. $\square$

Clearly, every Galois hyperplane is contained in A_0.

Proposition 6.5. *For every maximal Galois hyperplane* G *there is a unique maximal isotropic subspace* $U \subseteq C$ *such that* $U_X = U \cap (\zeta + W_0)$ *is equal to* $\zeta + \pi(G)$*.*

Proof. By Proposition 6.4, G has the form $x + V$ for a maximal 3-central space V. Take $U = F(\zeta + \pi(x)) + \pi(V)$ which is isotropic by computing the norm, and then $U_X = \zeta + \pi(G)$. Uniqueness follows from Corollary 5.4 applied to $\pi(V)$, which is contained only in the maximal isotropic spaces $F(\zeta + \pi(x)) + \pi(V)$ and $F(\zeta - \pi(x)) + \pi(V)$. $\square$

Corollary 6.6. *For every maximal* 3*-central space* V *of* A *there are exactly two translations which are Galois hyperplanes; they have the form* $x + V$ *and* $-x + V$ *for a suitable* $x \in A$*.*

Proof. By Corollary 5.4, $\pi(V)$ is contained in exactly two maximal isotropic subspaces of C, which have the form $F(\zeta + \pi(x)) + \pi(V)$ and $F(\zeta - \pi(x)) + \pi(V)$ for a suitable $x \in A$. Letting U denote one of the spaces $F(\zeta \pm \pi(x)) + \pi(V)$, the required translations are $\pi^{-1}(U_X - \zeta)$ for $U_X = U \cap (\zeta + W_0)$, namely $x + V$ and $-x + V$. $\square$

Theorem 6.7. *Every maximal Galois hyperplane in A is of one of the forms*

$$\begin{gathered} x+F+F[x]y, \\ x+F+F[x]y^2, \\ x+Fxy+F[y], \\ x+Fxy^2+F[y], \end{gathered}$$

for a suitable standard pair of generators x, y.

Proof. Apply Proposition 6.5 to Lemma 5.3, replacing x, y by $-x, y^2$ when necessary. □

7. A common slot lemma

Another approach to standard pairs of generators is via common slot lemmas, where the best known example is the common slot lemma for quaternion algebras: for every two 2-central elements x, x', there is an element y such that x, y and x', y are standard pairs of generators.

Let A be a central simple algebra of degree 3 over a field of characteristic 3. Let us define a graph structure on X_A, connecting x and x' by an edge iff there is an element y such that $(x,y),(x',y) \in \mathrm{XY}_A$.

The main result of [13] is that the diameter of X_A is at most 3: for every $x, x' \in \mathrm{X}_A$, there are $x_1, x_2 \in \mathrm{X}_A$ and $y_1, y_2, y_3 \in \mathrm{Y}_A$ such that every edge in the following diagram connects a standard pair:

$$\begin{tikzcd}[column sep=small] x \arrow[dr, dash] & & x_1 \arrow[dl, dash] \arrow[dr, dash] & & x_2 \arrow[dl, dash] \arrow[dr, dash] & & x' \arrow[dl, dash] \\ & y_1 & & y_2 & & y_3 & \end{tikzcd}$$

Noting that for $x \in \mathrm{X}_A$ we also have that $-x \in \mathrm{X}_A$, we prove here a similar statement:

Theorem 7.1. *Let A be as above. For every $x \in \mathrm{X}_A$, every element of X_A is at distance at most 2 from x or from $-x$.*

This follows immediately from Proposition 7.4 below.

Corollary 7.2. *Let $[\alpha, \beta)$ and $[\alpha', \beta')$ be two p-symbol presentations of the same algebra of degree 3. Then there are $\alpha_1 \in F$ and $\beta_1, \beta_2 \in F^\times$ such that*

$$[\alpha', \beta') \cong [\alpha', \beta_1) \cong [\alpha_1, \beta_1) \cong [\alpha_1, \beta_2) \cong [\alpha, \beta_2) \cong [\alpha, \beta)$$

or

$$[\alpha', \beta') \cong [\alpha', \beta_1) \cong [\alpha_1, \beta_1) \cong [\alpha_1, \beta_2) \cong [-\alpha, \beta_2) \cong [-\alpha, \beta^{-1}).$$

Remark 7.3. *Let x, x' be standard Galois elements in A. The following are equivalent:*

(1) *x, x' are neighbors in X_A;*
(2) *for some y for which x, y is a standard pair of generators, $x' - x \in F[y]$.*
(3) *for some y for which x, y is a standard pair of generators, and for some $w \in F[x]^\times$, $x' - x \in F[wy]$.*

Indeed if x, y form a standard pair of generators then x', y is a standard pair of generators iff $x' - x \in F[y]$, and x, y' is a standard pair of generators iff $y' \in F[x]^\times y$.

Proposition 7.4. *Let $x, x' \in \mathrm{X}_A$, so that $t = \zeta + \pi(x)$ and $t' = \zeta + \pi(x')$ are isotropic elements of separable type in $C^-(A) = F \oplus A_0/F$. We let* $\mathrm{dist}(\cdot, \cdot)$ *denote the distance in X_A.*

(1) *If $t \star t'$ is separable, then* $\mathrm{dist}(x, x') \leq 2$.
(2) *If $t \star t'$ is zero or nonseparable, then* $\mathrm{dist}(-x, x') \leq 2$.

Proof. (1) By Lemma 5.3 and Proposition 5.6, $t \star C = F(\zeta - \pi(x)) + \pi(F[x]y)$ and $C \star t' = F(\zeta - \pi(x')) + \pi(F[x']y'^2)$ intersect in the one-dimensional space spanned by the nonzero element $t \star t'$, which by assumption is separable. Therefore the coefficient of ζ in nonzero elements of the intersection is nonzero, and comparing coefficients there are some $w \in F[x]$ and $w' \in F[x']$ such that

$$x + wy \equiv x' + w'y'^2 \pmod{F}.$$

If $w = 0$ or $w' = 0$ then x, x' are neighbors by Remark 7.3: if $w = w' = 0$ this is clear; if $w' = 0$ and $w \neq 0$ then x, wy and x', wy are standard pairs; and if $w = 0$ and $w' \neq 0$ then $x, (w'y'^2)^2$ and $x', (w'y'^2)^2$ are. So we assume $w, w' \neq 0$. Then $x, x + wy$ are neighbors in X_A, so $x, x' + w'y'^2$ are neighbors as well, and likewise $x', x' + w'y'^2$ are neighbors, so finally $\mathrm{dist}(x, x') \leq \mathrm{dist}(x, x' + w'y'^2) + \mathrm{dist}(x' + w'y'^2, x') \leq 1 + 1 = 2$.

(2) The coefficient of ζ in $t \star t'$ is $1 - \mathrm{tr}(xx')$, and since by assumption this is zero, the coefficient in $(\zeta - \pi(x)) \star t'$, which is $1 + \mathrm{tr}(xx')$, is nonzero. Replacing x with $-x$, we are thus back in case (1). □

8. Strong orthogonality

In [8] we called isotropic elements t, t' in an arbitrary symmetric composition algebra C *strongly orthogonal*, denoted $t \rightsquigarrow t'$ or $t' \leftsquigarrow t$, if $t \star t' = 0$ and $\mathrm{B}(t, t') = 0$ (this relation is not symmetric). The motivation for this definition is that for C the algebra constructed by Okubo from a central simple algebra of degree 3, in characteristic not 3 and when the base field has third roots of unity ρ, x, x' are strongly orthogonal in C if and only if they form a standard pair of generators for A (namely $x'x = \rho x x'$).

Let A be a central simple algebra of degree 3 over a field of characteristic 3. Recall that $C^-(A) = F\zeta \oplus A_0/F$ and $\pi : A_0 \to A_0/F$ is the standard projection, where $A_0 = \{u \in A\colon \operatorname{tr}(u) = 0\}$.

Proposition 8.1. *Let t, t' be elements in $C^-(A)$. Then $t \rightsquigarrow t'$ if and only if there is a standard pair of generators x, y of A, such that one of the following cases holds (where $\sim$ denotes equality up to multiplication by a nonzero scalar from F):*

1. *$t \sim \zeta + \pi(x)$ and $t' = \pi(y^{-1})$.*
2. *$t = \pi(y)$ and $t' \sim \zeta + \pi(x)$.*
3. *$t = \pi(y)$ and $t' = \pi(y')$ where $\pi([y, y']) = 0$.*

Proof. We need to solve $t \star t' = 0$ with $\mathrm{B}(t, t') = 0$. Since $\mathrm{N}(t)\mathrm{N}(t') = 0$, at least one of the vectors is isotropic; assume t is isotropic, then $t' \in \operatorname{Ker}(L_t) = \operatorname{Im}(R_t)$ so t' is isotropic as well, and likewise if t' is isotropic, then t is isotropic as well. Thus both t and t' are isotropic.

By Remark 5.1 there are four options: each of t and t' is either separable or nonseparable. The two elements cannot both be separable, because if $t = \zeta + \pi(x)$ and $t' = \zeta + \pi(x')$ then $t \star t' = 0$ forces $\operatorname{tr}(xx') = 1$, while $\mathrm{B}(t, t') = 0$ forces $\operatorname{tr}(xx') = -1$. Assume t is separable, and write $t \sim \zeta + \pi(x)$ where $x \in \mathrm{X}_A$. Then $t' \in C \star t = F(\zeta - \pi(x)) + \pi(F[x]y^2)$ where x, y is a standard pair by Lemma 5.3, but since t' is nonseparable it belongs to $\pi(F[x]y^2) = \pi(F[x]y^{-1})$, so multiplying y by a suitable element of $F[x]^\times$ we get a new standard pair of generators x, y with $t' = \pi(y^{-1})$. Similarly assume t' is separable and write $t' = \zeta + \pi(x)$. Then $t \in t' \star C = F(\zeta - x) + \pi(F[x]y)$, and the same argument applies.

Finally if t, t' are both nonseparable then we can write $t = \pi(y)$ and $t' = \pi(y')$ for some 3-central elements y and y' in A, and $t \star t' = \operatorname{tr}(yy')\zeta + [y, y']$, but $[\pi(y), \pi(y')] = 0$ if and only if $y' \in F[y] + Fxy^2$ (where x, y are a standard pair of generators), and then always $\operatorname{tr}(yy') = 0$. $\square$

The cases in Proposition 8.1 can be distinguished on the outset: $t' \star t \sim t'$ in the first case, $t' \star t \sim t$ in the second case, and $t' \star t = 0$ in the third. In particular if t, t' are nonseparable isotropic elements, then $t \rightsquigarrow t'$ iff $t' \rightsquigarrow t$. This observation motivates the following notion, which we do not pursue here.

Definition 8.2. *For isotropic elements t, s in a symmetric composition algebra C, we denote $t \rightharpoonup s$ if $t = s \star t$.*

Remark 8.3.
(1) *If $t \rightharpoonup s$ then $t \rightsquigarrow s$ since $t \star s = (s \star t) \star s = \mathrm{N}(s)t = 0$ and $\mathrm{B}(t,s) = \mathrm{B}(s \star t, s) = \mathrm{N}(s)t = 0$.*
(2) *In $C = C^-(A)$, $t \rightharpoonup s$ iff $t = \pi(y)$ and $s = \zeta + \pi(x)$ for some standard pair of generators x, y in A. (By Proposition 8.1 and the observation on $t \star t'$ following it).*

Remark 8.4. *Lifting strong orthogonality in C to relations in A is somewhat tricky. In spite of Proposition 8.1, the fact that $y \rightsquigarrow \zeta + \pi(x)$ for $x \in \mathrm{X}_A$ and $y \in \mathrm{Y}_A$ does not imply that $(x, y) \in \mathrm{XY}_A$, because the computation is done in $\pi(A_0)$. What we can conclude is that an element $y' \in A$ exists such that $(x, y') \in \mathrm{XY}_A$, and $\pi(y') = \pi(y)$. (On the other hand $(x, y') \in \mathrm{XY}_A$ iff $(x', y') \in \mathrm{XY}_A$ for every x' with $\pi(x') = \pi(x)$).*

Proposition 8.5. *Let $y, y' \in \mathrm{Y}_A$. Then*

$$\pi(y) \rightsquigarrow * \rightsquigarrow \pi(y')$$

can be completed with a nonzero isotropic vector $y'' \in \mathrm{Y}_A$, iff $\mathrm{tr}(yy') = 0$.

(Note that the relation between elements of Y_A is symmetric, so the directions of the arrows is immaterial).

Proof. The condition for $\pi(y) \rightsquigarrow \pi(y'')$ is that $[\pi(y), \pi(y'')] = 0$, or equivalently $y'' \in F[y] + Fxy^2$ where $x \in \mathrm{X}_A$ completes y to a standard pair of generators. To solve $\pi(y'') \rightsquigarrow \pi(y')$ as well, we need $(F[y] + Fxy^2) \cap (F[y'] + Fx'y'^2) \neq 0$, where x', y' is a standard pair of generators. But this implies

$$\mathrm{span}\left\{\zeta - x, y, y^2, xy^2\right\} \cap \mathrm{span}\left\{\zeta - x', y', y'^2, x'y'^2\right\} \neq 0,$$

and is implied by the intersection having dimension 2, so we are done by the first line of Proposition 5.10. □

Corollary 8.6. *For any $y \in \mathrm{Y}_A$ and any isotropic element $t' \in C$ one can complete a chain of length 3 from y to t', in any direction of the arrows, with nonseparable isotropic vectors.*

Proof. First assume $t' \sim \zeta + \pi(x')$ for $x' \in \mathrm{X}_A$. Complete x' to a standard pair of generators x', y'. The space $F[x']y'y \subseteq A$ has dimension 3, so there is a nonzero element $f \in F[x']$ such that $\mathrm{tr}(fy'y) = 0$. Replacing y' by fy', we can complete

$$\pi(y) \rightsquigarrow * \rightsquigarrow \pi(y') \rightsquigarrow \zeta + \pi(x')$$

by Proposition 8.5. For the reverse direction of the final arrow consider $F[x']y'^{-1}y$ and replace y' by an element such that $\mathrm{tr}(y'^{-1}y) = 0$, to complete

$$\pi(y) \rightsquigarrow * \rightsquigarrow \pi(y'^{-1}) \leftsquigarrow \zeta + \pi(x').$$

If $t' = \pi(y'')$ for $y'' \in \Upsilon_A$, then choose $0 \neq y' \in F[y'']$ subject to the single linear condition $\mathrm{tr}(yy') = 0$; thus $\pi(y') \rightsquigarrow \pi(y'')$ and we finish by Proposition 8.5. As before, the direction of the arrows connecting elements of Υ_A is immaterial. ☐

Corollary 8.7. *Any two isotropic elements $t, t' \in C$ can be connected by a chain of length 4 of the form $t \rightsquigarrow * \rightsquigarrow * \rightsquigarrow * \rightsquigarrow t'$, and likewise for any direction of the arrows.*

Proof. Connect t' to a suitable nonseparable element and finish by Corollary 8.6. ☐

Remark 8.8. *In Theorem 8.8 of* [8] *we proved a similar result, for an arbitrary symmetric composition algebra, for six of the ten possible directions of the arrows (up to symmetry). Corollary 8.7 is a stronger statement for $C = C^-(A)$.*

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